CONSTRUCTIONS OF BETA ELEMENTS OF STABLE HOMOTOPY OF SPHERES

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Let p be a prime number p greater than three. We work in the stable homotopy category $\mathcal{S}_{(p)}$ of spectra localized at the prime p. Consider the Adams-Novikov spectral sequence converging to the stable homotopy groups $\pi_*(S^0)$ of spheres with E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*),$$

where BP denotes the Brown-Peterson spectrum. In [1], Miller, Ravenel and Wilson defined the generalized Greek letter elements in the E_2-term. In particular, we have the beta elements.

Theorem 1 ([1, Th. 2.6])**.** *There are generators*

$$\beta_{sp^n/j,i+1} \in E_2^{2,(sp^n(p+1)-j)q} \quad (q = 2p-2)$$

of order p^{i+1} for $n \ge 0$, $p \nmid s \ge 1$, $j \ge 1$, $i \ge 0$ subject to

1) $j \le p^n$ *if* $s = 1$,
2) $p^i \mid j \le p^{n-i} + p^{n-i-1} - 1$, *and*
3) $a_{n-i-1} < j$ *if* $p^{i+1} \mid j$.

We abbreviate $\beta_{s/t,1}$ and $\beta_{s/1,1}$ to $\beta_{s/t}$ and β_s.

It is an interesting problem which of them survives in the spectral sequence. So far, the following elements are known to be permanent cycles:

a) β_s for $s \ge 1$ in [12],
b) $\beta_{sp/t}$ for $s \ge 1$ and $t \le p$, and $t < p$ if $s = 1$ in [2], [3],
c) $\beta_{sp^2/t}$ for $s \ge 1$ and $t \le 2p$, and $t \le 2p-2$ if $s = 1$ in [2], [4],
d) $\beta_{sp^2/t}$ for $s \ge 1$ and $t \le p^2 - 2$ in [11],
e) $\beta_{sp^n/t}$ for $s \ge 1$, $n \ge 3$, $1 \le t \le 2^{n-2}p$, and $t \le 2^{n-2}(p-1)$ if $s = 1$, in [6], [7],
f) $\beta_{sp^2/p,2}$ for $s \ge 2$ in [4], and
g) $\beta_{sp^n/up,2}$ for $s \ge 1$, $n \ge 3$, $1 \le u \le 2^{n-2}$, and $up \le 2^{n-2}(p-1)$ if $s = 1$, in [6], [7].

Furthermore, Ravenel showed that β_{p^n/p^n} cannot be a permanent cycle for $n \ge 1$ (*cf.* [9, 6.4.2. Th.]). Thus, the beta elements $\beta_{sp^n/t}$ for $2^{n-2}p < t \le p^n + p^{n-1} - 1$ and $t < p^n$ if $s = 1$ are left undetermined.

Consider the subsets of the E_2-term:

$$\begin{aligned}
\mathfrak{B}_{Oka}(s,u) &= \mathfrak{B}^0_{Oka}(s,u) \cup \mathfrak{B}^1_{Oka}(s,u),\\
\mathfrak{B}^0_{Oka}(s,u) &= \{\beta_{skp^n/t} : k \ge 1,\ n \ge 0,\ 1 \le t \le 2^n u\}\\
\mathfrak{B}^1_{Oka}(s,u) &= \{\beta_{skp^n/tp,2} : k \ge 1,\ n \ge 1,\ t \ge 1,\ tp \le 2^{n-1}u\}.
\end{aligned}$$

Let K_u denotes the cofiber of $\alpha^u\colon \Sigma^{uq}M \to M$ for the mod p Moore spectrum M and the Adams map α. Then, $BP_*(K_u) = BP_*/(p, v_1^u)$, and we consider an element $f_{s,u} \in \pi_*(K_u)$ such that $BP_*(f_{s,u}) = v_2^s \in BP_*(K_u)$.

Theorem 2 ((Oka [6], [7]). *If $f_{s,u} \in \pi_*(K_u)$ exists, then every element of $\mathfrak{B}_{Oka}(s,u)$ survives to $\pi_*(S)$.*

We here consider the subsets:

$$\begin{array}{rcl} \mathfrak{B}(p^i,u) & = & \mathfrak{B}^0(p^i,u) \cup \mathfrak{B}^1(p^i,u), \\ \mathfrak{B}^0(p^i,u) & = & \{\overline{\beta}_{sp^{i+n}/t} : s \geq 1,\ n \geq 0,\ 1 \leq t \leq 2^n u - 2\} \\ \mathfrak{B}^1(p^i,u) & = & \{\overline{\beta}_{sp^{i+n}/tp,2} : s \geq 1,\ n \geq 1,\ t \geq 1,\ tp \leq 2^{n-1}u - p\} \end{array}$$

We further consider the cofiber W of the beta element $\beta_1 \in \pi_{pq-2}(S^0)$, and an element $f_{p^i,u} \in \pi_*(W \wedge K_u)$ such that $BP_*(f_{s,u}) = v_2^s \in BP_*(W \wedge K_u)$. Then, we have a similar theorem.

Theorem 3. *If $f_{p^i,u} \in \pi_*(W \wedge K_u)$ exists, then every element of $\mathfrak{B}(p^i,u)$ survives to $\pi_*(S)$.*

In [11, Th. 1.7], we showed the existence of $f_{p^2,p^2} \in \pi_*(W \wedge K_{p^2})$ for $p > 5$, though there does not exist $f_{p^2,p^2} \in \pi_*(K_{p^2})$ shown by Ravenel.

Corollary 4. *Let $p > 5$. Then, $\mathfrak{B}(p^2,p^2)$ yields a beta family of $\pi_*(S)$.*

In other words, we have following beta elements generating subgroups of $\pi_*(S)$:

e′) $\beta_{sp^n/t}$ for $s \geq 1$, $n \geq 2$, $1 \leq t \leq 2^{n-2}p^2 - 2$, and

g′) $\beta_{sp^n/up,2}$ for $s \geq 1$, $n \geq 3$, $1 \leq u \leq 2^{n-3}p - 1$.

This improves Oka's results if the prime number p is greater than five.

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