GENERALIZED BOUSFIELD LATTICES AND A GENERALIZED RETRACT CONJECTURE

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Let $\mathcal{M}$ be a closed symmetric monoidal category with zero object, and consider an object M of it. We call a full subcategory $\langle M \rangle$ of $\mathcal{M}$ a *Bousfield class* of M if it consists of objects A of $\mathcal{M}$ such that $MA = 0$ by its monoidal structure. Then we have a partial order on Bousfield classes by $\langle M \rangle \le \langle N \rangle$ if every object of $\langle N \rangle$ is an object of $\langle M \rangle$. Then the subcategories $\langle S \rangle$ and $\langle O \rangle$ of the unit S and the zero O are the greatest and the least ones in the order, respectively. We call a collection of Bousfield classes a *Bousfield lattice*, and denote it by $\mathbb{B}(\mathcal{M})$. In a case where a Bousfield lattice is a set, the partial order introduces a lattice structure to it, and we may investigate it algebraically.

In a sense, the stable homotopy theory is analyzing stable homotopy categories (*cf.* [4]). A stable homotopy category is a symmetric monoidal category, and so we may consider its Bousfield lattice. In particular, T. Ohkawa [6] (*cf.* [1]) showed that the Bousfield lattice $\mathbb{B}$ of the stable homotopy category of spectra is a set, and then Iyengar-Krause [5] generalized it to a stable homotopy category.

In order to investigate a category, we sometimes classify special subcategories of it. From a viewpoint of this, we study a Bousfield lattice by classifying localizing subcategories (see [4]). Indeed, every Bousfield class is a localizing subcategory.

In [3], Hovey and Palmieri studied the Bousfield lattice $\mathbb{B}$ deeply. Furthermore, they proposed many conjectures on the structure of $\mathbb{B}$. Among them, there is the retract conjecture, which is one of our main topic. Dwyer and Palmieri [2] constructed a stable homotopy category, where the conjecture does not hold. So far, there seems no nontrivial category in which the conjecture holds. Here, we give some examples of categories with the affirmative answer to the conjecture.

Let $\mathcal{L}_E$ for a spectrum E be the stable homotopy category of E-local spectra, and $\mathbb{B}(\mathcal{L}_E)$ denote the Bousfield lattice in the sense of Bousfield. We consider the Johnson-Wilson spectra $E(n)$ and the Morava K-theories $K(n)$ for $n \ge 0$. By the chromatic viewpoint, investigating the categories $\mathcal{L}_n (= \mathcal{L}_{E(n)})$ and $\mathcal{L}_{K(n)}$ is one of main targets of stable homotopy theory. We determine the Bousfield lattices of these categories.

Theorem 1. *Let $E = \bigvee_{i \in F} K(i)$ be a spectrum for a finite subset F of $\mathbb{Z}_{\ge 0}$. Then $\mathbb{B}(\mathcal{L}_E)$ is isomorphic to $\prod_{i \in F} \mathbb{Z}/2$.*

The chromatic tower $\mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \cdots$ induces the inverse system

$$\mathbb{B}(\mathcal{L}_0) \leftarrow \mathbb{B}(\mathcal{L}_1) \leftarrow \mathbb{B}(\mathcal{L}_2) \leftarrow \cdots .$$

Moreover, we notice that $B_\infty := \lim_n \mathbb{B}(\mathcal{L}_n) = \prod_n \mathbb{Z}/2$ as a lattice. We call a spectrum *harmonic* if it is $(\bigvee_{i \ge 0} K(i))$-local.

Theorem 2. *Let $\mathcal{H}$ be the stable homotopy category of harmonic spectra. Then $\mathbb{B}(\mathcal{H})$ is isomorphic to B_∞ as a lattice.*

References

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