On group actions with codimension one orbits

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Table of Contents

1 Introduction

2 General cases

3 Torus manifolds with codimension one extended actions

4 Applications
Definitions and Examples

Definition

We say $\varphi : G \times M \to M$ a $G$-action on $M$ if $\varphi$ satisfies the following two conditions:

1. $\varphi(e, x) = x$;
Definitions and Examples

Definition

We say \( \varphi : G \times M \rightarrow M \) a \( G \)-action on \( M \) if \( \varphi \) satisfies the following two conditions:

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2. \( \varphi(g, \varphi(h, x)) = \varphi(gh, x) \).
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1. $\varphi(e, x) = x$;
2. $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$.

Definition

We say the set $G(x) = \{\varphi(g, x) = gx \mid g \in G\}$ a $G$-orbit of $x$. The set of $G$-orbits is said to be an orbit space and denote it by $M/G$. 
Example

Let \((X, r) \in S^n \subset \mathbb{R}^n \oplus \mathbb{R}\) and \(A \in SO(n)\). The following map gives an \(SO(n)\)-action on \(S^n\):

\[
(X, r) \xrightarrow{A} (AX, r).
\]
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On group actions with codimension one orbits

Introduction

Codimension one group actions

In this talk, we assume \( \exists x \in M \) such that

\[
\dim M - \dim G(x) = 1.
\]

Classifications of such actions have been studied by many mathematicians: Wang (1960), Bredon(1972), Uchida(1979), Asoh(1981), Alekseevskii-Alekseevskii(1992), Hambleton-Hausmann(2003), etc.
Introduction

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Classifications of such actions have been studied by many mathematicians: Wang (1960), Bredon (1972), Uchida (1979), Asoh (1981), Alekseevskii-Alekseevskii (1992), Hambleton-Hausmann (2003), etc.

Today’s talk

I will introduce how to classify such actions (in particular, Wang-Uchids’s classification method for Hattori-Masuda’s torus manifolds).
On group actions with codimension one orbits

Basic facts

Theorem (slice theorem)

For every orbit \( G(x) \cong G/K \) of \( x \in M \), there is a \( G \)-invariant closed neighborhood \( X \) such that

\[
X \cong G \times_K D_x,
\]

where

- \( D_x \) is a closed \( k_x \)-dim disk, where \( k_x = \dim M - \dim G(x) \).
Basic facts

Theorem (slice theorem)

For every orbit $G(x) \cong G/K$ of $x \in M$, there is a $G$-invariant closed neighborhood $X$ such that

$$X \cong G \times_K D_x,$$

where

- $D_x$ is a closed $k_x$-dim disk, where $k_x = \dim M - \dim G(x)$;
- $K$ acts on $D_x$ by the slice representation $\sigma : K \to O(k_x)$. 

On group actions with codimension one orbits

- General cases

**Slice theorem**

For every $G(x) = G/K$, $\exists$ a tubular neighborhood $X \cong G \times_K D_x$. 

\[ \text{Diagram showing the slice theorem: } G/K \rightarrow X \]
On group actions with codimension one orbits

General cases

Four cases

Because

1. the slice theorem,
On group actions with codimension one orbits

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On group actions with codimension one orbits

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Case 2 \( M/G = \mathbb{R}^+ \);
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Case 1  $M/G = \mathbb{R}$;
Case 2  $M/G = \mathbb{R}^+$;
Case 3  $M/G = S^1$;
On group actions with codimension one orbits

General cases

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Case 1 \( M/G = \mathbb{R} \);
Case 2 \( M/G = \mathbb{R}^+ \);
Case 3 \( M/G = S^1 \);
Case 4 \( M/G = [0, 1] \).
Case 1; $M/G = \mathbb{R}$

Using the slice theorem, $M \cong G/K \times \mathbb{R}$. 
Case 1; \( M/G = \mathbb{R} \)

Using the slice theorem, \( M \cong G/K \times \mathbb{R} \).

Case 1

Classification of homogeneous spaces \( G/K \).
(Theory of Lie group).
Case 2; $M/G = \mathbb{R}^+$

Using the slice theorem, $M \cong G \times_K V$, where $K$ acts on $V$ via $\sigma: K \to O(V)$ such that $K$ acts on $S(V) \cong K/H$ transitively.
On group actions with codimension one orbits

General cases

Case 2; \( M/G = \mathbb{R}^+ \)

Using the slice theorem, \( M \cong G \times_K V \), where \( K \) acts on \( V \) via \( \sigma : K \to O(V) \) such that \( K \) acts on \( S(V) \cong K/H \) transitively.

