Two classifications of simply connected 6-dimensional torus manifolds with vanishing odd degree cohomology

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Dedicated to Professor Victor Buchstaber on his 70th birthday.

Abstract. Smooth actions of 3-dimensional torus T^3 on smooth closed, simply connected, 6-dimensional manifold M with fixed points are studied. Such a manifold M is called a simply connected 6-dimensional torus manifold. In this paper, we prove the following two different types of the classification of simply connected 6-dimensional torus manifolds with $H^{odd}(M) = 0$: (1) the Orlik-Raymond type classification, i.e., such a manifold is an equivariant connected sum of copies of the 6-dimensional sphere S^6, a 6-dimensional quasitoric manifold, and some equivariant S^4-bundles over S^2; (2) the generalization of Masuda's theorem for the case of 6-dimensional quasitoric manifolds, i.e., such 6-dimensional manifolds M_1 and M_2 are equivariantly diffeomorphic up to automorphisms of T^3 if and only if their equivariant cohomology algebras $H^*_T(M_1)$ and $H^*_T(M_2)$ are isomorphic up to automorphisms of $H^*(BT)$.

1. Introduction

A *torus manifold* M is a $2n$-dimensional, connected, oriented, manifold with n-dimensional torus $T^n(= T)$-action with fixed points. The purpose of this paper is to show two different classifications of simply connected, (in this paper we assume compact) 6-dimensional torus manifolds with $H^{odd}(M) = 0$ up to (weakly) equivariant diffeomorphism. Here, two torus manifolds M_1 and M_2 are *weakly equivariantly diffeomorphic* if they are equivariantly diffeomorphic up to automorphisms of T (see Section 2 for details). In this paper, we use the integer $\mathbb{Z}$-coefficient cohomology and the symbol $H^*(-)$ represents the cohomology ring with $\mathbb{Z}$-coefficient, unless otherwise noted.

The notion of a torus manifold is introduced by Hattori and Masuda in [**HaMa**] as the topological (in some sense, ultimate) generalization of non-singular toric varieties (i.e., toric manifolds from complex analytic point of view, see [**Fu, Od**]) and quasitoric manifolds (which are the topological counterpart of non-singular toric varieties, see [**BuPa, DaJa**] and Section 6.2). For example, when the dimension of torus manifold is two, such a torus manifold is equivariantly diffeomorphic to the 2-dimensional sphere S^2 with the standard T^1-action. If the dimension of a torus manifold M is four and M is simply connected, Orlik and Raymond in [**OrRa**] prove that M can be constructed by the data of orbit space (combinatorially, this is the ℓ-gon where ℓ is the cardinality $|M^T|$ of the set of fixed points M^T) and information of isotropy subgroups. Moreover, they prove that such a torus manifold M is the 4-sphere S^4 or an equivariant connected sum of copies of complex projective space $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ (reversed orientation) and Hirzebruch surfaces H_k, i.e., projectivization of the complex 2-dimensional vector bundle $\gamma^{\otimes k} \oplus \epsilon$ over $\mathbb{C}P^1$ where γ is the tautological line bundle and ϵ is the trivial line bundle over $\mathbb{C}P^1$. They prove this result by using a combinatorial argument.

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Note that S^2 and all simply connected 4-dimensional torus manifolds satisfy $H^{odd}(M) = 0$. In this situation, torus manifolds behave quite nicely. Masuda and Panov prove that the cohomology of torus manifolds concentrates in even degrees if and only if its equivariant cohomology $H^*_T(M)$ is free as an $H^*(BT)$-module in [**MaPa**] (see Section 3 about equivariant cohomology algebra). A manifold with torus action whose equivariant cohomology is free as $H^*(BT)$-module is called an *equivariantly formal* manifold in [**GKM**]. In this paper, a torus manifold with $H^{odd}(M) = 0$ is called an *equivariantly formal torus manifold*. An equivariantly formal torus manifold can be constructed by the orbit space and the information of isotropy subgroups such as simply connected 4-dimensional torus manifold studied in [**OrRa**] or quasitoric manifolds studied in [**DaJa**] (also see Section 5). Moreover, their topological invariants, such as (equivariant) cohomology rings, (equivariant) characteristic classes, are completely determined by the combinatorial data of their 0 and 1 dimensional orbit spaces (see [**MaPa, MMP**], also see [**GuZa**] for the other class of manifolds with torus actions, called *GKM manifolds*). So a natural next step is to study the (topological) classification of equivariantly formal torus manifolds. In this paper, we study the classification of 6-dimensional equivariantly formal torus manifolds.

