Cohomological non-rigidity of eight-dimensional complex projective towers

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Abstract. A complex projective tower or simply a $\mathbb{C}P$-tower is an iterated complex projective fibrations starting from a point. In this paper, we classify certain class of 8-dimensional $\mathbb{C}P$-towers up to diffeomorphism. As a consequence, we show that cohomological rigidity is not satisfied by the collection of 8-dimensional $\mathbb{C}P$-towers, i.e., there is a two distinct 8-dimensional $\mathbb{C}P$-towers which have the same cohomology rings.

1. Introduction

Let $\mathcal{M}$ be a collection of diffeomorphism classes of smooth manifolds and $\mathbf{H}^*\mathcal{M}$ be the isomorphism classes of cohomology rings of manifolds in $\mathcal{M}$. Let $H^* : \mathcal{M} \to \mathbf{H}^*\mathcal{M}$ be the map defined by $M \in \mathcal{M} \mapsto H^*(M;\mathbb{Z})$. In general, H^* is not bijective. However, if we restrict the class of manifolds then this map sometimes becomes a bijection; e.g., if $\mathcal{M}$ is a collection of orientable 2-dimensional manifolds then it is well-known that the map H^* is bijective. We say such collection $\mathcal{M}$ is *cohomologically rigid* or $\mathcal{M}$ satisfies *cohomological rigidity*. The problem asking whether the map $H^* : \mathcal{M} \to \mathbf{H}^*\mathcal{M}$ is bijective or not is called a *cohomological rigidity problem*. In this paper, we study the cohomological rigidity problem for *complex projective towers* (or simply a $\mathbb{C}P$-*tower*) introduced in [**8**].

A $\mathbb{C}P$-*tower* of height m is a sequence of complex projective fibrations

$$C_m \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0 = \{\text{a point}\} \tag{1}$$

where $C_i = P(\xi_{i-1})$ is the projectivization of a complex vector bundle ξ_{i-1} over C_{i-1}. We call each C_i the *ith stage* of the tower. If we forget the tower structure, then we call C_i an *(i-stage)* $\mathbb{C}P$-*manifolds*. In [**8**], we show that the diffeomorphism types of 6-dimensional $\mathbb{C}P$-manifolds are determined by their cohomology rings, i.e., the collection of 6-dimensional $\mathbb{C}P$-manifolds $\mathcal{CPM}^6$ is cohomologically rigid. This is the generalization of the fact that the collection $\mathcal{GBM}^6$ of 6-dimensional generalized Bott manifolds is cohomologically rigid in [**6**]. On the other hand, it is known that the collection $\mathcal{GBM}_2^{2n}$ of $2n$-dimensional 2-stage generalized Bott manifolds is also cohomologically rigid. The purpose of this paper is to show that the collection $\mathcal{CPM}_2^8$ of 8-dimensional 2-stage $\mathbb{C}P$-manifolds is not cohomologically rigid.

To state our main theorem, let us recall the theorem proved by Atiyah and Rees in [**?**, (2.8) Theorem]. Let $\mathcal{VECT}_2(\mathbb{C}P^3)$ be the collection of vector bundle isomorphism classes of complex 2-dimensional vector bundles over $\mathbb{C}P^3$.

Theorem 1.1 (Atiyah-Rees). *There exists an injective map* $\phi : \mathcal{VECT}_2(\mathbb{C}P^3) \to \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ *such that* $\phi(\xi) = (\alpha(\xi), c_1(\xi), c_2(\xi))$*, where* $c_1(\xi)$ *and* $c_2(\xi)$ *are the first and the second Chern classes of* ξ*, and* $\alpha(\xi)$ *is a mod* 2 *element which is* 0 *when* $c_1(\xi)$ *is odd.*

By Theorem 1.1, any element in $\mathcal{VECT}_2(\mathbb{C}P^3)$ can be denoted by $\eta_{(\alpha,c_1,c_2)}$, where $(\alpha, c_1, c_2) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ such that $\alpha \equiv 0 \pmod 2$ when $c_1 \equiv 1 \pmod 2$. On the other hand, it can be seen easily that $P(\eta_{(\alpha,c_1,c_2)})$ is diffeomorphic to $P(\eta_{(0,1,c_2-(c_1^2-1)/4)})$ if $c_1 \equiv 1 \mod 2$, and is diffeomorphic to $P(\eta_{(\alpha,0,c_2-c_1^2/4)})$ if $c_1 \equiv 0 \mod 2$, see Lemma 3.2.

Let $N(u) := P(\eta_{(0,1,u)})$, and let $\mathcal{N} := \{N(u) \mid u \in \mathbb{Z}\}$. Similarly, let $M_\alpha(u) := P(\eta_{(\alpha,0,u)})$, and let $\mathcal{M} := \{M_\alpha(u) \mid \alpha \in \{0,1\}, u \in \mathbb{Z}\}$. We now state the main result of the paper (see Theorem 4.2 for (1) and see Theorem 5.2 for more precise statement of (2)).

THEOREM 1.2. *For the classes $\mathcal{M}$ and $\mathcal{N}$, we have the following.*

(1) *The class $\mathcal{N}$ is cohomologically rigid. In fact, the following are equivalent:*
   (a) *$N(u)$ is diffeomorphic to $N(u')$;*
   (b) *$u = u'$;*
   (c) *$H^*(N(u);\mathbb{Z}) \cong H^*(N(u');\mathbb{Z})$ as graded rings.*

(2) *The class $\mathcal{M}$ is not cohomologically rigid. In fact, $H^*(M_0(u);\mathbb{Z}) \cong H^*(M_1(u);\mathbb{Z})$ as graded rings for all u, but if $\frac{u(u+1)}{12} \in \mathbb{Z}$ then $M_0(u)$ is not diffeomorphic, actually not homotopic, to $M_1(u)$.*

The second part of the theorem is proved in Proposition 5.4 by showing that $\pi_6(M_0(u)) \ncong \pi_6(M_1(u))$ when $\frac{u(u+1)}{12} \in \mathbb{Z}$.

