On projective bundles over small covers

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1. Basic facts of small covers
Definition and Characterization of small cover

**Small cover** (Davis-Januszkiewicz, 1990)

\[ M^n \text{def} \leftrightarrow \text{a compact } n\text{-dimensional manifold } M^n \text{ with the following two conditions:} \]

1. \( M^n \) has an effective, locally standard \((\mathbb{Z}_2)^n\)-action, i.e., locally looks like the standard \((\mathbb{Z}_2)^n \cong \mathbb{R}^n\);

2. the orbit space is an \( n\)-dimensional simple polytope \( M^n/(\mathbb{Z}_2)^n = P^n \), i.e., each vertex is constructed by the intersection of just \( n \) facets.

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\end{array} \quad \text{Simple polytope} \]

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\end{array} \quad \text{Non-simple polytope} \]
Small covers can be reconstructed from the following pair \((P^n, \lambda)\).

1. \(P^n\): an \(n\)-dimensional simple polytope,

2. \(\lambda : \mathcal{F} \to \{0, 1\}^n\): a characteristic function such that

\[
\det(\lambda(F_1), \ldots, \lambda(F_n)) \equiv 1 \pmod{2} \text{ for } \bigcap_{i=1}^n F_i = \{v\},
\]

where \(\mathcal{F} = \{F_1, \ldots, F_m\}\) denotes the set of all facets (codimension-one faces) of \(P\).

The function \(\lambda\) is also denoted by the following matrix

\[
(\lambda(F_1), \ldots, \lambda(F_m)) = \begin{pmatrix} I_n & \Lambda \end{pmatrix} \in M(n, m; \mathbb{Z}_2),
\]

where \(\Lambda \in M(n, m-n; \mathbb{Z}_2)\). We call \((I_n \Lambda)\) a characteristic matrix.
Then

\[ M(P, \lambda) = (\mathbb{Z}_2)^n \times P^n / \sim_\lambda \]

is small cover, where

\[(t, p) \sim_\lambda (t', p) \iff t't^{-1} \in (-1)^{\lambda(F)} \mid p \in F \supset (\mathbb{Z}_2)^n.\]
Example 1. In the following figure, the left and right pair are called by \((\Delta^2, \lambda_2)\) and \((I^2, \lambda_1^2)\) respectively (where \(e_1\) and \(e_2\) are standard basis in \(\{0, 1\}^2\)).

\[
\begin{align*}
M(\Delta^2, \lambda_2) &= \mathbb{R}P(2) \\
M(I^2, \lambda_1^2) &= T^2
\end{align*}
\]
In summary, we have the following correspondence:

Small Covers $\leftrightarrow$ Polytopes and functions $(P, \lambda)$
2. Motivation
Cohomological rigidity problem for small cover

Assume $H^*(M; \mathbb{Z}_2) \simeq H^*(M'; \mathbb{Z}_2)$ for two small covers $M$ and $M'$.

**Question:** Are $M$ and $M'$ homeomorphic?

\[\downarrow\]

**Answer:** No!

There are counter examples in the above class.
Masuda’s counter examples

\[ M(q) = P(q\gamma \oplus (b-q)\epsilon) \]: the projective bundle over \( \mathbb{R}P(a) \), where \( \gamma \) is the canonical line bundle, \( \epsilon \) is the trivial bundle and \( 0 \leq q \leq b \).

**Theorem 1** (Masuda). *The following two statements hold:*

1. \( H^*(M(q); \mathbb{Z}_2) \cong H^*(M(q'); \mathbb{Z}_2) \iff q' \equiv q \text{ or } b-q \mod 2^{h(a)} \),
   where \( h(a) = \min \{ n \in \mathbb{N} \cup \{0\} \mid 2^n \geq a \} \);

2. \( M(q) \cong M(q') \iff q' \equiv q \text{ or } b-q \mod 2^{k(a)} \),
   where \( k(a) = \# \{ n \in \mathbb{N} \mid 0 < n < a \text{ and } n \equiv 0, 1, 2, 4 \mod 8 \} \).
Put $a = 10$, then we have $h(10) = 4$, $k(10) = 5$.

Put $b = 17$ and $q = 1$ and $q' = 0$.

Then $H^*(M(1)) \cong H^*(M(0))$ (by $q' \equiv 17 - q \mod 2^{h(10)} = 16$), but $M(1) \not\cong M(0)$ (by $q' \not\equiv 17 - q \mod 2^{k(10)} = 32$).

**Problem:** Characterize (or classify) the topological types of projective bundles over small covers.
3. Projective bundles over small covers
Let $\xi = (E(\xi), \pi, M, \mathbb{R}^k)$ be an equivariant $k$-dimensional vector bundle over a small cover $M^n$.

Put $\sigma_0(M)$ is the image of the zero section and

$$ P(\xi) = E(\xi) - \sigma_0(M) / \mathbb{R}^*, $$

then $P(\xi)$ is the $\mathbb{R}P^{k-1}$-bundle over $M$.

