Weighted Stanley-Reisner ring and Equivariant Cohomology Ring of a Singular Toric Variety

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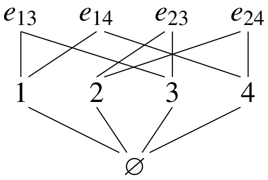
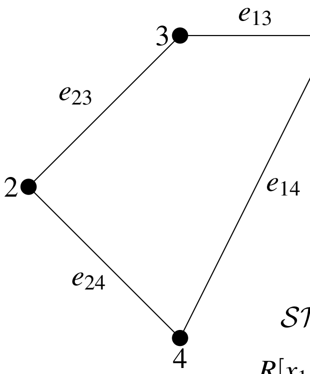
Simplicial complex

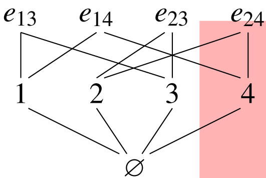




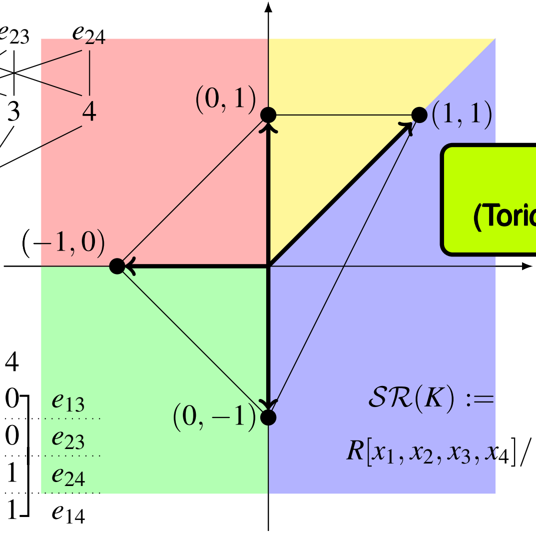
$$\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \hline 1 & 0 & 1 & 0 & e_{13} \\ 0 & 1 & 1 & 0 & e_{23} \\ 0 & 1 & 0 & 1 & e_{24} \\ 1 & 0 & 0 & 1 & e_{14} \end{array}$$

$$\mathcal{SR}(K) := R[x_1, x_2, x_3, x_4]/\langle x_1x_2, x_3x_4\rangle$$

Simplicial complex + Geometry



$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} e_{13} \\ e_{23} \\ e_{24} \\ e_{14} \end{matrix}$$



X_Σ
(Toric variety)

$\mathcal{SR}(K) :=$

$R[x_1, x_2, x_3, x_4]/\langle x_1x_2, x_3x_4\rangle$

Toric Variety

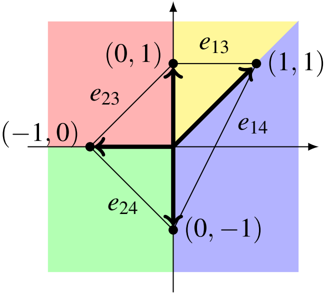
Definition

A toric variety is a normal complex algebraic variety with algebraic $(\mathbb{C}^*)^n$-action having a dense orbit.

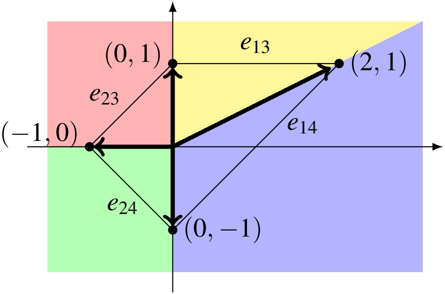
Theorem (Fundamental theorem for toric varieties)

The category of toric varieties is equivalent to the category of fans.

$$X_\Sigma \longleftrightarrow \Sigma_X$$



X_Σ: smooth

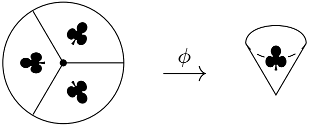


X_Σ: singular

Orbifold: a very rough introduction

A topological space, locally homeomorphic to $\tilde{U}/G$,

1. $\tilde{U}$: open subset in $\mathbb{R}^n$,
2. G: finite subgroup of $O(n)$, and $G \curvearrowright \tilde{U}$.
3. G acts (effectively) on $\tilde{U}$ via the action of $O(n)$ on $\mathbb{R}^n$.



- $(\tilde{U}, G, \phi)$: an orbifold chart around $\phi(0)$,
- G: the local group around $\phi(0)$

A natural source for making an orbifold

- M: a smooth manifold,
- G: a compact Lie group acting smoothly, effectively, and almost freely[1] on M.

