MAPPING SPACES FROM PROJECTIVE SPACES

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Abstract. We denote the n-th projective space of a topological monoid G by \( B_nG \) and the classifying space by \( BG \). Let \( G \) be a topological monoid of pointed homotopy type of a CW complex and \( G' \) a grouplike topological monoid. We prove the weak equivalence between the pointed mapping space \( \text{Map}_0(B_nG, BG) \) and the space of all \( A_n \)-maps from \( G \) to \( G' \). This fact has several applications. As the first application, we show that the connecting map \( G \to \text{Map}_0(B_nG, BG) \) of the evaluation fiber sequence \( \text{Map}_0(B_nG, BG) \to \text{Map}(B_nG, BG) \to BG \) is delooped. As other applications, we consider higher homotopy commutativity, \( A_n \)-types of gauge groups and \( T^k \)-spaces by Iwase–Mimura–Oda–Yoon.

We define the topological categories \( \mathcal{A}_n \) of topological monoids and \( A_n \)-maps, \( \mathcal{A}_n^R \) of spaces equipped with right actions of topological monoids and \( A_n \)-equivariant maps. Similarly, \( \mathcal{A}_n^L \) is defined. We will give the continuous bar construction functor
\[
B_n : \mathcal{A}_n^R \times \mathcal{A}_n \to \mathcal{A}_n^L \to \mathcal{CG},
\]
where \( \mathcal{CG} \) is the category of compactly generated spaces. There is the natural inclusion \( \iota_n : B_n(X, G, Y) \to B(X, G, Y) = B_\infty(X, G, Y) \). We also denote the n-th projective space functor by
\[
B_n : \mathcal{A}_n \to \mathcal{CG}_*,
\]
where \( B_nG = B_n(\ast, G, \ast) \) and \( \mathcal{CG}_* \) is the category of pointed compactly generated spaces.

Our main result is the following theorem.

**Theorem 1.** Let \( G \) be a well-pointed topological monoid of homotopy type of a CW complex and \( G' \) a well-pointed grouplike topological monoid. Then the following composite is a weak equivalence.
\[
\mathcal{A}_n(G, G') \xrightarrow{B_n} \text{Map}_0(B_nG, B_nG') \xrightarrow{(\iota_n)^\#} \text{Map}_0(B_nG, BG').
\]

This result refines the Stasheff’s recognition of \( A_n \)-maps between topological monoids (1963). As an application of this result, we obtain the following result.

**Theorem 2.** Let \( G \) be a well-pointed topological group of homotopy type of a CW complex. Then the connecting map \( \delta \) of the evaluation fiber sequence
\[
G \to \text{Map}_0(B_nG, BG) \to \text{Map}(B_nG, BG) \to BG
\]
is equivalent to the map \( G \to \mathcal{A}_n(G, G) \) given by the conjugation. Moreover, \( \delta \) is delooped and the above evaluation fiber sequence extends to the right.

As other applications, we investigate the relations with higher homotopy commutativity, \( A_n \)-types of gauge groups and \( T^k \)-spaces by Iwase–Mimura–Oda–Yoon.

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