The organization of this paper is as follows. In Section 2, as examples of $\mathbb{C}P$-towers, we explain when flag manifolds admit the structure of $\mathbb{C}P$-tower. In Section 3, we recall some basic facts from [**8**]. In Section 4, we show that $\mathcal{N}$ satisfies the cohomological rigidity. In Section 5, we compute the 6-dimensional homotopy group of the elements in some class of $\mathcal{M}$ and show that $\mathcal{M}$ does not satisfy the cohomological rigidity.

2. Flag manifolds of type A and C

The $\mathbb{C}P$-towers contain many interesting classes of manifolds. In the previous paper [**8**], we introduce that generalized Bott manifolds or the Milnor surface admits the structure of $\mathbb{C}P$-towers. We first introduce the other two examples of $\mathbb{C}P$-towers. Let $\mathcal{CPM}^{2n}_m$ be the collection of $2n$-dimensional m-stage $\mathbb{C}P$-manifolds up to diffeomorphism.

EXAMPLE 2.1. A *partial flag manifold* $\mathcal{F}(d_1, d_2, \ldots, d_k)$, for $0 = d_0 < d_1 < d_2 < \cdots d_{k-1} < d_k = n+1$, is defined by the set of the following partial flags:

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{C}^{n+1},$$

where V_i is a complex subspace of $\mathbb{C}^{n+1}$ with the complex dimension d_i. This is well-known to be diffeomorphic to the homogeneous space $U(n+1)/(U(n_1) \times \cdots \times U(n_k))$, where $n_i = d_i - d_{i-1}$ for $i = 1, \ldots, k$. Denote the partial flag manifold $\mathcal{F}(i, i+1, \ldots, n+1)$ by $\mathcal{F}_i$. In particular, we call $\mathcal{F}_1 = \mathcal{F}(1, 2, \ldots, n+1)$ a *flag manifold of type A* (or a *complete flag manifold*), and denote it by $\mathcal{F}l(\mathbb{C}^{n+1})$. We will show that the flag manifold of type A has the structure of a $\mathbb{C}P$-tower with height n. We first define the map $p_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ by

$$p_i : \{0\} \subset V_i \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1} \mapsto \{0\} \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}.$$

Because the pull-back of a point in $\mathcal{F}_{i+1}$ by p_i can be regarded as the set of codimension one subspaces $V_i \subset V_{i+1}$, $\mathcal{F}_i$ is a $Gr_i(V_{i+1})$-bundle over $\mathcal{F}_{i+1}$. Here, $Gr_i(V_{i+1})$ is the complex Grassmaniann of i-dimensional subspaces in V_{i+1}, i.e., $\mathcal{F}(i, i+1)$. Because the normal subspace of a codimension one subspace $V_i \subset V_{i+1}$ is just the line going through the origin, the complex Grassmaniann $Gr_i(V_{i+1})$ may be regarded as the i-dimensional complex projective space $P(V_{i+1}) = (V_{i+1} \setminus \{0\})/\mathbb{C}^*$. By using this fact, it is easy to check that $\mathcal{F}_i$ is the projectivization of the tautological bundle over $\mathcal{F}_{i+1}$, i.e., $\mathcal{F}_i = P(\eta_{i+1})$, where the tautological bundle η_{i+1} is the complex $(i+1)$-dimensional vector bundle defined by the following subset in $\mathcal{F}_{i+1} \times \mathbb{C}^{n+1}$:

$$\{(\{0\} \subset V_{i+1} \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}, x) \mid x \in V_{i+1}\}.$$

Therefore, $\mathcal{F}l(\mathbb{C}^{n+1})$ has the structure of a $\mathbb{C}P$-tower as follows:

$$\mathcal{F}l(\mathbb{C}^{n+1}) = P(\eta_2) \xrightarrow{\mathbb{C}P^1} \mathcal{F}_2 = P(\eta_3) \xrightarrow{\mathbb{C}P^2} \cdots \xrightarrow{\mathbb{C}P^{n-1}} \mathcal{F}_n \simeq \mathbb{C}P^n \longrightarrow \{*\}.$$

Hence, the flag manifold of type A is an element of $\mathcal{CPM}^{n^2+n}_n$.

Example 2.2. Let $(\mathbb{C}^{2n}, \omega)$ be a complex vector space with a symplectic structure ω given by the skew-symmetric bilinear form defined by the following matrix:

$$\Omega = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix},$$

where O is the $(n \times n)$-zero matrix and I_n is the $(n \times n)$-identity matrix. Let V be a complex linear subspace in $\mathbb{C}^{2n}$. Define the *ω-perpendicular space* of V to be the subspace

$$V^{\omega} = \{w \in \mathbb{C}^{2n} \mid \omega(v, w) = v^T \Omega w = 0 \text{ for all } v \in V\}.$$

Note that $(V^{\omega})^{\omega} = V$ and $\dim_{\mathbb{C}} V + \dim_{\mathbb{C}} V^{\omega} = 2n$. We call V is *isotropic* (resp. *coisotropic*) if $V \subset V^{\omega}$ (resp. $V^{\omega} \subset V$). A *symplectic partial flag manifold* $Sp^n\mathcal{F}(d_1, d_2, \ldots, d_k)$, for $0 = d_0 < d_1 < d_2 < \cdots d_{k-1} < d_k \leq n$, is defined by the set of (isotropic) partial flags