$\Rightarrow$ M/G: an orbifold with the following orbifold chart near $[x] \in M/G$,

$$(x \in U \underset{\text{open}}{\subset} M,\ G_x,\ \phi\colon U \to U/G_x)$$

[1] A point may have a finite stabilizer

Toric Orbifold

- A fan Σ is called a simplicial fan,
  if for each $cone(\lambda_{i_1}, \ldots, \lambda_{i_n}) \in \Sigma$, $\{\lambda_{i_1} \ldots, \lambda_{i_n}\} \subset \mathbb{Z}^n$ is linearly independent.
- The toric variety X_Σ associated to a simplicial fan is called a toric orbifold.

Question

What are "M" and "G" in this setting?

About M

$$K \rightsquigarrow \mathcal{Z}_K = \bigcup_{\sigma \in K} (D^2, S^1)^v \subset \mathbb{C}^m,$$

where if $\sigma = \{i_1 \cup \cdots \cup i_n\} \in K$,

$$(D^2, S^1)^\sigma = \prod_{j=1}^{m} A_j, \ \ A_j = \begin{cases} D^2 & j \in \{i_1, \ldots, i_n\} \\ S^1 & j \notin \{i_1, \ldots, i_n\} \end{cases}$$

Proposition

1. *$\mathcal{Z}_K$ is an $(m+n)$-dimensional smooth manifold.*
2. *T^m acts on $\mathcal{Z}_K$ by coordinate multiplication.*

About G

1. $\Lambda := [\lambda_1 \mid \cdots \mid \lambda_m] : \mathbb{Z}^m \twoheadrightarrow \mathbb{Z}^n$
2. $\Lambda \otimes \mathbb{R} : \mathbb{Z}^m \otimes \mathbb{R} \twoheadrightarrow \mathbb{Z}^n \otimes \mathbb{R}$
3. $0 \longrightarrow \ker \tilde{\Lambda} \hookrightarrow T^m \xrightarrow{\tilde{\Lambda}} T^n \longrightarrow 0$

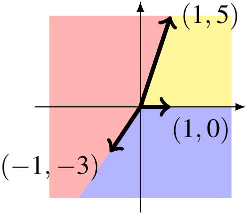
Proposition

$\ker \tilde{\Lambda}$ *acts on* $\mathcal{Z}_K$ *almost freely.*

Definition (Toric orbifold)

$$X_\Sigma := \mathcal{Z}_K / \ker \tilde{\Lambda}$$

Example



- $\mathcal{Z}_K = (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2)$
  $= \partial(D^2 \times D^2 \times D^2) = S^5$
- $2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \ker \tilde{\Lambda} = \{(t^2, t^3, t^5) \mid t \in S^1\} \subset T^3.$
- $X_\Sigma = S^5 / \ker \tilde{\Lambda} = \mathbb{C}P^2_{(2,3,5)}.$

Question: $\mathbb{C}P^n_{(a_0,\ldots,a_n)} \overset{?}{\cong} \mathbb{C}P^n$

Theorem (Kawasaki, '73)

For $(a_0,\ldots,a_n) \in \mathbb{N}^{n+1}$ *with* $\gcd(a_0,\ldots,a_n) = 1$,

$$H^i(w\mathbb{C}P^n_{(a_0,\ldots,a_n)};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i\text{=even} \\ 0 & \text{if } i\text{=odd} \end{cases}.$$

Moreover, if $\langle \gamma_k \rangle = H^{2k}(w\mathbb{C}P^n_{(a_0,\ldots,a_n)};\mathbb{Z})$,

$$\gamma_i \cup \gamma_j = \frac{\ell_i \ell_j}{\ell_{i+j}} \gamma_{i+j},$$

where $\ell_k = \operatorname{lcm}\left\{ \frac{a_{i_0}\cdots a_{i_k}}{\gcd(a_{i_0},\ldots,a_{i_k})} \mid 0 \leq i_0 < \cdots < i_k \leq n \right\}$.

After Kawasaki..

1. (Al. Amrani, 1994) K-theory of $w\mathbb{C}P$.
2. (Nishimura–Yoshimura, 1997) KO-theory of $w\mathbb{C}P$.
3. (Bahri–Franz–Ray, 2009) Equivariant cohomology of $w\mathbb{C}P$.
4. (Bahri–Franz–Notbohm-Ray, 2013) The classification of $w\mathbb{C}P$, up to homeomorphism and homotopy, in terms of weights.

