On classification of certain 8-dimensional torus manifolds

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### §1. Motivation and Main result

**Torus manifold**

**Definition (torus manifold)**

A compact orientable $T^n$-mfd $M^{2n}$ is called a **torus manifold** if the $T$-action on $M$ has at least one fixed point.

**Example**

- $\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{O\})/\mathbb{C}^*$ with the standard $T^n$-action.
- $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ with the standard $T^n$-action.
- A product $M_1 \times M_2$ of two torus manifolds.
- An equivariant connected sum $M_1 \# M_2$ of two ($n$-dimensional) torus manifolds along fixed points $p_1 \in M_1$ and $p_2 \in M_2$. 
More examples: well-studied classes in torus manifolds

Smooth toric variety A smooth normal variety $V$ with $(\mathbb{C}^*)^n$-action such that there exists a dense orbit (the restricted $T^n \subset (\mathbb{C}^*)^n$ action becomes a torus manifold).
More examples: well-studied classes in torus manifolds

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**Quasitoric manifold (Davis-Januszkiewicz, 1990)**  
A torus manifold $M$ whose $T$-action is locally standard and orbit space $M/T$ is a simple convex polytope.
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Quasitoric manifold (Davis-Januszkiewicz, 1990) A torus manifold $M$ whose $T$-action is locally standard and orbit space $M/T$ is a simple convex polytope.

Topological toric manifold (Ishida-Fukukawa-Masuda, 2010) A $2n$-dim mfd $M$ with smooth $(\mathbb{C}^*)^n$-action with a dense orbit such that locally looks like smooth $(\mathbb{C}^*)^n$-representation on $\mathbb{C}^n$. 
Motivation and Main result

Cohomological properties

Cohomology rings of all the previous (well-studied) classes are generated by $H^2(M)$.

Example

- $\mathbb{C}P^n$ is a topological toric; $H^*(\mathbb{C}P^n) \simeq \mathbb{Z}[x]/\langle x^{n+1} \rangle$, $\deg x = 2$.
- $S^{2n}$ ($n \geq 2$) is **NOT** a topological toric; $H^*(S^{2n}) \simeq \mathbb{Z}[y]/\langle y^2 \rangle$, $\deg y = 2n$.
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Theorem (Masuda-Panov)

The cohomology ring of a torus manifold $M$ is generated by $H^2(M)$ iff its $T$-action is locally standard and $M/T$ is a homology polytope (i.e., manifold with corner whose all faces and their intersections are acyclic).
Motivation

Question

What can we say about the compliment of torus manifolds generated by $H^2(M)$ (topological toric manifolds)?
Main result

Let $\mathcal{M}_{b_2=0}^{2n}$ be the set of all $2n$-dim simply connected torus manifolds with $H^{\text{odd}}(M) = 0$ and $H^2(M) = 0$.

Theorem

Let $M \in \mathcal{M}_{b_2=0}^8$, i.e., $n = 4$. Then, $(M, T^4)$ is (weak) equivariantly homeomorphic to

$$(S^8, T^4) \quad \text{or} \quad \#_{i=1}^\ell (S^4 \times S^4, T^2 \times T^2).$$
Remarks

Fact

1. $\mathcal{M}_{b_2=0}^2 = \emptyset$, i.e., $n = 1$ (because $(M^2, T^1) \cong (S^2, T^1)$);
2. $\mathcal{M}_{b_2=0}^4 = \{(S^4, T^2)\}$, i.e., $n = 2$ (Orlik-Raymond);
3. $\mathcal{M}_{b_2=0}^6 = \{(S^6, T^3)\}$, i.e., $n = 3$ (Wall, Jupp).

Remark

1. **The quaternionic projective space** $\mathbb{H}P^2$ is **NOT** a torus manifold.
2. **If an $S^4$-bundle over $S^4$ is a torus manifold, then it must be** $(S^4 \times S^4, T^2 \times T^2)$ (note that there are **infinitely many** topological types of $S^4$-bundles over $S^4$);
Corollary

Let $M_1, M_2 \in \mathcal{M}^8_{b_2=0}$. Then, the following five statements are equivalent:

1. $b_4(M_1) = b_4(M_2) \in 2\mathbb{Z}_{\geq 0}$;
2. $M_1 \cong M_2$ (up to homeo);
3. $(M_1, T^4) \cong_w (M_2, T^4)$ (up to weak $T^4$-equivariant homeo);
4. $H^*(M_1; \mathbb{Z}) \cong H^*(M_2; \mathbb{Z})$ (up to graded ring isomorphism);
5. $H^*_T(M_1; \mathbb{Z}) \cong_w H^*_T(M_2; \mathbb{Z})$ (up to weakly $H^*(BT^4; \mathbb{Z})$-algebraic isomorphism).