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k \subset \mathbb{C}^{2n},$$

where V_i is a complex isotropic subspace of $(\mathbb{C}^{2n}, \omega)$ with the complex dimension d_i. It is easy to check that this is equivalent to considering the following set of partial flags:

$$\{0\} \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k \subset V_k^{\omega} \subset V_{k-1}^{\omega} \subset \cdots \subset V_1^{\omega} \subset \mathbb{C}^{2n}.$$

This is well-known to be diffeomorphic to the homogeneous space $Sp(n)/(U(n_1) \times \cdots \times U(n_k) \times Sp(n_{k+1}))$, where $n_i = d_i - d_{i-1}$ for $i = 1, \ldots, k$ and $n_{k+1} = \frac{1}{2}(\dim V_k^{\omega} - \dim V_k) = n - d_k$. If $d_k = \dim V_k = n$, i.e., $V_k = V_k^{\omega}$ (called *Lagrangian*), then $Sp^n\mathcal{F}(d_1, d_2, \ldots, d_{k-1}, n)$ is diffeomorphic to $Sp(n)/(U(n_1) \times \cdots \times U(n_k))$. Denote the symplectic partial flag manifold $Sp^n\mathcal{F}(1, 2, \ldots, i)$ by $Sp^n\mathcal{F}_i$ for $i \geq 1$. In particular, we call $Sp^n\mathcal{F}_n = Sp^n\mathcal{F}(1, 2, \ldots, n)$ a *flag manifold of type C* (or a *symplectic flag manifold*), and denote it by $Sp\mathcal{F}l(\mathbb{C}^{2n})$. We will show that the flag manifold of type C has the structure of a $\mathbb{C}P$-tower with height n. We first define the map $q_i : Sp^n\mathcal{F}_{i+1} \to Sp^n\mathcal{F}_i$ by

$$\begin{aligned} q_i : \{0\} &\subset V_1 \subset \cdots \subset V_i \subset V_{i+1} \subset V_{i+1}^{\omega} \subset V_i^{\omega} \subset \cdots \subset V_1^{\omega} \subset \mathbb{C}^{2n} \\ &\mapsto \{0\} \subset V_1 \subset \cdots \subset V_i \subset V_i^{\omega} \subset \cdots \subset V_1^{\omega} \subset \mathbb{C}^{2n}. \end{aligned}$$

The pull-back of a point in $Sp^n\mathcal{F}_i$ by q_i can be regarded as the set of isotropic subspaces V_{i+1} in $\mathbb{C}^{2n}$ which contains the isotropic subspace V_i as a codimension one subspace. Note that, for any vectors $v \in V_i^{\omega} \setminus V_i$, the subspace $V_i \oplus \mathrm{span}_{\mathbb{C}}(v)$ is an isotropic subspace which contains V_i as a codimension one subspace. Therefore, there exists a one-to-one correspondence between the pull-back of a point in $Sp^n\mathcal{F}_i$ by q_i and all complex lines in the quotient vector space $V_i^{\omega}/V_i \simeq \mathbb{C}^{2n-2i}$, i.e., $Sp^n\mathcal{F}_{i+1}$ is a $\mathbb{C}P^{2n-2i-1}$-bundle over $Sp^n\mathcal{F}_i$. By using this fact, it is easy to check that $Sp^n\mathcal{F}_{i+1}$ is the projectivization of the quotient bundle over $Sp^n\mathcal{F}_i$, i.e., $Sp^n\mathcal{F}_{i+1} = P(\zeta_i^{\omega}/\zeta_i)$, where two tautological bundles ζ_i^{ω} and ζ_i are defined by the following subsets in $Sp^n\mathcal{F}_i \times \mathbb{C}^{2n}$, respectively:

$$\begin{aligned} &\{(\{0\} \subset V_1 \subset \cdots \subset V_i \subset V_i^{\omega} \subset \cdots \subset V_1^{\omega} \subset \mathbb{C}^{2n}, x) \mid x \in V_i^{\omega}\}, \\ &\{(\{0\} \subset V_1 \subset \cdots \subset V_i \subset V_i^{\omega} \subset \cdots \subset V_1^{\omega} \subset \mathbb{C}^{2n}, x) \mid x \in V_i\}. \end{aligned}$$

Note that ζ_i^{ω} is the $\mathbb{C}^{2n-i}$-vector bundle and ζ_i is the $\mathbb{C}^i$-vector bundle; therefore, the quotient bundle ζ_i^{ω}/ζ_i is the $\mathbb{C}^{2n-2i}$-vector bundle. Therefore, $Sp\mathcal{F}l(\mathbb{C}^{2n})$ has the structure of a $\mathbb{C}P$-tower as follows:

$$\begin{aligned} &Sp\mathcal{F}l(\mathbb{C}^{2n}) = P(\zeta_{n-1}^{\omega}/\zeta_{n-1}) \xrightarrow{\mathbb{C}P^1} Sp^n\mathcal{F}_{n-1} = P(\zeta_{n-2}^{\omega}/\zeta_{n-2}) \xrightarrow{\mathbb{C}P^3} \cdots \\ &\cdots \xrightarrow{\mathbb{C}P^{2n-3}} Sp^n\mathcal{F}_1 \simeq \mathbb{C}P^{2n-1} \longrightarrow \{*\}. \end{aligned}$$

Hence, the flag manifold of type C is an element of $\mathcal{CPM}_n^{2n^2}$.

Remark. As is well-known, both of the flag manifolds $\mathcal{F}l(\mathbb{C}^{n+1}) \simeq U(n+1)/T^{n+1}$ and $Sp\mathcal{F}l(\mathbb{C}^{2n}) \simeq Sp(n)/T^n$ with $n \geq 2$ do not admit the structure of a *toric manifold* (see e.g. [**2**]). On the other hand, $U(2)/T^2 \cong Sp(1)/T^1 \cong \mathbb{C}P^1$ is a toric manifold.