---

**Case 2**

Classification of transitive, linear actions on spheres.

(Borel, Montgomery-Samelson, Poncet in 1940’s–1950’s).
Case 3; $M/G = S^1$

In this case, there is the following fibre bundle:

$$G/K \hookrightarrow M \twoheadrightarrow S^1,$$

s.t. its structure group is $N_G(K)/K = \{f : G/K \xrightarrow{G} G/K\}$. 
On group actions with codimension one orbits

General cases

Case 3; \( M/G = S^1 \)

In this case, there is the following fibre bundle:

\[
G/K \leftrightarrow M \rightarrow S^1,
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s.t. its structure group is \( N_G(K)/K = \{ f : G/K \rightarrow G/K \} \).

Case 3

Classification of fibre bundles.
(Theory of fibre bundle)
On group actions with codimension one orbits

Case 4; $M/G = [0, 1]$

In this case, $M$ can be constructed by attaching $X_1$ and $X_2$ s.t. $X_i/G = \mathbb{R}^+$ (Case 2).

The well-known classification cannot be used for Case 4.
Case 4; $M/G = [0, 1]$

In this case, $M$ can be constructed by attaching $X_1$ and $X_2$ s.t. $X_i/G = \mathbb{R}^+$ (Case 2).

The well-known classification cannot be used for Case 4.

Case 4

The essential case!
Torus manifolds

Definition

Let $M$ be a $2n$-dimensional, compact, connected, oriented manifold and $T$ an $n$-dimensional torus. We say $(M, T)$ a torus manifold if $(M, T)$ satisfies the following two conditions:

1. $T$ acts on $M$ effectively;
Torus manifolds

Definition

Let $M$ be a $2n$-dimensional, compact, connected, oriented manifold and $T$ an $n$-dimensional torus. We say $(M, T)$ a torus manifold if $(M, T)$ satisfies the following two conditions:

1. $T$ acts on $M$ effectively;
2. $M^T \neq \emptyset$. 
Let \((z_1, \ldots, z_n, r) \in S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}\) and \((t_1, \ldots, t_n) \in T^n\). The following map gives a \(T^n\)-action on \(S^{2n}\):

\[
(z_1, \ldots, z_n, r) \mapsto (t_1 z_1, \ldots, t_n z_n, r).
\]
Example

1. Let \((z_1, \ldots, z_n, r) \in S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}\) and \((t_1, \ldots, t_n) \in T^n\). The following map gives a \(T^n\)-action on \(S^{2n}\):

\[
(z_1, \ldots, z_n, r) \mapsto (t_1z_1, \ldots, t_nz, n, r).
\]

2. Let \([z_0 : \cdots : z_n] \in \mathbb{C}P(n) = \mathbb{C}^{n+1} - \{o\}/\mathbb{C}^*\). The following map gives a \(T^n\)-action on \(\mathbb{C}P^n\):

\[
[z_0 : z_1 : \cdots : z_n] \mapsto [z_0 : t_1z_1 : \cdots : t_nz_n].
\]
The previous examples are induced by the following actions:

1. $X \in U(n)$ (or $X \in SO(2n)$) acts on $S^{2n} \cap \mathbb{C}^n$ (or $S^{2n} \cap \mathbb{R}^{2n}$), i.e.,

$$S^{2n} \ni (z, r) \mapsto (Xz, r) \in S^{2n}.$$
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2. $X \in PU(n + 1)$ acts on $\mathbb{C}P^n$ naturally, i.e.,

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Remark

The 1st action has codimension one orbits $S^{2n} - 1$;

The 2nd action has the codimension zero orbit $\mathbb{C}P^n$. 
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The previous examples are induced by the following actions:

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   \[ S^{2n} \ni (z, r) \mapsto (Xz, r) \in S^{2n} \].

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Remark

1. The 1st action has \textit{codimension one orbits} \( S^{2n-1} \);
2. The 2nd action has the \textit{codimension zero orbit} \( \mathbb{C}P^n \).
Problem

*Classify torus manifolds \((M, T)\) with codimension 0 or 1 extended actions \((M, G)\).*
Problem

Classify torus manifolds \((M, T)\) with codimension 0 or 1 extended actions \((M, G)\).

