Determination of the 2-primary components in 32-stem unstable homotopy groups of spheres

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We work in the 2-primary components of homotopy groups of spheres. Let η, ν, σ, ε, $\bar{\nu}$ μ, ζ, κ, ρ, ω, η^*, $\bar{\mu}$, ν^*, $\bar{\zeta}$, $\bar{\sigma}$, $\bar{\kappa}$, ϕ, $\bar{\rho}$, δ, μ_3, ζ_3, θ', κ^*, ω^*, ρ_3 be the generators of π_n^s $(1 \leq n \leq 31)$. This work begins with Toda: *Composition methods in the homotopy groups of spheres, Ann. of Math. Studies, 49(1962)* and was relayed by Mimura, Mori and Oda. Oda: *Unstable homotopy groups of spheres, Inst. Adv. Res. Fukuoka Univ. 44(1979)* determined π_{n+32}^n for $2 \leq n \leq 8$. The purpose of this talk is to determine it for $n \geq 9$. The tool is the EHP sequence $\cdots \longrightarrow \pi_{n+k-1}^{n-1} \xrightarrow{E} \pi_{n+k}^n \xrightarrow{H} \pi_{n+k}^{2n-1} \xrightarrow{P} \pi_{n+k-2}^{n-1} \longrightarrow \cdots$.

The stable group π_{32}^s is determined by Barratt-Mahowald-Tangora: *Some differentials in the Adams spectral sequence I, II, Topology, 6, 9(1967, 70)* $\pi_{32}^s = \{\eta_4, [[q]], d_1, \eta J_{31}\} \cong (\mathbb{Z}_2)^4$, $\langle \sigma^2, \eta, \sigma^2, \eta, \sigma^2 \rangle \subset d_1$.

The key step is to find the unstable elements corresponding to $[[q]], d_1$. According to Kochman: *Homotopy groups of stable homotopy groups, Lecture Note on Math. 1423(1990)*, $\langle \bar{\sigma}, \sigma, \nu, \eta \rangle$ is taken as a representative of d_1.

Lemma. *1. There exists an element $\overline{\nu\bar{\kappa}_{10}} \in \pi_{42}^{10}$ such that $H(\overline{\nu\bar{\kappa}_{10}}) = \nu_{19}\bar{\kappa}_{22}$ and $\overline{\nu\bar{\kappa}}$ corresponds to $[[q]]$. 2. There exist the relations $\omega_{14}\sigma_{30}^2 = \psi_{14}\sigma_{37}$, $\psi_{13}\sigma_{36} \in \{\bar{\sigma}_{13}, \sigma_{32}, \nu_{39}\} \mod \zeta_{3,13}\nu_{40} = 8\sigma_{13}\bar{\rho}_{20}$ and an element $\bar{\bar{\sigma}}_{14}$ such that*

$$\bar{\bar{\sigma}}_{14} \in \left\{ \begin{matrix} \omega_{14} & \sigma_{30}^2 \\ & , \\ E\bar{\bar{\sigma}}_{13} & E\tilde{\nu}_{39} \end{matrix} , \eta_{44} \right\}_1 \ (\textit{matrix Toda bracket}),$$

$H(\bar{\bar{\sigma}}_{14}) \equiv \bar{\sigma}_{27} \mod 4\bar{\zeta}_{27}$ and $\bar{\bar{\sigma}} \in \langle \bar{\sigma}, \sigma, \nu, \eta \rangle$. Here the extension $\bar{\bar{\sigma}}_{13}$ and coextenstion $\tilde{\nu}_{39}$ with respect to σ_{32} are taken satisfying $\bar{\bar{\sigma}}_{13} \circ \tilde{\nu}_{39} = \psi_{13}\sigma_{36}$ and $\tilde{\nu}_{39}\eta_{43} = 0$.

In the following result, the notation $\alpha \longrightarrow \beta$ stands for $H(\alpha) = \beta$.

Theorem. $\pi_{34}^2 = \{\eta_2\delta_3\sigma_{27}, \eta_2\varepsilon_3\nu_{11}\bar{\kappa}_{14}, \eta_2\nu'\varepsilon_6\bar{\kappa}_{14}, \eta_2\phi'\nu_{28}^2\} \cong (\mathbb{Z}_2)^4$; $\delta_3 \longrightarrow \nu_5\bar{\sigma}_8 \mod \nu_5\bar{\zeta}_8$, $\phi' \longrightarrow \phi_5 \mod 4\nu_5\bar{\kappa}_8$, $\bar{\rho}'''$; $\phi_5 \longrightarrow \bar{\sigma}_9$.