Moreover, by computing the generators of flag manifolds of other types (B_n ($n \geq 3$), D_n ($n \geq 4$), G_2, F_4, E_6, E_7, E_8), they do not admit the structure of $\mathbb{C}P$-towers, see [**1**] (or [**7**] for classical types). Namely, we have the following proposition:

PROPOSITION 2.3. *Let M be a flag manifold denoted by G/T, where G is a compact simple Lie group and T is its maximal torus. If M admits the structure of a $\mathbb{C}P$-tower, then G must be a compact Lie group of type A or C.*

The following problem also naturally arises (also see Remark 5).

PROBLEM 1. Let $H^* : \mathcal{CPM} \to \mathbf{H}^*\mathcal{CPM}$ be the map defined by taking the cohomology rings. Classify diffeomorphism types of all manifolds in the class $(H^*)^{-1}(H^*(U(n+1)/T^{n+1}))$ and $(H^*)^{-1}(H^*(Sp(n)/T^n))$.

3. Some preliminaries

In this section, we recall some basic facts.

3.1. Preliminaries from [**8**]. We first recall some basic facts from [**8**, Section 2].

Let ξ be an n-dimensional complex vector bundle over a topological space X, and let $P(\xi)$ denote its projectivization. Then, the following formula holds (see [**8**]):

$$H^*(P(\xi);\mathbb{Z}) \cong H^*(X;\mathbb{Z})[x]/\langle x^{n+1} + \sum_{i=1}^{n}(-1)^i c_i(\pi^*\xi)x^{n+1-i}\rangle \tag{2}$$

where $\pi^*\xi$ is the pull-back of ξ along $\pi : P(\xi) \to X$ and $c_i(\pi^*\xi)$ is the ith Chern class of $\pi^*\xi$. Here x can be viewed as the first Chern class of the canonical line bundle over $P(\xi)$, i.e., the complex 1-dimensional sub-bundle γ_ξ in $\pi^*\xi \to P(\xi)$ such that the restriction $\gamma_\xi|_{\pi^{-1}(a)}$ is the canonical line bundle over $\pi^{-1}(a) \cong \mathbb{C}P^{n-1}$ for all $a \in X$. Therefore $\deg x = 2$. Since it is well-known that the induced homomorphism $\pi^* : H^*(X;\mathbb{Z}) \to H^*(P(\xi);\mathbb{Z})$ is injective, we often abuse the notation $c_i(\pi^*\xi)$ by $c_i(\xi)$. The formula (2) is called the *Borel-Hirzebruch formula*.

In order to prove the main theorem, we often use the following two lemmas.

LEMMA 3.1. *Let γ be any complex line bundle over M, and let $P(\xi)$ be the projectivization of a complex vector bundle ξ over M. Then, $P(\xi)$ is diffeomorphic to $P(\xi \otimes \gamma)$.*

LEMMA 3.2. *Let γ be a complex line bundle, and let ξ be a 2-dimensional complex vector bundle over a manifold M. Then the Chern classes of the tensor product $\xi \otimes \gamma$ are as follows.*

$$\begin{aligned} c_1(\xi \otimes \gamma) &= c_1(\xi) + 2c_1(\gamma); \\ c_2(\xi \otimes \gamma) &= c_1(\gamma)^2 + c_1(\gamma)c_1(\xi) + c_2(\xi). \end{aligned}$$

3.2. Atiyah-Rees's theorem. By Theorem 1.1, all of the complex 2-plane bundles over $\mathbb{C}P^3$ can be denoted by $\eta_{(\alpha,c_1,c_2)}$ for some $(\alpha, c_1, c_2) \in \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$. Using Lemma 3.1, its projectivization $P(\eta_{(\alpha,c_1,c_2)})$ is diffeomorphic to $P(\eta_{(\alpha,c_1,c_2)} \otimes \gamma)$ for any complex line bundle γ over $\mathbb{C}P^3$. Moreover, by Lemma 3.2 and the proof of Theorem 1.1 in [**?**], we also have

$$\eta_{(\alpha,c_1,c_2)} \otimes \gamma \equiv \eta_{(\alpha,c_1+2c_1(\gamma),c_1(\gamma)^2+c_1(\gamma)c_1+c_2)}.$$

Therefore, we may assume $c_1 \in \{0, 1\}$. Consequently, in order to classify all $P(\eta_{(\alpha,c_1,c_2)})$ up to diffeomorphisms, it is enough to classify the following:

$$\begin{aligned} M_0(u) &= P(\eta_{(0,0,u)}); \\ M_1(u) &= P(\eta_{(1,0,u)}); \\ N(u) &= P(\eta_{(0,1,u)}), \end{aligned}$$

where $u \in \mathbb{Z}$. We denote the class of $M_0(u)$, $M_1(u)$ up to diffeomorphism by $\mathcal{M}$ and that of $N(u)$ by $\mathcal{N}$. Then, both classes $\mathcal{M}$ and $\mathcal{N}$ are the subclasses of $\mathcal{CPM}_2^8$ consisting of 8-dimensional 2-stage $\mathbb{C}P$-manifolds.