History

Theorem (Danilov '78, Jurkiewicz '85)

For a smooth toric variety X_Σ,

$$H^*(X_\Sigma;\mathbb{Z}) \cong \mathcal{SR}(K;\mathbb{Z})/\mathcal{J},$$

where $\mathcal{J} = \langle \sum_{i=1}^m \langle \lambda_i, e_j \rangle x_i = 0 \mid j = 1, \ldots, n \rangle$

Theorem (Danilov '78)

For a toric orbifold X_Σ,

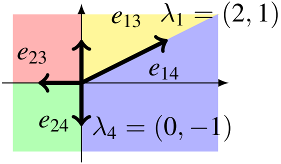
$$H_T^*(X_\Sigma;\mathbb{Q}) \cong \mathcal{SR}(K;\mathbb{Q})/\mathcal{J}.$$

Summary

- Σ: a smooth/simplicial fan.
- K: underlying simplicial complex.
- X_Σ: associated toric variety.

	$H^*(X_\Sigma;\mathbb{Q})$	$H^*(X_\Sigma;\mathbb{Z})$
Toric manifolds	$\mathcal{SR}(K;\mathbb{Q})/\mathcal{J}$	$\mathcal{SR}(K;\mathbb{Z})/\mathcal{J}$
Toric orbifolds	$\mathcal{SR}(K;\mathbb{Q})/\mathcal{J}$	??

Revisit the Stanley–Reisner ring



$$\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \frac{1}{2}u_1 & 0 & -\frac{1}{2}u_1 + u_2 & 0 & e_{13} \\ 0 & u_2 & -u_1 & 0 & e_{23} \\ 0 & -u_1 & 0 & -u_2 & e_{24} \\ \frac{1}{2}u_1 & 0 & 0 & \frac{1}{2}u_1 - u_2 & e_{14} \end{array}$$

$$\begin{bmatrix} \lambda_1 & \lambda_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u_1 \\ -\frac{1}{2}u_1 + u_2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & \lambda_4 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u_1 \\ \frac{1}{2}u_1 - u_2 \end{bmatrix}$$

In general...

$$^{\forall}\sigma = cone\{\lambda_{i_1}, \ldots, \lambda_{i_n}\} \in \Sigma^{(n)},$$

$$\rightsquigarrow \quad z^{\sigma} := (z_1^{\sigma}, \ldots, z_m^{\sigma}) \in \bigoplus_m \mathbb{Q}[u_1, \ldots, u_n],$$ by

(C1) $z_j^{\sigma} = 0$ if $j \notin \{i_1, \ldots, i_n\}$,

(C2) $$\begin{bmatrix} z_{i_1}^{\sigma} \\ \vdots \\ z_{i_n}^{\sigma} \end{bmatrix} = \begin{bmatrix} \lambda_{i_1} & \Big| & \cdots & \Big| & \lambda_{i_n} \end{bmatrix}^{-1} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

Definition

$h(x_1, \ldots, x_m) \in \mathbb{Z}[x_1, \ldots x_m]$ satisfies the integrality condition with respect to Σ, if $h(z^{\sigma}) \in \mathbb{Z}[u_1, \ldots, u_n]$ for all $\sigma \in \Sigma^{(n)}$.

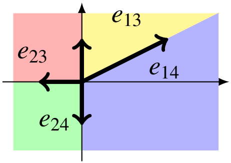
Weighted Stanley–Reisner Ring

$$w\mathcal{SR}[\Sigma] := \{h \in \mathbb{Z}[x_1, \dots, x_m] \mid h \text{ satisfies integrality condition}\}/\mathcal{I}.$$

Remark

- *When the fan Σ is smooth, $w\mathcal{SR}[\Sigma] = \mathcal{SR}[\Sigma]$.*
- *In general, $w\mathcal{SR}[\Sigma] \underset{subring}{\subset} \mathcal{SR}[\Sigma]$.*

Example



$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{matrix} 1 & 2 & 3 & 4 \end{matrix}$$

$$\begin{bmatrix} \frac{1}{2}u_1 & 0 & -\frac{1}{2}u_1 + u_2 & 0 \\ 0 & u_2 & -u_1 & 0 \\ 0 & -u_1 & 0 & -u_2 \\ \frac{1}{2}u_1 & 0 & 0 & \frac{1}{2}u_1 - u_2 \end{bmatrix} \begin{matrix} e_{13} \\ e_{23} \\ e_{24} \\ e_{14} \end{matrix}$$

- Degree 2 elements are...

$$2x_1,\ x_2,\ 2x_3,\ 2x_4,\ 2x_1 - x_2 \text{ and } x_1 + x_3 - x_4.$$

- Degree 4 elements are...