Before we state our main theorem, we introduce Wiemeler's theorem in [**Wi**] for 6-dimensional equivariantly formal torus manifolds (not necessarily simply connected).

THEOREM 1.1 (Wiemeler). *Let W be an equivariantly formal 6-dimensional torus manifold. Then, there are a simply connected, equivariantly formal 6-dimensional torus manifold M and a homology 3-sphere hS^3 such that*

$$W \cong M \#_T (hS^3 \times T^3) \quad \textit{up to equivariant diffeomorphism.}$$

Here, in Theorem 1.1, $hS^3 \times T^3$ is a product manifold with the free T^3-action on the 2nd factor, and the symbol $\#_T$ represents the equivariant gluing along two free orbits of M and $hS^3 \times T^3$ (note that there are always free T^3-orbit in M because M has a fixed point). Note that if we take non-standard sphere as homology 3-sphere in Theorem 1.1 then this provides non-simply connected equivariantly formal 6-dimensional torus manifolds, because their fundamental groups are isomorphic, i.e., $\pi_1(W) \simeq \pi_1(hS^3)$ (see [**Wi**]).

The goal of this paper is to classify M in Wiemeler's theorem, i.e., simply connected, equivariantly formal 6-dimensional torus manifolds. More precisely, we show the following two results (see Section 6 (Theorem 6.10) and Section 7 (Theorem 7.1) for detailed notations):

THEOREM 1.2. *Let M be a simply connected, equivariantly formal 6-dimensional torus manifold. Then, M is equivariantly diffeomorphic to one of the following manifolds:*

(1) *an $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$ with the torus action induced from a T^3-action on $\mathbb{C}^3$.;*
(2) *a 6-dimensional quasitoric manifold X;*
(3) *an S^4-bundle over S^2 equipped with the structure of a torus manifold,*

or otherwise, there is a 6-dimensional quasitoric manifold X and S^4-bundles over S^2, say S_i for some $i = 1, \ldots, \ell$, such that M has the following equivariant connected sum decomposition:

$$X \# S_1 \# \cdots \# S_\ell.$$

This theorem can be regarded as the 6-dimensional analogue of Orlik-Raymond's type classification (also see [**Mc**] and [**Ku08**]). Moreover, we classify such torus manifolds by using equivariant cohomology algebras.

THEOREM 1.3. *Let M_1 and M_2 be simply connected, equivariantly formal 6-dimensional torus manifolds. Then, the following two statements are equivalent:*

(1) *$(M_1, T^3) \cong_w (M_2, T^3)$, i.e., weakly equivariantly diffeomorphic;*
(2) *$H^*_T(M_1) \simeq_w H^*_T(M_2)$, i.e., weakly isomorphic as the $H^*(BT)$-algebras.*

This can be regarded as the generalization of Masuda's theorem, i.e., the equivariant "homeomorphism" types of (quasi)toric manifolds are determined by their equivariant cohomology algebras up to weakly isomorphisms in [**Ma08**], in the case of dimension 6. By the classification of 2 and 4-dimensional simply connected torus manifolds, the similar facts in Theorem 1.3 are also true for the case when the dimension is 2 or 4.