3.3. Intersection of two classes $\mathcal{M}$ and $\mathcal{N}$ are empty. Finally, in this section, we prove $\mathcal{M} \cap \mathcal{N} = \emptyset$ by comparing their cohomology rings. Namely, we prove the following lemma:

LEMMA 3.3. *Two cohomology rings $H^*(M_\alpha(u))$ and $H^*(N(u'))$ are not isomorphic for any $u,\ u' \in \mathbb{Z}$.*

PROOF. By using the Borel-Hirzebruch formula (2), we also have the cohomology rings with $\mathbb{Z}_2$-coefficient as follows:

$$H^*(M_\alpha(u); \mathbb{Z}_2) \cong \mathbb{Z}_2[X, Y]/\langle X^4,\ uX^2 + Y^2\rangle, \quad \text{and}$$
$$H^*(N(u'); \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/\langle x^4,\ u'x^2 + xy + y^2\rangle.$$

Now, the element $uX + Y$ in $H^2(M_\alpha(u); \mathbb{Z}_2)$ satisfies that

$$(uX + Y)^2 = u^2X^2 + 2uXY + Y^2 \equiv uX^2 + Y^2 (= 0) \mod 2$$

However, the squares of all non-zero elements $x,\ y,\ x + y$ in $H^2(N(u'); \mathbb{Z}_2)$ are not zero because of its ring structure. Hence, $H^*(M_\alpha(u)) \not\cong H^*(N(u'))$ for all $u,\ u' \in \mathbb{Z}$. $\square$

Hence, we have the following corollary:

COROLLARY 3.4. *There are no intersections between two classes $\mathcal{M}$ and $\mathcal{N}$.*

4. Cohomological rigidity of $\mathcal{N}$

In this section, we shall prove the cohomological rigidity of the class $\mathcal{N}$. To show that, it is enough to prove the following lemma.

LEMMA 4.1. *The following two statements are equivalent.*

(1) $H^*(N(u)) \cong H^*(N(u'))$
(2) $u = u' \in \mathbb{Z}$

PROOF. Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : H^*(N(u)) \cong H^*(N(u'))$ where

$$H^*(N(u)) \cong \mathbb{Z}[X, Y]/\langle X^4,\ uX^2 + xy + Y^2\rangle;$$
$$H^*(N(u')) \cong \mathbb{Z}[x, y]/\langle x^4,\ u'x^2 + xy + y^2\rangle.$$

Here, we may put

$$f(X) = ax + by \quad \text{and} \quad f(Y) = cx + dy,$$

for some $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = \epsilon = \pm 1$. By taking its inverse, we also have that

$$f^{-1}(x) = d\epsilon X - b\epsilon Y \quad \text{and} \quad f^{-1}(y) = -c\epsilon X + a\epsilon Y.$$

Because $f(Y^2 + XY + uX^2) = 0$ and $f^{-1}(y^2 + xy + u'x^2) = 0$, we have that

$$c^2 - d^2u' = -ua^2 + b^2uu' - ac + bdu'; \tag{3}$$
$$2cd - d^2 = -2abu + b^2u - ad - bc + bd. \tag{4}$$

Because $f(X^4) = 0$ and $f^{-1}(x^4) = 0$, there are the following two cases:

(1) $b = 0$;
(2) $b \neq 0$ and $4a^3 - 6a^2b + 4ab^2(1-u') + b^3(2u'-1) = -4d^3 - 6d^2b - 4db^2(1-u) + b^3(2u-1) = 0$.

If $b = 0$, then $|a| = |d| = 1$. Therefore, by (4), $2c = d - a$, i.e., $c = 0$ if $d = a$ or $c = -a$ if $d = -a$. Because $c^2 - u' = -u - ac$ by (3), we have that $u = u'$.

Assume $b \neq 0$. By the equation $4a^3 - 6a^2b + 4ab^2(1 - u') + b^3(2u' - 1) = 0$, we have b is even. Therefore, together with $ad - bc = \pm 1$, we also have a is odd. We note that the equation $4a^3 - 6a^2b + 4ab^2(1 - u') + b^3(2u' - 1) = 0$ can be written as

$$(2a - b)(2a^2 - 2ab + b^2 - 2b^2u') = 0.$$

Because a is odd and b is even, the 2nd factor is not zero; therefore, we have that

$$b = 2a.$$

Since $ad - bc = \pm 1$, we conclude $(a, b) = \pm(1, 2)$. The same argument applied to the equation $-4d^3 - 6d^2b - 4db^2(1-u) + b^3(2u-1) = 0$ shows that $-b = 2d$ and $(d, b) = \pm(-1, 2)$. Therefore, (a, b, d) must be either $(1, 2, -1)$ or $(-1, -2, 1)$. Then, $c = 0, -1$ in the former case while $c = 0, 1$ in the latter case because $ad - bc = \pm 1$. In any case, it follows from (3) that $u' + u = 4uu'$ and this identity holds only when $u = u' = 0$ since $u, u' \in \mathbb{Z}$. This completes the case where $b \neq 0$. $\square$

Therefore, by Theorem 1.1 and Lemma 4.1, we have the following theorem.

THEOREM 4.2. *The following three statements are equivalent.*

(1) *Two spaces $N(u)$ and $N(u')$ are diffeomorphic.*
(2) *Two cohomology rings $H^*(N(u))$ and $H^*(N(u'))$ are isomorphic.*
(3) *$u = u' \in \mathbb{Z}$.*

In particular, the class $\mathcal{N}$ is cohomologically rigid.

This establishes Theorem 1.2 (1).

5. Cohomological non-rigidity of $\mathcal{CPM}_2^8$

In this section, we prove that $\mathcal{M}$ is not cohomologically rigid. We first show the following fact about the cohomology rings of elements in $\mathcal{M}$.