**Lemma 1.** $P(\xi)$ is a small cover $\iff \xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_k$ where $\gamma_i$ is a line bundle.

We call such $P(\xi)$ a projective bundle over small cover (or projective bundle).
Lemma 2. $P(\xi)$ has the following two properties:

1. **the orbit space** is $P^n \times \Delta^{k-1}$ (where $M/\mathbb{Z}_2^n = P^n$);

2. **the characteristic matrix** of $P(\xi)$ can be denoted by

\[
\begin{pmatrix}
I_n & O & \Lambda & 0 \\
O & I_{k-1} & \Lambda' & 1
\end{pmatrix}
\]

Therefore, in order to consider the projective bundle over small cover, we may only consider the following matrix:

\[
\begin{pmatrix}
I_n & \Lambda \\
O & \Lambda'
\end{pmatrix} \in M(n + k - 1, m; \mathbb{Z}_2)
\]
Characterization of projective bundles

Idea: Attach this matrix to the facets of $P^n$ directly

For example, for $r = (r_1, \ldots, r_{k-1}) \in \{0, 1\}^{k-1}$,

The following matrix

$$
\begin{pmatrix}
I_2 & 1 \\
0 & r
\end{pmatrix} \in M(k + 1, 3; \mathbb{Z}_2),
$$

corresponds with

$$P(\gamma^{r_1} \oplus \cdots \oplus \gamma^{r_{k-1}} \oplus \epsilon),$$

where $\gamma^0 = \epsilon$ and $\gamma^1 = \gamma$ over $\mathbb{R}P(2)$.

$r = r_1 e'_1 + \cdots + r_{k-1} e'_{k-1}$. 
Projective characteristic functions

$\lambda_P : \mathcal{F}_P \to \{0, 1\}^n \times \{0, 1\}^{k-1}$: **projective characteristic functions** such that

$$\det(\lambda_P(F_{i_1}), \ldots, \lambda_P(F_{i_n}), X_1, \ldots, X_{k-1}) = 1$$

for $F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset$ and $\{X_1, \ldots, X_{k-1}\} \subset \{e'_1, \ldots, e'_{k-1}, 1\}$, where $e'_i$ is the standard basis of $\{0, 1\}^{k-1}$.

Then $(P, \lambda_P)$ characterizes the projective bundle over small cover.
4. Main Theorem
In order to characterize projective bundles over 2-dim small covers, we introduce a operation $\#\Delta^{k-1}$ on the projective characteristic functions as follows.

**Remark:** This operation corresponds with the fibre some of projective bundles (gluing along the fibres).
**Main Theorem**

**Theorem 2.** Let $P(\xi)$ be a projective bundle over 2-dimensional small cover $M^2$. Then $P(\xi)$ can be constructed from projective bundles $P(\zeta)$ over the real projective space $\mathbb{R}P^2$ and $P(\kappa)$ over the torus $T^2$ by using $\#\Delta^{k-1}$.

\[
P(\zeta) = P(\gamma r_1 \oplus \cdots \oplus \gamma r_{k-1} \oplus \epsilon) \quad \text{and} \quad P(\kappa) = P(\gamma r_1 \otimes \gamma r'_1 \oplus \cdots \oplus \gamma r_{k-1} \otimes \gamma r'_{k-1} \oplus \epsilon)
\]
Outline of proof

Step 1: Prove there are two edges $F_i, F_j$ such that
$$\det(\lambda_P(F_i), \lambda_P(F_j), X_1, \ldots, X_{k-1}) = 1.$$ 

Step 2: Then we can do the converse of the operation $\#\Delta^{k-1}$ along $F_i$ and $F_j$.

Step 3: Iterating the above argument, finally $P$ decomposes into the sum of $\Delta^2$'s and $I^2$'s.
Finally we list up all topological types of projective bundles over $\mathbb{R}P(2)$ and $T^2$.

**Proposition 1.** The topological type of $P(\zeta)$ is one of the following 4 topological types:

$$S^2 \times_{\mathbb{Z}_2} P(q\mathbb{R} \oplus (k-q)\mathbb{R}),$$

for $q = 0, 1, 2, 3$.

**Proposition 2.** The topological type of $P(\kappa)$ is one of the following 4 topological types:

$$T^2 \times_{\mathbb{Z}_2} P(R_1 \oplus R_2 \oplus (k-2)\mathbb{R});$$

$$T^2 \times_{\mathbb{Z}_2} P(R_1 \oplus (k-1)\mathbb{R});$$

$$T^2 \times_{\mathbb{Z}_2} P(R_2 \oplus (k-1)\mathbb{R});$$

$$T^2 \times \mathbb{R}P(k-1),$$

where $T^2 \times_{\mathbb{Z}_2} R_i$ is the canonical bundle of the $i$-th $S^1 \subset T^2$ ($i = 1, 2$).