$\pi_{35}^3 = \{\phi'\sigma_{28}, \nu'\eta_6\varepsilon_7\bar{\kappa}_{15}, \mu_{3,3}\sigma_{28}\} \cong \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$; $\mu_{3,3} \longrightarrow \bar{\rho}''' \longrightarrow 4\bar{\zeta}_9$.

$\pi_{36}^4 = \{\nu_4\delta'', \nu_4\sigma'\varepsilon_{14}\kappa_{22}, \nu_4\sigma'\omega_{14}\nu_{30}^2, \nu_4\phi_7\nu_{30}^2, \nu_4\eta_7\varepsilon_8\bar{\kappa}_{16}, (E\phi')\sigma_{29}, (E\nu')\eta_7\varepsilon_8\bar{\kappa}_{16}, \mu_{3,4}\sigma_{29}\} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$; $\delta'' \longrightarrow \phi_{13} \mod 4\nu_{13}\bar{\kappa}_{16}$.

$\pi_{37}^5 = \{\phi''\sigma_{30}, \nu_5\eta_8\varepsilon_9\bar{\kappa}_{17}, \mu_{3,5}\sigma_{30}\} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^2$; $\phi'' \longrightarrow \sigma_9^3$.

$\pi^6_{38} = \{G^{(2)}_0, P(\sigma_{13}\bar\kappa_{20}), P(EA^{\mathrm{II}}_1), P((EA^{(1)}_1)\sigma_{33}), (E\phi'')\sigma_{31}, \mu_{3,6}\sigma_{31}\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$; $G^{(2)}_0 \longrightarrow \zeta_{3,11} \mod 2\zeta_{3,11}$; $\zeta_{3,5} \longrightarrow 8\bar\rho_9$; $\bar\rho_9 \longrightarrow 16\rho_{17}$; $A^{\mathrm{II}}_1 \longrightarrow \omega_{23} \mod \sigma_{23}\mu_{30}$, $A^{(1)}_1 \longrightarrow \mu_{23} \mod \nu^3_{23}, \eta_{23}\varepsilon_{24}$.

$\pi^7_{39} = \{\sigma'\mu_{3,14}, \sigma'\eta_{14}\sigma_{15}\bar\mu_{22}, (E^2\phi'')\sigma_{32}, \mu_{3,7}\sigma_{32}\} \cong (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$.

$\pi^8_{40} = \{\sigma_8\xi_{15}\sigma_{33}, \sigma_8\eta^{*'}\mu_{31}, \sigma_8\mu_{3,15}, \sigma_8\eta_{15}\bar\mu_{16}\sigma_{33}, (E\sigma')\mu_{3,15}, (E\sigma')\eta_{15}\sigma_{16}\bar\mu_{23}, (E^3\phi'')\sigma_{33}, \mu_{3,8}\sigma_{33}\} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^5 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$.

$\pi^9_{41} = \{\sigma_9\xi_{16}\sigma_{34}, \sigma_9\mu_{3,16}, \sigma^2_9\eta_{23}\bar\mu_{24}, \mu_{3,9}\sigma_{34}\} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^3$.

$\pi^{10}_{42} = \{[\iota_{10}, \bar\rho_{10}], \overline{\nu\bar\kappa}_{10}, \sigma_{10}\xi_{17}\sigma_{35}, \sigma_{10}\mu_{3,17}\} \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

$\pi^{11}_{43} = \{\overline{\nu\bar\kappa}_{11}, \sigma_{11}\mu_{3,18}, \sigma_{11}\xi_{18}\sigma_{36}, \lambda'\kappa_{29}\} \cong (\mathbb{Z}_2)^4$, $\lambda' \longrightarrow \varepsilon_{21}$.

$\pi^{12}_{44} = \{\theta\bar\kappa_{24}, \xi_{12}\sigma^2_{30}, \overline{\nu\bar\kappa}_{12}, \sigma_{12}\mu_{3,19}, (E\lambda')\kappa_{30}\} \cong (\mathbb{Z}_2)^5$.