LEMMA 5.1. *The following two statements are equivalent.*

(1) *$H^*(M_\alpha(u)) \cong H^*(M_{\alpha'}(u'))$ where $\alpha,\ \alpha' \in \{0, 1\}$.*
(2) *$u = u' \in \mathbb{Z}$*

PROOF. Because (2) $\Rightarrow$ (1) is trivial, it is enough to show (1) $\Rightarrow$ (2). Assume there is an isomorphism $f : H^*(M_\alpha(u)) \cong H^*(M_{\alpha'}(u'))$ where

$$H^*(M_\alpha(u)) \cong \mathbb{Z}[X, Y]/\langle X^4,\ uX^2 + Y^2\rangle;$$
$$H^*(M_{\alpha'}(u')) \cong \mathbb{Z}[x, y]/\langle x^4,\ u'x^2 + y^2\rangle.$$

We may use the same representation for f as in the proof of Lemma 4.1. Note that $f(uX^2+Y^2) = 0$ and $f^{-1}(u'x^2 + y^2) = 0$. By using the representation of f, we have the following equations:

$$ua^2 - uu'b^2 + c^2 - u'd^2 = 0; \tag{5}$$
$$u'd^2 - uu'b^2 + c^2 - a^2u = 0. \tag{6}$$

By (5) and (6), we have

$$c^2 = b^2uu'; \tag{7}$$
$$ua^2 = u'd^2. \tag{8}$$

Because $X^4 = 0$, we also have that

$$ab(a^2 - b^2u') = 0.$$

We first assume $ab \neq 0$. Then

$$a^2 = b^2u'$$

by this equation. Together with (7) and (8), we have that

$$c^2b^2 = b^4uu' = b^2a^2u = b^2d^2u' = a^2d^2.$$

This implies that

$$(ad - bc)(ad + bc) = \epsilon(ad + bc) = 0.$$

Hence, $ad = -bc$. However this gives a contradiction because $ad - bc = 2ad = \epsilon = \pm 1$. Consequently, we have $ab = 0$. Since $ad - bc = \epsilon$, if $a = 0$ then $|b| = |c| = 1$; therefore, we have $u = u' = \pm 1$ by (7); if $b = 0$ then $|a| = |d| = 1$; therefore, we have $u = u'$ by (8). This establishes the statement. $\square$

Lemma 5.1 says that cohomology rings of $\mathcal{M}$ are not affected by $\alpha \in \mathbb{Z}_2$. On the other hand, the goal of this section is to prove the following theorem, i.e., some topological types of $\mathcal{M}$ are affected by $\alpha \in \mathbb{Z}_2$.

THEOREM 5.2. *Assume $u(u+1)/12 \in \mathbb{Z}$. The following three statements are equivalent.*

(1) *Two spaces $M_\alpha(u)$ and $M_\beta(u')$ are diffeomorphic.*
(2) $(\alpha, u) = (\beta, u') \in \mathbb{Z}_2 \times \mathbb{Z}$.
(3) *Two spaces $M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent.*

In order to prove Theorem 5.2, we first compute the 6-dimensional homotopy group of $M_\alpha(u)$ in Proposition 5.4. Now $M_\alpha(u)$ can be defined by the following pull-back diagram:

$$\begin{tikzcd} M_\alpha(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\ \mathbb{C}P^3 \arrow[r, "\mu_{\alpha,u}"] & BU(2) \end{tikzcd}$$

Let $p : S^7 \to \mathbb{C}P^3$ be the canonical S^1-fibration and $P(\xi_{\alpha,u})$ be the pull-back of $M_\alpha(u)$ along p. Namely, we have the following diagram:

$$\begin{tikzcd} P(\xi_{\alpha,u}) \arrow[r] \arrow[d] & M_\alpha(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\ S^7 \arrow[r, "p"] & \mathbb{C}P^3 \arrow[r, "\mu_{\alpha,u}"] & BU(2) \end{tikzcd} \tag{9}$$

Then, we have the following lemma.

LEMMA 5.3. *For $* \geq 3$, $\pi_*(P(\xi_{\alpha,u})) \cong \pi_*(M_\alpha(u))$.*

PROOF. Because $P(\xi_{\alpha,u})$ is the pull-back of $M_\alpha(u)$, the homotopy exact sequences of $P(\xi_{\alpha,u})$ and $M_\alpha(u)$ satisfy the following commutative diagram:

$$\begin{tikzcd} \pi_{*+1}(S^7) \arrow[r] \arrow[d] & \pi_*(\mathbb{C}P^1) \arrow[r] \arrow[d] & \pi_*(P(\xi_{\alpha,u})) \arrow[r] \arrow[d] & \pi_*(S^7) \arrow[r] \arrow[d] & \pi_{*-1}(\mathbb{C}P^1) \arrow[d] \\ \pi_{*+1}(\mathbb{C}P^3) \arrow[r] & \pi_*(\mathbb{C}P^1) \arrow[r] & \pi_*(M_\alpha(u)) \arrow[r] & \pi_*(\mathbb{C}P^3) \arrow[r] & \pi_{*-1}(\mathbb{C}P^1) \end{tikzcd}$$

From the homotopy exact sequence of the fibration $S^1 \to S^7 \to \mathbb{C}P^3$, we have $\pi_*(S^7) \cong \pi_*(\mathbb{C}P^3)$ for $* \geq 3$. Therefore, by using the 5 lemma, we have the statement. □

Now we may prove the following proposition.