Definition

An extended action \((M, G)\) of \((M, T)\) is an action which commutes the following diagram:

\[
\begin{array}{ccc}
G \times M & \xrightarrow{\bar{\varphi}} & M \\
\cup & \rightarrow & \varphi \\
T \times M & &
\end{array}
\]

where \(T\) is a maximal torus of a compact, connected Lie group \(G\).
Wang-Uchida’s method for $M/G = [0, 1]$

**Assumption**

A $2n$-dimensional torus manifold $(M, T)$ extends to an $(M, G)$ with codimension one orbits $G/K$, i.e., $\dim G/K = 2n - 1$. 
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- Let $p \in M^T$ (fixed point).
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- Let $p \in M^T$ (fixed point).
- The orbit $G(p) \cong G/G_p$ satisfies that $\mathcal{T} \subset G_p \subset G$. 

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On group actions with codimension one orbits

Torus manifolds with codimension one extended actions

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- Let $p \in M^T$ (fixed point).
- The orbit $G(p) \cong G/G_p$ satisfies that
  
  $T \subset G_p \subset G$.

- Therefore, $\dim G(p)$ is even, i.e.,
  
  $\dim G(p) < 2n - 1$. 


Wang-Uchida’s method for $M/G = [0, 1]$

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- Let $p \in M^T$ (fixed point).
- The orbit $G(p) \cong G/G_p$ satisfies that $T \subset G_p \subset G$.
- Therefore, $\dim G(p)$ is even, i.e., $\dim G(p) < 2n - 1$.
- Because $M$ is compact, we have $M/G = [0, 1]$.
Moreover, we have $G(p)$ is a homogeneous torus manifold, i.e., a torus manifold with codimension 0 extended action.
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**Assumption**

To simplify the argument, we assume two singular orbits $G/K_i$ $(i = 1, 2)$ are (simply connected) homogeneous torus manifolds.
Step 1 – Two singular orbits –

Moreover, we have $G(p)$ is a homogeneous torus manifold, i.e., a torus manifold with codimension 0 extended action.

Assumption

To simplify the argument, we assume two singular orbits $G/K_i$ ($i = 1, 2$) are (simply connected) homogeneous torus manifolds.
Homogeneous torus manifolds

Using Lie group theory (Borel-deSiebental), we have

**Theorem**

Let $G/K_i$ ($i = 1, 2$) be a (simply connected) homogeneous torus manifold. Then $G$ and $K_i$ are one of the followings:

$$G \approx \prod_{j=1}^{a} PU(\ell_j + 1) \times \prod_{h=1}^{b} SO(2m_h + 1) \times G'_i,$$

$$K_i \approx \prod_{j=1}^{a} P(U(\ell_j) \times U(1)) \times \prod_{h=1}^{b} SO(2m_h) \times G'_i.$$

Hence, we have

$$G/K_i \cong \prod_{j=1}^{a} \mathbb{C}P(\ell_j) \times \prod_{h=1}^{b} S^{2m_h}.$$
Step 2 – Two tubular neighborhoods –

Next, we compute slice representations $\sigma_i : K_i \rightarrow SO(2k_i)$ s.t. $G'_i (\subset K_i)$ acts transitively on $S^{2k_i-1}$, where $2k_i = 2n - \dim G/K_i$.

Then, there are three cases:

1. $G'_i \simeq SU(k_i) \times T^1$;

Moreover, we have

for each case.
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Next, we compute slice representations \( \sigma_i : K_i \rightarrow SO(2k_i) \) s.t. \( G'_i(\subset K_i) \) acts transitively on \( S^{2k_i-1} \), where 
\[
2k_i = 2n - \dim G / K_i.
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Then, there are three cases:

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2. \( G'_i \simeq SO(2k_i). \)

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1. $G'_i \simeq SU(k_i) \times T^1$;
2. $G'_i \simeq SO(2k_i)$.

Moreover, we have

- $K \simeq \sigma_i^{-1}(SO(2k_i - 1))$;

for each case.
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Then, there are three cases:

1. $G'_i \cong SU(k_i) \times T^1$;
2. $G'_i \cong SO(2k_i)$.

Moreover, we have

- $K \cong \sigma_i^{-1}(SO(2k_i - 1))$;
- $X_i \cong G \times_{K_i} D^{k_i}$,

for each case.
Step 3 – Attaching map and construction—

Next, we compute attaching maps

\[ \tau : \partial X_1 \cong G/K \xrightarrow{G} \partial X_2 \cong G/K, \]

i.e., we compute \( \tau \in N_G(K)/K \).