Remark 1.4. Due to [**Wi**], there are infinitely many distinct T^n-actions on S^{2n} for $n \geq 4$. Moreover, we can easily check that the induced *torus graphs* (see Section 4) of them are weakly isomorphic to that of the standard T^n-action on $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$. Therefore, by [**MMP**], equivariant cohomology algebras of distinct T^n-actions on S^{2n} for $n \geq 4$ are weakly isomorphic to that of the standard T^n-action on S^{2n} (also see Section 4). This implies that the similar facts in Theorem 1.3 are not true for the case when the dimension of simply connected equivariantly formal torus manifolds is greater than 6.

We also note that the equivariant cohomology algebra of an equivariantly formal 6-dimensional torus manifold forms $W \cong M\#_T(hS^3 \times T^3)$ is isomorphic to that of M. Namely, by Wiemeler's theorem, Theorem 1.3 does not hold for general equivariantly formal 6-dimensional manifold.

The organization of this paper is as follows. In Section 2, we recall the basic facts about torus manifolds. In particular, we recall the tangential representation T_pM on a fixed point $p \in M^T$ and the structure of a manifold with corners of the orbit space M/T. In Section 3, the equivariantly formal torus manifold is introduced. In Section 4, we introduce two combinatorial objects called a characteristic function and a torus graph, and discuss the relations of them. Then, in Section 5, we discuss the construction of 6-dimensional torus manifolds from these combinatorial objects. The construction of the torus manifold from the given combinatorial objects is one of the key points to show the main theorems. Finally, we prove Theorem 1.2 (Theorem 6.10) in Section 6 by using the combinatorial arguments on torus graphs, and Theorem 1.3 (Theorem 7.1) in Section 7 by using the zero-length arguments in [**Ma08, Ku11-2**] and using the construction of the torus manifold discussed in Section 5.

2. Torus manifolds and their orbit spaces

In this section, we recall some basic facts about torus manifolds. We first recall the basic notions from transformation group theory (see [**Br, Hs, Ka**]).

Let (M_1, G, φ_1) and (M_2, G, φ_2) be G-manifolds. We call (M_1, G, φ_1) and (M_2, G, φ_2) are *weakly (or ρ-)equivariantly diffeomorphic (resp. homeomorphic)* if there are a diffeomorphism (resp. homeomorphism) $f : M_1 \to M_2$ and an automorphism $\rho : G \to G$ such that the following diagram commutes

$$\begin{array}{ccc} G \times M_1 & \xrightarrow{\varphi_1} & M_1 \\ \downarrow{\scriptstyle \rho\times f} & & \downarrow{\scriptstyle f} \\ G \times M_2 & \xrightarrow{\varphi_2} & M_2 \end{array}$$

and we denote them by $(M_1, G, \varphi_1) \cong_w (M_2, G, \varphi_2)$ (resp. $(M_1, G, \varphi_1) \cong_w^{C^0} (M_2, G, \varphi_2)$). If ρ is the identity, they are called an *equivariantly diffeomorphic* (resp. *homeomorphic*) and denote by $(M_1, G, \varphi_1) \cong (M_2, G, \varphi_2)$ (resp. $(M_1, G, \varphi_1) \cong^{C^0} (M_2, G, \varphi_2)$).

We denote the isotropy subgroup of $x \in M$ as G_x. The set $\cap_{x\in M} G_x$ is called the *kernel* of a G-action on M. A G-action on M is said to be *almost effective* if this action has a finite kernel, i.e., $\cap_{x\in M} G_x$ is finite. If the kernel is trivial, we call such a G-action on M is *effective*. It is easy to check that for the given G-manifold M the induced action $G/\cap_{x\in M} G_x$ is effective. The symbol M^G represents *the set of fixed point*.

A torus manifold is defined as follows.

Definition 2.1 (torus manifold [**HaMa, Ma99**]). Let M be an oriented, compact, connected $2n$-dimensional smooth manifold. The manifold M is called a *torus manifold*, if there is an (almost) effective n-dimensional torus T^n-action on M with fixed points, i.e., $M^T \neq \emptyset$.