PROPOSITION 5.4. *Assume $u(u+1)/12 \in \mathbb{Z}$. The following two isomorphisms hold.*

(1) $\pi_6(P(\xi_{\alpha,u})) \cong \pi_6(M_\alpha(u)) \cong \mathbb{Z}_{12}$ *if* $\alpha \equiv u(u+1)/12 \pmod 2$
(2) $\pi_6(P(\xi_{\beta,u})) \cong \pi_6(M_\beta(u)) \cong \mathbb{Z}_6$ *if* $\beta \not\equiv u(u+1)/12 \pmod 2$

PROOF. We first prove the 1st statement. If $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod 2$, then it follows from [**?**] that $\xi_{\alpha,u}$ is induced from the rank 2 complex vector bundle over $\mathbb{C}P^4$. Namely, there is the following commutative diagram:

$$\begin{tikzcd} \xi_{\alpha,u} \arrow[r] \arrow[d] & \eta_{(\alpha,0,u)} \arrow[r] \arrow[d] & \widetilde{\mu}_{\alpha,u} \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}^2 \arrow[d] \\ S^7 \arrow[r, "p"] & \mathbb{C}P^3 \arrow[r] & \mathbb{C}P^4 \arrow[r] & BU(2) \end{tikzcd} \tag{10}$$

On the other hand, we have that $\pi_7(\mathbb{C}P^4) \cong \pi_7(S^9) = \{0\}$, by using the homotopy exact sequence for the fibration $S^1 \to S^9 \to \mathbb{C}P^4$. This implies that $\xi_{\alpha,u}$ is the trivial $\mathbb{C}^2$-bundle over S^7. Therefore,

$$P(\xi_{\alpha,u}) = S^7 \times \mathbb{C}P^1$$

when $u(u+1)/12 \in \mathbb{Z}$ and $\alpha \equiv u(u+1)/12 \pmod 2$. Hence, we also have that

$$\pi_6(M_\alpha(u)) \cong \pi_6(S^7 \times \mathbb{C}P^1) \cong \pi_6(\mathbb{C}P^1) \cong \mathbb{Z}_{12}.$$

Next we prove the 2nd statement. Let $\mu_{\alpha,u} : \mathbb{C}P^3 \to BU(2)$ be a continuous map which induces the above $\eta_{(\alpha,0,u)}$, and β be the element in $\mathbb{Z}_2$ which is not equal to α. Let $x \in \mathbb{C}P^3$ and $s = \mu_{\alpha,u}(x) \in BU(2)$ be base points. Take a disk neighborhood around $x \in \mathbb{C}P^3$ and pinch its boundary to a point, i.e., the boundary of $D^6 \subset \mathbb{C}P^3$ pinches to a point, then we obtain the surjective map

$$\rho : \mathbb{C}P^3 \to \mathbb{C}P^3 \vee S^6,$$

where $\mathbb{C}P^3 \vee S^6$ may be regarded as the wedge sum with respect to the base points $x \in \mathbb{C}P^3$ and $y \in S^6$. Due to theorem of Atiyah-Rees [**?**], we have $\eta_{(\beta,0,u)} \not\equiv \eta_{(\alpha,0,u)}$. This implies that the vector bundle $\eta_{(\beta,0,u)}$ is induced from the following continuous map:

(11)
$$\mu_{\beta,u} : \mathbb{C}P^3 \xrightarrow{\ \rho\ } \mathbb{C}P^3 \vee S^6 \xrightarrow{\ \nu_\alpha\ } BU(2)$$

where $\nu_\alpha = \mu_{\alpha,u} \vee \kappa$ for the generator $\kappa \in \pi_6(BU(2), s) \cong \mathbb{Z}_2$.[1] Hence, we have the following commutative diagram.

(12)
$$\begin{tikzcd}
P(\xi_{\beta,u}) \arrow[r] \arrow[d] & M_\beta(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\
S^7 \arrow[r, "p"] \arrow[dr] & \mathbb{C}P^3 \arrow[r, "\mu_{\beta,u}"] \arrow[d, "\rho"] & BU(2) \\
 & \mathbb{C}P^3 \vee S^6 \arrow[ur, "\nu_\alpha"'] &
\end{tikzcd}$$

From the $\mathbb{C}P^1$-fibrations $\mathbb{C}P^1 \to P(\xi_{\beta,u}) \to S^7$ and $\mathbb{C}P^1 \to EU(2) \times_{U(2)} \mathbb{C}P^1 \cong BT^2 \to BU(2)$ in the above diagram (12), there is the following commutative diagram.

$$\begin{tikzcd}
\pi_7(S^7) \cong \mathbb{Z} \arrow[r] \arrow[d] & \pi_6(\mathbb{C}P^1) \arrow[r] \arrow[d, "\cong"] & \pi_6(P(\xi_{\beta,u})) \arrow[r] \arrow[d] & \pi_6(S^7) = \{0\} \arrow[d] \\
\pi_7(BU(2)) \cong \mathbb{Z}_{12} \arrow[r, "\cong"] & \pi_6(\mathbb{C}P^1) \arrow[r] & \pi_6(BT^2) = \{0\} \arrow[r] & \pi_6(BU(2)) \cong \mathbb{Z}_2
\end{tikzcd}$$

This diagram shows that the following exact sequence:

(13)
$$\mathbb{Z} \cong \pi_7(S^7) \to \pi_7(BU(2))(\cong \mathbb{Z}_{12}) \to \pi_6(P(\xi_{\beta,u})) \to \{0\}.$$

In this diagram, the left homomorphism is induced from $\widetilde{\mu} := \mu_{\beta,u} \circ p \colon S^7 \to BU(2)$, say $\widetilde{\mu}_\# : \mathbb{Z} \to \mathbb{Z}_{12}$. We claim $\widetilde{\mu}_\#(1) = [6]_{12} \in \mathbb{Z}_{12}$. Because the diagram (12) is commutative, we may regard that $\widetilde{\mu} := \mu_{\beta,u} \circ p \colon S^7 \to BU(2)$ can be defined by passing through the map $\nu_\alpha : \mathbb{C}P^3 \vee S^6 \to BU(2)$, i.e., $\widetilde{\mu} = \nu_\alpha \circ \rho \circ p$. Because $\nu_\alpha = \mu_{\alpha,u} \vee \kappa$, we also have

$$\widetilde{\mu} = (\mu_{\alpha,u} \vee \kappa) \circ \rho \circ p = (\mu_{\alpha,u} \circ \rho \circ p) \vee (\kappa \circ \rho \circ p).$$