Namely, the set $\mathcal{M}^8_{b_2=0}$ satisfies (equivariant) cohomological rigidity.
Cohomological rigidity (another motivation)

Definition (cohomological rigidity)
A class of the manifolds $\mathcal{M}$ satisfies **cohomological rigidity** if $M_1, M_2 \in \mathcal{M}$ satisfies the following: if $H^*(M_1) \cong H^*(M_2)$ then $M_1 \cong M_2$.

Problem (cohomological rigidity problem; Masuda-Suh, 2006)

*Does the class of (quasi)toric manifolds satisfy cohomological rigidity?*

Remark

*This problem is still open, though there are many partial affirmative answers.*

*Simply connected torus manifolds with $H^{odd}(M)$ do NOT satisfy cohomological rigidity; however $\mathcal{M}_{b_2=0}^8$ satisfies cohomological rigidity.*
§2. Proof of Main result

Key Theorem

Theorem (Schmitt, 2002)

Let $\mathcal{N}^{8}_{b_2=0}$ be the set of all simply connected 8-dimensional manifold with $H^{\text{odd}}(M) = H^2(M) = 0$. Then, the homeomorphism types of $\mathcal{N}^{8}_{b_2=0}$ are determined by

1. $H^*(M)$ (cohomology ring);
2. $p_1(M)$ (the 1st Pontrjagin class).

From now, $(M, T)$ is a simply connected torus manifold with $H^{\text{odd}}(M) = H^2(M) = 0$ (i.e., $M \in \mathcal{M}^{8}_{b_2=0} \subset \mathcal{N}^{8}_{b_2=0}$).

Problem

What kinds of ring structures can $H^*(M)$ have?
Torus graph and Cohomology ring

The torus graph \((\Gamma, \mathcal{A})\) of \((M, T)\) is defined (roughly) as follows:

1. The set of vertices \(V(\Gamma)\) is \(M^T\);
2. The set of edges \(E(\Gamma)\) is the set of 1-dim orbits connecting two fixed points \(p, q \in M^T\);
3. \(\mathcal{A} : E(\Gamma) \rightarrow H^2(BT)\) is called an axial function defined by the tangential representation on \(T_p(M)\) for all \(p \in M^T\).
Examples of torus graphs

The left graph is induced from \((S^8, T^4)\). The right graph is induced from \((\mathbb{C}P^2, T^2)\).

Here, \(\alpha, \beta, \gamma, \delta\) are the generators of \(H^*(BT^4)\).
Cohomology rings

Let $H^*_T(\Gamma, \mathcal{A})$ be the following ring:

$$\{ f : V(\Gamma) \to H^*(BT) \mid f(p) - f(q) \equiv 0 \mod \mathcal{A}(pq) \}.$$

**Remark**

$H^*_T(\Gamma, \mathcal{A})$ has the $H^*(BT)$-algebraic structure induced from $\pi^* : H^*(BT) \to H^*_T(\Gamma, \mathcal{A})$ such that $\pi^*(\alpha) = \alpha$ (constant map).

**Theorem ((essentially) Masuda-Panov)**

- $H^*_T(M) (= H^*(ET \times_T M)) \simeq H^*_T(\Gamma, \mathcal{A})$ as $H^*(BT)$-algebra;
- $H^*(M) \simeq H^*(\Gamma, \mathcal{A})$, where $H^*(\Gamma, \mathcal{A}) = H^*_T(\Gamma, \mathcal{A})/\text{Im } \pi^{>0}$. 
Maeda-Masuda-Panov’s formula

Let $F = (\Gamma', \mathcal{A}|_{E(\Gamma')})$ be a $k$-face ($k$-valent torus subgraph) of a torus graph $(\Gamma, \mathcal{A})$.