Note that in the original definition of the torus actions on torus manifold are effective. In this paper, by the technical reason, we assume that the torus actions on torus manifolds are almost effective. We often denote a torus manifold M by (M, T) or (M, T, φ) if we emphasis the action $\varphi : T \times M \to M$ (we also denote T^n as T).

A torus manifold M is said to be *locally standard* if every point in M has a T-invariant open neighborhood U which is weakly equivariantly homeomorphic to an open subset $\widetilde{U} \subset \mathbb{C}^n$ invariant under the standard T^n-action on $\mathbb{C}^n$, i.e.,

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n),$$

where $(t_1, \ldots, t_n) \in T$ and $(z_1, \ldots, z_n) \in \mathbb{C}^n$.

2.1. Tangential representation and omniorientations. Let M be a $2n$-dimensional torus manifold and $p \in M^T$ be its fixed point. We next introduce the omniorientation on M (see [**BuPa, HaMa**]). Using the differentiable slice theorem (see [**Br, Ka**]), the induced T-action on the tangent space $T_pM \simeq \mathbb{R}^{2n}$ of p is nothing but a "real" T-representation space, i.e., an action induced from the (non-degenerate) linear map $\iota : T \to GL(T_pM)$ (i.e., $\ker \iota$ is a finite set). This T-representation or the T-representation space T_pM itself is called a *tangential representation on* p. In order to define some combinatorial objects from torus manifolds in Section 4 (a *characteristic function* on the orbit space or a *torus graph*), we need to regard the tangential representation on each p as a complex representation (and the normal bundle of a *characteristic submanifold*). (Note that there is no canonical way to regard T_pM as a "complex" T-representation space, unless M has a T-invariant almost complex structure.) In order to do that, we introduce an *omniorientation* on the torus manifold M. Let M_i be a codimension-2 torus submanifold in M which fixed by some circle subgroup T_i in T. In other words, M_i is a fixed pointwise component of M^{T_i}. Such M_i is a $(2n-2)$-dimensional torus manifold with T/T_i-action, and call it a *characteristic submanifold*. An *omniorientation* $\mathcal{O}$ of M is a set of the fixed orientation of M and of characteristic submanifolds M_i, i.e., $\mathcal{O} = \{[M], [M_1], \ldots, [M_m]\}$ where $[M] \in H_{2n}(M)$ and $[M_i] \in H_{2n-2}(M_i)$ $(i = 1, \ldots, m)$ are the fundamental classes. If there are just m characteristic submanifolds in M, we can choose 2^{m+1} omniorientations. Note that if M has a T-invariant almost complex structure J, then the canonical omniorientation $\mathcal{O}_J$ is induced by the almost complex structure.

Let $(M, \mathcal{O})$ be an omnioriented torus manifold and M_i be a characteristic submanifold, $i = 1, \ldots, m$. Because $\dim M = 2n$, a fixed point p is the intersection of exactly n characteristic submanifolds. Namely, if we set $I_p = \{i \in [m] \mid p \in M_i\}$, where $[m] = \{1, \ldots, m\}$, then $|I_p| = n$ for all $p \in M^T$ and

$$\{p\} = \cap_{i \in I_p} M_i.$$

By using the orientations of M_i and M, we can define the invariant complex structure on the T-invariant (real 2-dimensional) normal bundle over M_i; therefore, the normal bundle of M_i is a (T-invariant) complex line bundle. This implies that the quotient space

$$T_pM / T_pM_i \simeq N_pM_i = \mathbb{C}, \quad i \in I_p$$

is the complex 1-dimensional (irreducible) T-representation space. So, there is the irreducible decomposition of T_pM to 1-dimensional complex T-representations as follows:

$$T_p(M) \simeq T_p(M, \mathcal{O}) = \oplus_{i \in I_p} N_pM_i.$$

In summary, if we choose an omniorientation on M, the tangent space on a fixed point p decomposes into irreducible representations as follows:

