On classifications of a certain class of 6-dimensional torus manifolds

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§0. Contents of this talk

§1. Torus manifolds
§2. Classification of (low dimensional) torus manifolds
§3. Main Result 1 (geometric topological classification)
§4. Main Result 2 (algebraic topological classification)
§1. Torus manifolds

Definition (Hattori-Masuda)
Let $M$ be a cpt, con, orie, $2n$-dim manifold with $n$-dim torus $T$-action. Then, $M$ (or $(M, T)$) is called a torus manifold if $M^T \neq \emptyset$.

Example
(1) $T^n \times S^2 \subset C^n \oplus \mathbb{R}$
(2) $T^n \times \mathbb{C}P^n = (\mathbb{C}^n + 1 - \{0\}) = \mathbb{C}^\ast$ (on last $n$ coord)
(3) An equivariant connected sum $M_1 \# M_2$ of two ($2n$-dim) torus mfds along fixed points $p_1 \in M_1$ and $p_2 \in M_2$.
(4) A gluing along free orbits $M \# T(N \times T)$ for any $n$-dim cpt orie mfd $N$ (⇒ torus mfds are very huge class!).
(5) (Quasi)toric (next slide)
§ 1. Torus manifolds

Definition (Hattori-Masuda)

Let $M$ be a cpt, con, orie, $2n$-dim manifold with $n$-dim torus $T$-action. Then, $M$ (or $(M, T)$) is called a torus manifold if $M^T \neq \emptyset$.

Example

1. $T^n \sim S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R} \Rightarrow M^T = \{(0, 1), (0, -1)\}$.
2. $T^n \sim \mathbb{C}P^n = (\mathbb{C}^{n+1} - \{O\})/\mathbb{C}^*$ (on last $n$ coord) $\Rightarrow M^T = \{[1 : 0 : \cdots : 0], \ldots, [0 : \cdots : 0 : 1]\}$.
3. An equivariant connected sum $M_1 \# M_2$ of two ($2n$-dim) torus mfds along fixed points $p_1 \in M_1$ and $p_2 \in M_2$.
4. A gluing along free orbits $M \#_T (N^n \times T)$ of with $N \times T$ for any $n$-dim cpt orie mfd $N$ ($\Rightarrow$ torus mfds are very huge class!).
5. (Quasi)toric. (next slide)
Definition (toric manifold)

A torus mfd $M^{2n}$ is called a (smooth) toric manifold if there is a $T^n$-invariant complex structure. (by Ishida-Karshon)

Definition (quasitoric manifold)

A torus mfd $M^{2n}$ is called a quasitoric manifold if

1. $T^n$-action is locally isomorphic to $T^n \sim \mathbb{C}^n$;
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Example

1. $\mathbb{C}P^n$ with the standard $T^n$-action is (quasi)toric.
2. $\mathbb{C}P^2 \# \mathbb{C}P^2$ (this is known as quasitoric but not toric)
3. $S^{2n}$ for $n \geq 2$ is NOT (quasi)toric.
FACT

$2n$-dim smooth toric $\Rightarrow$ $2n$-dim quasitoric for $n \leq 3$. 
§2. Classification of low dimensional torus mfds

Problem

*How can we classify torus manifolds?*

Proposition (when $n = 1$)

*Every 2-dim torus mfd is (equivariantly diffeomorphic to) $S^2$.***

$(\therefore M^T = \chi(M) > 0.)$
Theorem (Orlik-Raymond (1970, when $n = 2$))

Let $M$ be a simply connected 4-dim torus mfd. Then,

$$M \cong M_1 \# \cdots \# M_\ell$$

where each $M_i$ is $S^4$, $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ or Hirzebruch surfaces ($\mathbb{C}P^1$-bdl over $\mathbb{C}P^1$).
Theorem (Orlik-Raymond (1970, when \( n = 2 \)))

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Problem

How can we generalize Orlik-Raymond’s thm to the higher dimension? (What kind of assumptions do we need?)
 Sect. Main Result 1

Theorem (K (when $n = 3$))

Let $M$ be a simply con, 6-dim torus mfd with $H^{\text{odd}}(M) = 0$. Then,

$$M \cong M_1 \# \cdots \# M_\ell$$

where each $M_i$ is one of the followings:

1. $S^6$;
2. 6-dim quasitoric manifold;
3. $S^4$-bundle over $S^2$ which is the form

$$S^3 \times S^1 S(\mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{R})$$

where $S^1$ acts on $S^3$ freely ($S^3/S^1 \cong S^2$) and $t \in S^1$ acts on $z \in \mathbb{C}_k = \mathbb{C}$ by $z \mapsto zt^k$ for $k \in \mathbb{Z}$. 
Outline of proof: Step 1 \((H^{odd}(M) = 0)\)

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Assume \(H^{\text{odd}}(M) = 0\).

Theorem (Masuda-Panov)

\[ H^{\text{odd}}(M) = 0 \iff \]

(1) \(T \curvearrowright M\) is locally \(T \curvearrowright \mathbb{C}^n\) (locally standard);

(2) \(M/T\) is face acyclic \((H_*(F) \simeq H_*(\text{pt})\) for each face \(F\)).

In particular, \(\dim M = 6 \Rightarrow M/T\) is a homology 3-disk.
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Proposition

Let \(W\) be a 6-dim torus mfd with \(H^{odd}(W) = 0\). Then, there are a simply con, 6-dim torus mfd \(M\) (with \(H^{odd}(M) = 0\)) and a homology 3-sphere \(hS^3\) such that

\[ W \cong M \#_T(hS^3 \times T^3). \]
**Step 2 (simply connected)**

Assume $H^{odd}(M) = 0$ and $M$ is simply connected. 