Definition (Thom class)

The element $\tau_F : V(\Gamma) \to H^{2n-2k}(BT) \in H^*_T(\Gamma, \mathcal{A})$ is defined by

$$\tau_F(p) = \begin{cases} 
\prod_{e \in N_p(F)} \mathcal{A}(e) & \text{if } p \in V(\Gamma') \\
0 & \text{otherwise}
\end{cases}$$
Example of cohomology ring

**Definition (Cohomology ring of torus graph)**

\[
\mathbb{Z}[\Gamma] = \mathbb{Z}[[\tau_F \mid \text{F is a face}]]/\langle \tau_G \tau_H - \tau_G \wedge H \sum_{E \in G \cap H} \tau_E \rangle.
\]

**Theorem (Maeda-Masuda-Panov)**

\[
H_T^*(\Gamma, \mathcal{A}) \simeq \mathbb{Z}[\Gamma].
\]

**Example**

For the previous torus graph, we denote the Thom classes of faces as \(\tau_1, \tau_2, \tau_p, \tau_q\) respectively from left. Then,

\[
H_T^*(\Gamma, \mathcal{A}) \simeq \mathbb{Z}[\tau_1, \tau_2, \tau_p, \tau_q]/\langle \tau_1 \tau_2 - \tau_p - \tau_q, \ \tau_p \tau_q \rangle.
\]
Sketch of Proofs

Let \((\Gamma, A)\) be the torus graph induced from \((M, T) \in \mathcal{M}_{b_2=0}^8\).

Using Maeda-Masuda-Panov’s formula and the \(H^*(BT)\)-algebraic structure of \(H^*_T(\Gamma, A)\), we have the following lemma:

**Lemma**

*There exists exactly four 3-faces \(\Gamma_\alpha, \Gamma_\beta, \Gamma_\gamma, \Gamma_\delta\) such that their Thom classes are generators \(\alpha, \beta, \gamma, \delta\) of \(H^*(BT) \subset H^*_T(\Gamma, A)\) and \(V(\Gamma) = V(\Gamma_\alpha) = V(\Gamma_\beta) = V(\Gamma_\gamma) = V(\Gamma_\delta)\).*
Construction of our torus graphs

Using this lemma, there exists 2-faces $\Gamma_{\alpha,\gamma}$ and $\Gamma_{\beta,\delta}$ (one-skelton of $2k$-gon) such that $\Gamma = \Gamma_{\alpha,\gamma} \cup \bigcup V \Gamma_{\beta,\delta}$, where $\# V = 2k$. 

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Proof of Main result

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When is $H^*(\Gamma, \mathcal{A})$ induced from $M \in \mathcal{M}_{b_2=0}^8$?

Let $(\Gamma_e, \mathcal{A})$ be the following torus graph:

![Diagram](image)

Then,

$$\dim H^0(\Gamma_e, \mathcal{A}) = 1; \dim H^2(\Gamma_e, \mathcal{A}) = 0; \dim H^4(\Gamma_e, \mathcal{A}) = 1;$$

$$\dim H^6(\Gamma_e, \mathcal{A}) = 2; \dim H^8(\Gamma_e, \mathcal{A}) = 1.$$

This is NOT induced from $\mathcal{M}_{b_2=0}^8$. 
Combinatorial structure of our torus graphs

By using the inductive argument and the Mayer-Vietoris type theorem, we have the following theorem:

**Theorem**

Assume $H^*(\Gamma, \mathcal{A})$ is induced from $M \in \mathcal{M}_{b_2=0}^8$. Then, $(\Gamma, \mathcal{A})$ is the torus graph induced from $(S^8, T^4)$ or $\#_{i=1}^k(S^4 \times S^4, T^2 \times T^2)$. 
Final step

Using Schmitt’s theorem and the known result of Pontrjagin class of torus manifolds (proved by Hattori-Masuda), we have the main result.

**Theorem**

Let $M \in \mathcal{M}_{b_2=0}$, i.e., $n = 4$. Then, $(M, T^4)$ is (weak) equivariantly homeomorphic to the standard

$$ (S^8, T^4) \quad \text{or} \quad \#_{i=1}^\ell (S^4 \times S^4, T^2 \times T^2). $$
Problem

Let $M \in \mathcal{M}^{2n}_{b_2=0}$, i.e., 2n-dim simply connected torus manifolds with $H^2(M) = H^{odd}(M) = 0$.

Problem

Is $(M, T^n)$ (weak) equivariantly homeomorphic (or diffeomorphic) to the standard $T^n$-action of some connected sums of products of even (not 2) dimensional spheres? In other words,

$$M \cong \#_{i=1}^\ell S^{2n_{1i}} \times \ldots S^{2n_{ki}}$$

(where $\sum_{j=1}^{k_i} n_{ji} = n$ and $n_{ji} \geq 2$)?

How about the torus graph induced from $(M, T^n)$?
Thank you for your kind attention.