$$4x_1^2,\ x_2^2,\ 4x_3^2,\ 4x_4^2,\ 4x_1x_3,\ 4x_1x_4,\ x_2x_3 \text{ and } x_2x_4.$$

- $\vdots$

Main theorem

Theorem (Bahri-Sarkar-S, arXiv:1509.03228)

For a toric orbifold X_Σ with $H^{odd}(X_\Sigma) = 0$,

$$H^*(X_\Sigma; \mathbb{Z}) \cong w\mathcal{SR}(\Sigma; \mathbb{Z})/\mathcal{J}.$$

	$H^*(X_\Sigma; \mathbb{Q})$	$H^*(X_\Sigma; \mathbb{Z})$
Toric manifolds	$\mathcal{SR}(K; \mathbb{Q})/\mathcal{J}$	$\mathcal{SR}(K; \mathbb{Z})/\mathcal{J}$
Toric orbifolds	$\mathcal{SR}(K; \mathbb{Q})/\mathcal{J}$	$w\mathcal{SR}(\Sigma; \mathbb{Z})/\mathcal{J}$

Proof

1. Chang–Skjelbred sequence
$$0 \to H_T^*(X_\Sigma; \mathbb{Z}) \to H_T^*(X_\Sigma^0; \mathbb{Z}) \to H_T^*(X_\Sigma^1, X_\Sigma^0; \mathbb{Z}) \to \cdots$$
is exact.

2. $H_T^*(X_\Sigma; \mathbb{Z}) \cong \mathcal{PP}(\Sigma; \mathbb{Z})$, the ring of piecewise polynomials.

3. $\mathcal{PP}(\Sigma; \mathbb{Z}) \xrightarrow{\ \phi\ } \mathcal{SR}(K)$.

4. $\mathrm{im}\phi \cong w\mathcal{SR}(\Sigma; \mathbb{Z})$.

Questions

- (Non-mathematrically...)
  Are you happy with the assumption $H^{odd}(X) = 0$?
- (Mathematically...)
  Are there any necessary or sufficient conditions for $H^{odd}(X) = 0$?

Partial answer

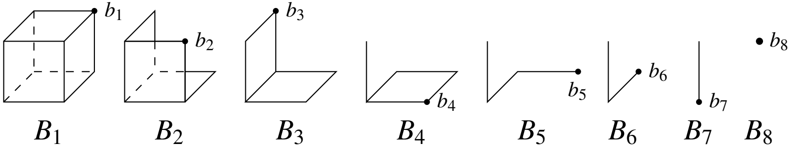
[Kuwata, Zeng, Masuda arXiv:1604.03138] Torsions in the cohomology of torus orbifolds.

- The complete answer for 4-dimensional torus orbifolds.
- A necessary condition for arbitrary dimensional torus orbifolds.

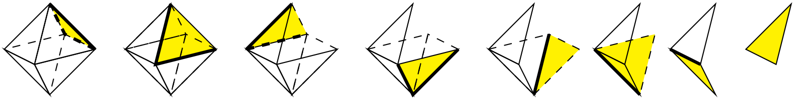
A retraction sequence

$$(\mathbb{C}^*)^n \curvearrowright X_\Sigma \quad \rightsquigarrow \quad T^n \curvearrowright X_\Sigma \quad \rightsquigarrow \quad \pi : X \to Q$$

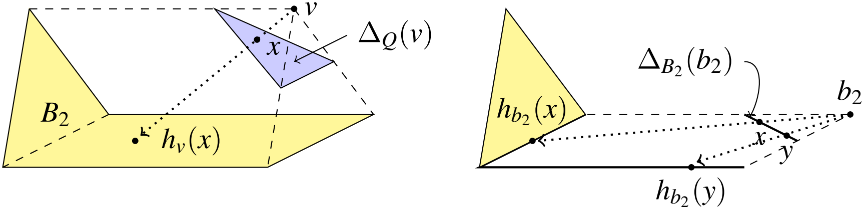
A retraction sequence is, for instance,



In terms of simplicial complex, (special case of) shelling.



Why retraction..?



- $\pi^{-1}(Q) = X_\Sigma$
- $\pi^{-1}(B_2) = \bigcup_{E\text{: face of } B_2} \pi^{-1}(E)$, where $\pi^{-1}(E) = X_\Sigma$.
- $\pi^{-1}(\Delta_Q(v)) = S^{2n-1}/G_v$, where $|G_v| = |\det \Lambda_v|$

How are they related?