Because we have \( X_1 \) and \( X_2 \) in Step 2, i.e., \( X_i = G \times K_i V_i \), a \( G \)-manifold \( M \) can be constructed by \( \tau \):

\[ M = X_1 \cup_{\tau} X_2. \]
Uchida’s criterion

To determine equivariant diffeomorphism types of $M(\tau) = X_1 \cup_{\tau} X_2$, it is enough to use the following Uchida’s criterion:

**Lemma (Uchida’s criterion)**

$M(\tau)$ is equivariantly diffeomorphic to $M(\tau')$ for $\tau, \tau' \in N_G(K)/K$ if one of the followings is satisfied:
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**Lemma (Uchida’s criterion)**

$M(\tau)$ is equivariantly diffeomorphic to $M(\tau')$ for $\tau, \tau' \in N_G(K)/K$ if one of the followings is satisfied:

1. $\tau$ and $\tau'$ are $G$-diffeotopic;
2. $\tau^{-1}\tau' : \partial X_1 \to \partial X_1$ or $\tau'\tau^{-1} : \partial X_2 \to \partial X_2$ is extendable to an equivariantly diffeomorphic on $X_i$. 
Main theorem

If the torus manifold \((M, T)\) extends to \((M, G)\) with codimension one orbits under the assumptions above, \((M, G)\) is one of the following manifolds:

\[
\left( \prod_h S^{2m_h} \times N, \prod_h SO(2m_h + 1) \times H \right),
\]

where \((N, H)\) satisfies one of the followings:

<table>
<thead>
<tr>
<th>(N)</th>
<th>(H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\prod_j S^{2\ell_j+1} \times T^a S(\mathbb{C}_\rho^k \oplus \mathbb{R}))</td>
<td>(\prod_j PU(\ell_j + 1) \times U(k))</td>
</tr>
<tr>
<td>(\prod_j S^{2\ell_j+1} \times T^a P(\mathbb{C}<em>\rho^{k_1} \oplus \mathbb{C}</em>\rho^{k_2}))</td>
<td>(\prod_j PU(\ell_j + 1) \times P(U(k_1) \times U(k_2)))</td>
</tr>
<tr>
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<td>(\prod_j PU(\ell_j + 1) \times SO(2k))</td>
</tr>
</tbody>
</table>

where \(\rho : T^a \rightarrow S^1\).
Corollary

If a (quasi) toric manifold \((M, T)\) extends to \((M, G)\) with codimension one orbits, then \((M, G)\) is

\[
\left( \prod_{j=1}^{a} S^{2\ell_j + 1} \times T^a P(\mathbb{C}^{k_1}_\rho \oplus \mathbb{C}^{k_2}), \prod_{j=1}^{a} PU(\ell_j + 1) \times P(U(k_1) \times U(k_2)) \right)
\]
Cohomological rigidity problem

Problem (Masuda-Suh ’06)

Let \((M_1, T^n)\) and \((M_2, T^n)\) be (quasi)toric manifolds (or more generally torus manifolds s.t. \(M_i/T^n\) is homotopy cell). Then,

\[
M_1 \simeq M_2 \iff H^*(M_1) \simeq H^*(M_2).
\]

This problem is still open for (quasi)toric manifolds and many partial affirmative answers are known (Choi-Masuda-Suh, Masuda-Panov, etc).
A compact manifold with corner is **nice** if there are exactly $n$ codimension one faces meeting at each vertex, and is a **homotopy cell** if it is nice and all of its faces are contractible.
Lemma

Our manifolds appeared in Main theorem are torus manifolds whose orbit spaces are homotopy cells.

Figure: $\mathbb{C}P^3/T^3$ and $S^4/T^2$. 
Theorem (Choi-K)

Two stage torus manifolds with codimension one extended actions, $S^{2\ell+1} \times S^1 N$, are classified by their cohomology rings, Pointrjagin classes and Stiefel-Whitney classes.
Theorem (Choi-K)

Two stage torus manifolds with codimension one extended actions, $S^{2\ell+1} \times s_1 N$, are classified by their cohomology rings, Pointrjagin classes and Stiefel-Whitney classes.

Answer

The answer of cohomological rigidity problem for torus manifolds whose orbit spaces are homotopy cells is NO.