By the argument when we proved the 1st statement, we see that $\mu_{\alpha,u} \circ \rho \circ p$ induces the trivial bundle over S^7, i.e., $\mu_{\alpha,u} \circ \rho \circ p$ is homotopic to the trivial map. This also implies that there is the following decomposition up to homotopy:

$$\widetilde{\mu} : S^7 \xrightarrow{p} \mathbb{C}P^3 \xrightarrow{\rho} \mathbb{C}P^3 \vee S^6 \xrightarrow{\pi} S^6 \xrightarrow{\kappa} BU(2),$$

where π is the collapsing map of $\mathbb{C}P^3$ to a point. Therefore, we have the following decomposition for the induced map

$$\widetilde{\mu}_\# : \pi_7(S^7) \xrightarrow{\Psi_\#} \pi_7(S^6) \cong \mathbb{Z}_2 \xrightarrow{\kappa_\#} \pi_7(BU(2)) \cong \mathbb{Z}_{12},$$

where the 1st map is induced from the surjective map $\Psi = \pi \circ \rho \circ p$. Because Ψ is surjective, i.e., not homotopic to the trivial map, we have $\Psi_\#(1) = [1]_2$ (the generator of $\pi_7(S^6) \cong \mathbb{Z}_2$). Moreover,

[1] This construction induces the free $\pi_6(BU(2)) \cong \pi_5(U(2)) \cong \mathbb{Z}_2$ action on $\widetilde{KSp}(\mathbb{C}P^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ (see [**?**]).

because $\kappa \in \pi_6(BU(2)) \cong \mathbb{Z}_2$ is the generator, i.e., non-trivial map, we have $\kappa_\#([1]_2) = [6]_{12} \in \mathbb{Z}_{12}$. This shows that $\widetilde{\mu}_\#(1) = [6]_{12}$; therefore, $\widetilde{\mu}_\#(\pi_7(S^7)) = \{[0]_{12},\ [6]_{12}\} \subset \mathbb{Z}_{12}$.

Consequently, by the exact sequence (13), we have that

$$\pi_6(P(\xi_{\beta,u})) \cong \pi_7(BU(2))/\widetilde{\mu}_\#(\pi_7(S^7)) \cong \mathbb{Z}_{12}/\{[0]_{12}, [6]_{12}\} \cong \mathbb{Z}_6.$$

By Lemma 5.3, we have the statement. $\square$

Remark. For example, the condition $u(u+1)/12 \in \mathbb{Z}$ is satisfied when $u = 0$ and $u = 3$. In these cases, by using Proposition 5.4, we have

$$\pi_6(M_\alpha(0)) \cong \begin{cases} \mathbb{Z}_{12} & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_6 & \text{for } \alpha \equiv 1 \end{cases}$$

and

$$\pi_6(M_\alpha(3)) \cong \begin{cases} \mathbb{Z}_6 & \text{for } \alpha \equiv 0 \\ \mathbb{Z}_{12} & \text{for } \alpha \equiv 1 \end{cases}$$

On the other hand, the case when $u = 1$ does not satisfy the condition $u(u+1)/12 \in \mathbb{Z}$. It follows from the cohomology ring of the flag manifold of type C (see e.g. [**1**] or [**7**]) that the flag manifold $Sp(2)/T^2$ is one of this case, i.e., $M_0(1)$ or $M_1(1)$. However, by using the homotopy exact sequence for the fibration $T^2 \to Sp(2) \to Sp(2)/T^2$ and the computation in [**12**], we have that

$$\pi_6(Sp(2)/T^2) \cong \pi_6(Sp(2)) = 0.$$

Therefore, Proposition 5.4 is not true for the case when $u(u+1)/12 \notin \mathbb{Z}$.

Let us prove Theorem 5.2

Proof of Theorem 5.2. By using Theorem 1.1, (2) $\Rightarrow$ (1) is trivial. The statement (1) $\Rightarrow$ (3) is also trivial. We claim (3) $\Rightarrow$ (2). Assume $M_\alpha(u)$ and $M_\beta(u')$ are homotopy equivalent. Then, $H^*(M_\alpha(u)) \cong H^*(M_\beta(u'))$. Therefore, it follows from Lemma 5.1 that $u = u'$. Moreover, in this case, $\pi_6(M_\alpha(u)) \cong \pi_6(M_\beta(u))$. If $\alpha \not\equiv \beta \mod 2$, then this gives a contradiction to Proposition 5.4. Hence, $\alpha \equiv \beta \mod 2$. We have (3) $\Rightarrow$ (2). This establishes Theorem 5.2. $\square$

In summary, by Lemma 5.1 and Theorem 5.2, we have the following corollary:

Corollary 5.5. *The set of 8-dimensional $\mathbb{C}P$-manifolds does not satisfy the cohomological rigidity.*

This establishes Theorem 1.2 (2).

Note that if we restrict the class of 8-dimensional $\mathbb{C}P$-manifolds to the 8-dimensional generalized Bott manifolds with height 2, then cohomological rigidity holds by [**5**]. On the other hand, the following question seems to be natural to ask for the class of $\mathbb{C}P$-manifolds $\mathcal{CPM}$ instead of the cohomological rigidity problem.

Problem 2. Is the class of $\mathbb{C}P$-manifolds $\mathcal{CPM}$ (up to diffeomorphism) determined by their homotopy types? More precisely, are $M_1, M_2 \in \mathcal{CPM}$ diffeomorphic if they have the same homotopy types?

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