$\pi^{13}_{45} = \{\lambda\kappa_{31}, (E\theta)\bar\kappa_{25}, \xi_{13}\sigma^2_{31}, \overline{\nu\bar\kappa}_{13}, \sigma_{13}\mu_{3,20}\} \cong (\mathbb{Z}_2)^5$, $\lambda \longrightarrow \nu^2_{25}$.

$\pi^{14}_{46} = \{\bar\zeta^*, \bar{\bar\sigma}_{14}, \overline{\nu\bar\kappa}_{14}, (E\lambda)\kappa_{32}, \xi_{14}\sigma^2_{32}, \sigma_{14}\mu_{3,21}\} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^5$; $\{\zeta^*, 8\iota_{38}, 2\sigma_{38}\}_1 \ni \bar\zeta^* \longrightarrow \bar\zeta_{27} \mod 2\bar\zeta_{27}$; $\zeta^* \longrightarrow \zeta_{27} \mod 2\zeta_{27}$.

$\pi^{15}_{47} = \{\overline{\nu\bar\kappa}_{15}, \bar{\bar\sigma}_{15}, \eta^{*'}\eta^*_{31}, \mu^{*''}, E\bar\zeta^*, (E^2\lambda)\kappa_{32}, \xi_{15}\sigma^2_{33}, \sigma_{15}\mu_{3,22}\} \cong (\mathbb{Z}_2)^8$; $\{\mu^{*'}, 2\iota_{39}, 8\sigma_{39}\}_1 \ni \mu^{*''} \longrightarrow \eta_{29}\bar\mu_{30}$; $\mu^{*'} \longrightarrow \eta_{29}\mu_{30}$.

$\pi^{16}_{48} = \{\overline{\nu\bar\kappa}_{16}, \bar{\bar\sigma}_{16}, \eta^{*2}_{16}, \mu^{**}_{16}, \nu^*_{16}\kappa_{34}, \eta^*_{16}\eta_{32}\rho_{33}, (E\eta^{*'})\eta^*_{32}, E\mu^{*''}, E^2\bar\zeta^*, (E^3\lambda)\kappa_{33}, \xi_{16}\sigma^2_{34}, \sigma_{16}\mu_{3,23}\} \cong (\mathbb{Z}_2)^{12}$; $\{\mu^*_{16}, 2\iota_{40}, 8\sigma_{40}\}_1 \ni \mu^{**}_{16} \longrightarrow \bar\mu_{31}$; $\mu^*_{16} \longrightarrow \mu_{31}$.

$\pi^{17}_{49} = \{\overline{\nu\bar\kappa}_{17}, \bar{\bar\sigma}_{17}, \eta^{*2}_{17}, \mu^{**}_{17}, \nu^*_{17}\kappa_{35}, \eta^*_{17}\eta_{33}\rho_{34}, \xi_{17}\sigma^2_{35}, \sigma_{17}\mu_{3,24}\} \cong (\mathbb{Z}_2)^8$.

$\pi^{18}_{50} = \{[\iota_{18}, \rho_{18}], \bar{\bar\varepsilon}_{18}, \overline{\nu\bar\kappa}_{18}, \bar{\bar\sigma}_{18}, \xi_{18}\sigma^2_{36}, \sigma_{18}\mu_{3,25}\} \cong \mathbb{Z}_{32} \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^4$; $\bar{\bar\varepsilon}_{18} \longrightarrow \bar\varepsilon_{35}$.

$\pi^{19}_{51} = \{\bar{\bar\varepsilon}_{19}, \overline{\nu\bar\kappa}_{19}, \bar{\bar\sigma}_{19}, \xi_{19}\sigma^2_{37}, \sigma_{19}\mu_{3,26}\} \cong \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^4$.

$\pi^n_{n+32} = \{\bar{\bar\varepsilon}_n, \overline{\nu\bar\kappa}_n, \bar{\bar\sigma}_n, \sigma_n\mu_{3,n+7}\} \cong (\mathbb{Z}_2)^4$ $(n = 20, 21)$.

$\pi^{22}_{54} = \{\bar\zeta^{*'}, \overline{\nu\bar\kappa}_{22}, \bar{\bar\varepsilon}_{22}, \bar{\bar\sigma}_{22}, \sigma_{22}\mu_{3,29}\} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^4$; $\{[\iota_{22}, \iota_{22}], \eta_{43}, \eta^2_{44}\sigma_{46}\}_1 \ni \bar\zeta^{*'} \longrightarrow \zeta_{43} \mod 2\zeta_{43}$.