$\Rightarrow M/T$ is the standard disk (with the structure of mfd with faces).
Main thm 1 and Proof

Step 2 (simply connected)

Assume $H^{odd}(M) = 0$ and $M$ is simply connected.

$\Rightarrow$ $M/T$ is the standard disk (with the structure of mfd with faces).

Therefore,

$(M^6, T^3)$ is determined by $(M/T, \lambda)$, where $\lambda$ is the information of isotropy subgroups of codim-2 (4-dim) torus submfd (characteristic function).
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Lemma

Such $(M^6, T^3)$ is determined by 3-valent torus graphs $(\Gamma, A)$, i.e., fixed points, 1-dim orbits $\Gamma$ and tangential representations $A$. 

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Lemma

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Problem (GOAL)

Classification of all 3-valent torus graphs!
Step 3 (classification of 3-valent torus graphs)

A torus graph \((\Gamma, A)\) of \((M, T)\) is defined by

1. \(V(\Gamma)\) is \(M^T\);
2. \(E(\Gamma)\) is invariant \(S^2\)'s;
3. \(A : E(\Gamma) \rightarrow H^2(BT)\) is tangential representation on \(T_p M\) for all \(p \in M^T\).
Torus graph of $T^2 \cong \mathbb{C}P^2$ by $[x : y : z] \mapsto [x : t_1y : t_2z]$.

Tangential rep. on $q = [0 : 1 : 0]$

$T_q M \cong \{ [x : 1 : z] \}$

$[x : 1 : z] \mapsto [x : t_1 : t_2z]$

$= [t_1^{-1}x : 1 : t_1^{-1}t_2z].$

$\therefore T_q M \cong V(-\alpha) \oplus V(\beta - \alpha).$

Here, $\langle \alpha, \beta \rangle = t^*_\mathbb{Z} \cong H^2(BT)$. 
Basic torus graphs

(1) \((S^6, T^3)\) where \(S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}\):

(2) 6-dim quasitoric, i.e., 2-connected planner, e.g. \((\mathbb{C}P^3, T^3)\):

(3) \(S^4\)-bdl over \(S^2\):
3-valent torus graphs

Theorem

$(\Gamma, \mathcal{A})$ (induced from 6-dim simply con torus mfds with $H^{\text{odd}}(M) = 0$) can be constructed from the connected sum of previous three graphs.
3-valent torus graphs

**Theorem**

\((\Gamma, A)\) (induced from 6-dim simply con torus mfds with \(H^{odd}(M) = 0\)) can be constructed from the connected sum of previous three graphs.

**Key step of the proof.**

\((\Gamma, A)\) has multiple edges \(\Leftrightarrow (\Gamma, A)\) can be split into (2) and (3):

![Diagram](image-url)
Therefore, we have

**Corollary**

Let $M$ be a simply connected, 6-dim torus manifold with $H^{odd}(M) = 0$. Then,

$$M \cong M_1 \# \cdots \# M_\ell$$

where $M_i$ is one of the followings:

1. $S^6$;
2. 6-dim quasitoric manifold;
3. $S^4$-bundle over $S^2$. 
4. Main Result 2

By using the classification results as above (geometric topological classification), we also have the following algebraic topological classification:

**Theorem**

Let $M, M'$ be simply con. $2n$-dim torus manifolds (for $n \leq 3$) with $H^{odd}(M) = 0$. Then, $H^*_T(M) \simeq_{alg}^w H^*_T(M') \iff (M, T) \simeq_{diff}^w (M', T)$.

Here, $H^*_T(M)$ is an equivariant cohomology $H^*(BT)$-algebra defined by taking the cohomology of $ET \times_T M \to ET / T = BT$, i.e.,

$$H^*(BT) \to H^*(ET \times_T M) = H^*_T(M).$$
Remarks on Main thm 2

Remark

Let $W = M \#_T (hS^3 \times T^3)$. Then, $H^*_T (M) \cong_{alg} H^*_T (W)$, however $M \not\cong W$ unless $hS^3 = S^3$. Therefore, Main thm 2 does not hold if we do not assume simply connected.
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Main thm 2 does not hold if we do not assume simply connected.

Remark

There are infinitely many \( T^n \)-actions on \( S^{2n} \) if \( n \geq 4 \) (distinguished by their orbit spaces \( S^{2n}/T^n \) (by Wiemeler)).
However, all \( H^*_T(S^{2n}) \) are isomorphic (because torus graphs are the same).
Therefore, Main thm 2 does not hold for the case when \( \dim M \geq 8 \).
Proposition (Top. classification)

Let \( M(a, b) = S^3 \times_{S^1} S(\mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{R}) \). Then,

1. \( H^*(M(a, b)) \cong H^*(S^2 \times S^4) \);
2. \( p(M(a, b)) = 1 \);
3. \( w(M(a, b)) = 1 + (a + b)x \), where \( x \in H^2(M(a, b); \mathbb{Z}_2) \).

Therefore, (by using Jupp’s result) there are only two topological types of \( M(a, b) \). More precisely,

- \( M(a, b) \cong S^2 \times S^4 \) if \( a + b \equiv_2 0 \);
- \( M(a, b) \) is a non-trivial \( S^4 \)-bdl over \( S^2 \) if \( a + b \equiv_2 1 \).
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Let $M(a, b) = S^3 \times_{S^1} S(C_a \oplus C_b \oplus \mathbb{R})$. Then,

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Remark

This is NOT cohomologically rigid.
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Problem (Cohomological rigidity problem for dim= 6 (Masuda-Suh))

Are 6-dim quasitoric mfds cohomologically rigid?
Happy 60th birthday to Prof. Yamazaki and Prof. Nagata!