Proposition

The composition

$$\pi^{-1}(\Delta_Q(v)) \xrightarrow{f} \bigcup_{E:\ \text{face of}\ B_2} \pi^{-1}(E) \hookrightarrow X_\Sigma$$

is a cofiber sequence. i.e., $X_\Sigma \simeq c(f)$

Corollary

$$\begin{aligned} H_*(X_\Sigma, \pi^{-1}(B_2)) &\cong H_*(C(L(\Delta_Q(v), \xi_v)), \pi^{-1}(B_2)) \\ &\cong \tilde{H}_{*-1}(\pi^{-1}(\Delta_Q(v))). \end{aligned}$$

Since $\pi^{-1}(E)$ is another toric variety...

Proposition

For each face $E \subset Q$, the composition

$$\pi^{-1}(\Delta_E(v)) \xrightarrow{f_E} \bigcup_{F:\ \text{face of } B_3} \pi^{-1}(F) \hookrightarrow X_E$$

is a cofiber sequence. i.e., $X_E \simeq c(f_E)$

Corollary

$$\begin{aligned} H_*(X_E, \pi^{-1}(B_3)) &\cong H_*(C(L(\Delta_E(b_2), \xi_v)), \pi^{-1}(B_3)) \\ &\cong \tilde{H}_{*-1}(\pi^{-1}(\Delta_E(b_2))). \end{aligned}$$

The long exact sequence of pair $(X_\Sigma, \pi^{-1}(B_2))$

$$\begin{array}{ccccccc}
\to H_{j+1}(X_\Sigma, \pi^{-1}(B_2)) & \to & H_j(\pi^{-1}(B_2)) & \to & H_j(X_\Sigma) & \longrightarrow & H_j(X_\Sigma, \pi^{-1}(B_2)) \longrightarrow \\
\| & & & & & & \| \\
\tilde{H}_j(\pi^{-1}(\Delta_Q(v))) & & & & & & \tilde{H}_{j-1}(\pi^{-1}(\Delta_Q(v))) \\
\| & & & & & & \| \\
|G_v|\text{-torsion} & & & & & & |G_v|\text{-torsion}
\end{array}$$

Main Theorem: A sufficient condition for $H^{odd} = 0$.

Theorem

Let X_Σ be a toric orbifold with $\pi : X \to Q$ the orbit map. Assume that for each $B \in \mathfrak{B}(Q)$ with $\dim B \geq 1$,

$$\gcd\big\{|G_E(v)| : v \in FV(B)\big\} = 1,$$

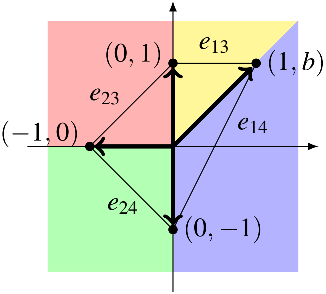
Then, the homology $H_*(X)$ is torsion free and concentrated in even degrees.

Theorem

Under the above assumption,

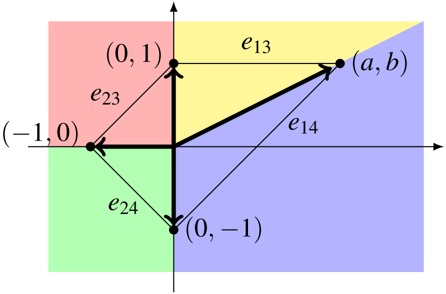
$$H^*(X_\Sigma; \mathbb{Z}) \cong w\mathcal{SR}[\Sigma]/\mathcal{J}.$$

Example: Orbifold Hirzebruch surface



$\mathfrak{H}_b$

Hirzebruch surface



$\mathfrak{H}_{(a,b)}$

Orbifold Hirzebruch surface

$H^*(\mathfrak{H}_b)$ VS $H^*(\mathfrak{H}_{(a,b)})$

$$H^j(\mathfrak{H}_{(a,b)}) = w\mathcal{SR}[\Sigma]/\mathcal{J} = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z}\langle w_1 \rangle \oplus \mathbb{Z}\langle w_2 \rangle & \text{if } j = 2 \\ \mathbb{Z}\langle w_3 \rangle & \text{if } j = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Multiplication structure is given by..

- $w_1^2 = 0$,
- $w_1 w_2 = a w_3$.
- $w_2^2 = ab w_3$,
- $w_1 w_3 = w_2 w_3 = w_3^2 = 0$.

Remark

$$H^*(\mathfrak{H}_b) = \mathbb{Z}[w_1, w_2]/\langle w_1^2, w_2^2 - b w_1 w_2 \rangle.$$

THANK YOU FOR YOUR ATTENTION!