$\pi^{23}_{55} = \{\tilde\eta^{*'}, \theta'_{23}\eta^2_{53}, \bar{\bar\varepsilon}_{23}, \overline{\nu\bar\kappa}_{23}, \bar{\bar\sigma}_{23}, \sigma_{23}\mu_{3,30}\} \cong (\mathbb{Z}_2)^6$; $\{\tilde\eta', 2\iota_{47}, 8\sigma_{47}\}_1 \ni \tilde\eta^{*'} \longrightarrow \eta_{45}\mu_{46}$.

$\pi^{24}_{56} = \{\theta'_{24}\eta^2_{54}, E\tilde\eta^{*'}, \tilde\eta\eta_{48}\sigma_{49}, \tilde\eta\bar\nu_{48}, \tilde\eta^*, \bar{\bar\varepsilon}_{24}, \overline{\nu\bar\kappa}_{24}, \bar{\bar\sigma}_{24}, \sigma_{24}\mu_{3,31}\} \cong (\mathbb{Z}_2)^9$; $\tilde\eta \longrightarrow \eta_{47}$, $\{\tilde\eta, 2\iota_{48}, 8\sigma_{48}\}_1 \ni \tilde\eta^* \longrightarrow \mu_{47} \mod \eta^2_{47}\sigma_{49}, \nu^3_{47}$.

$\pi^{25}_{57} = \{(E\tilde\eta)\eta_{49}\sigma_{50}, (E\tilde\eta)\bar\nu_{49}, E\tilde\eta^*, \overline{\nu\bar\kappa}_{25}, \bar{\bar\sigma}_{25}, \sigma_{25}\mu_{3,32}\} \cong (\mathbb{Z}_2)^6$.

$\pi^{26}_{58} = \{[\iota_{26}, \sigma_{26}], \overline{\nu\bar\kappa}_{26}, \bar{\bar\sigma}_{26}, \sigma_{26}\mu_{3,33}\} \cong \mathbb{Z}_{16} \oplus (\mathbb{Z}_2)^3$.

$\pi^n_{n+32} = \{\overline{\nu\bar\kappa}_n, \bar{\bar\sigma}_n, \sigma_n\mu_{3,n+7}\} \cong (\mathbb{Z}_2)^3$ $(n = 27, 28, 29)$.

$\pi^{30}_{62} = \{\eta_{4,30}, \overline{\nu\bar\kappa}_{30}, \bar{\bar\sigma}_{30}, \sigma_{30}\mu_{3,37}\} \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^3$; $\eta_{4,30} \longrightarrow \nu_{59}$.

$\pi^{31}_{63} = \{\eta_{4,31}, \theta'^*, \overline{\nu\bar\kappa}_{31}, \bar{\bar\sigma}_{31}, \sigma_{31}\mu_{3,38}\} \cong (\mathbb{Z}_2)^5$; $\{\theta'_{31}, 4\iota_{61}, \eta_{61}\}_1 \ni \theta'^* \longrightarrow \eta^2_{61}$.

$\pi^{32}_{64} = \{\eta_{4,32}, \theta''_{32}, [\iota_{32}, \eta_{32}], \overline{\nu\bar\kappa}_{32}, \bar{\bar\sigma}_{32}, \sigma_{32}\mu_{3,39}\} \cong (\mathbb{Z}_2)^6$; $\{\theta'_{32}, 2\iota_{62}, \eta_{62}\}_1 \ni \theta''_{32} \longrightarrow \eta_{63}$.

$\pi^{33}_{65} = \{\eta_{4,33}, \theta''_{33}, \overline{\nu\bar\kappa}_{33}, \bar{\bar\sigma}_{33}, \sigma_{33}\mu_{3,40}\} \cong (\mathbb{Z}_2)^5$.

$\pi^n_{n+32} = \{\eta_{4,n}, \overline{\nu\bar\kappa}_n, \bar{\bar\sigma}_n, \sigma_n\mu_{3,n+7}\} \cong (\mathbb{Z}_2)^4$; $\theta''_n \equiv \eta_{4,n} \mod \overline{\nu\bar\kappa}_n, \bar{\bar\sigma}_n, \sigma_n\mu_{3,n+7}$; $n \geq 34$.