$$T_pM \simeq \oplus_{i \in I_p} V(\alpha_{i,p}). \tag{2.1}$$

where $V(\alpha_{i,p}) \simeq \mathbb{C}$ is the 1-dimensional (complex) irreducible representation space with the non-trivial representation $\alpha_{i,p} : T \to S^1$. By taking its differential, we may regard $\alpha_{i,p} \in (\mathfrak{t}^*)_{\mathbb{Z}}$, i.e., an element in the lattice of the dual of Lie algebra $\mathfrak{t}(= \mathrm{Lie}(T) \simeq \mathbb{R}^n)$ of T.

Since the T-action on a torus manifold M is (resp. almost) effective, the tangential representation at p must be (resp. almost) faithful, i.e., $\cap_{i=1}^n \ker \alpha_i = \{0\}$ (resp. finite) (see [**Ku11-1**, Lemma 3.3]). Therefore, M^T is a discrete set in M, and M^T is a finite set because M is compact.

Let $f : M \to M'$ be a weakly ρ-equivariant diffeomorphism for two torus manifolds M and M'. Note that, for a characteristic submanifold M_i in M, its image $f(M_i)$ is also a characteristic submanifold M'_i in M'. We call f *preserves omniorientations* if the induced homomorphism $f_* : H_{2n}(M) \to H_{2n}(M')$ and $(f|_{M_i})_* : H_{2n-2}(M_i) \to H_{2n-2}(M'_i)$ preserves the fundamental classes, i.e., $f_*([M]) = [M']$ and $(f|_{M_i})_*([M_i]) = [M'_i]$.

2.2. Orbit spaces of locally standard torus manifolds. In this section, we recall the structure of orbit spaces of locally standard torus manifolds.

2.2.1. *Manifold with corners.* We first recall the smooth manifold with corners (see e.g. [**Le**]). In this section, the following notation is often used:

$$[n] = \{0,\ 1,\ \dots,\ n\},$$

and

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = 1, \dots, n\}.$$

Let Q^n be an n-dimensional (topological) manifold with boundary. A *chart with corners* for Q^n is a pair $\{(U, \psi_U)\}$, where U is an open subset of Q^n and

$$\psi_U : U \to \mathbb{R}^n_+$$

is homeomorphic from U to a (relatively) open subset $\widetilde{U} \subset \mathbb{R}^n_+$. Two charts with corners (U, ψ_U), (V, ψ_V) are said to be *smoothly compatible* if the composite function $\psi_U \circ \psi_V^{-1} : \psi_V(U \cap V) \to \psi_U(U \cap V)$ has a smooth extension, i.e., there is an open neighborhood $W \subset \mathbb{R}^n$ of $\psi_V(U \cap V)$ and a smooth function $F : W \to \mathbb{R}^n$ such that the restriction satisfies $F|_{\psi_V(U\cap V)} = \psi_U \circ \psi_V^{-1}$. A *smooth structure with corners* of Q^n is a maximal collection of smoothly compatible charts with corners whose domains cover Q^n. A topological manifold with boundary together with a smooth structure with corners is called a *smooth manifold with corners*.

Let $p \in Q^n$ be a point of an n-dimensional smooth manifold with corners Q^n. A smooth chart (U, ψ_U) of p, i.e., an open set U containing the element $p \in U$, defines the number $d(p) \in [n]$ by the number of zero-components of $\psi_U(p) \in \mathbb{R}^n_+$. By the smooth compatibility of charts, this number is independent on the choice of a smooth chart of p. Therefore, the map $d : Q^n \to [n]$ is well-defined and $d(p)$ is called *depth* of p. We call the closure of a connected component of $d^{-1}(k)$ $(0 \leq k \leq n)$ a *codimension-k face*. In particular, the codimension-0 face is Q^n itself. Moreover, a codimension-1, $(n-1)$ and n face is called a *facet*, *edge* and *vertex*, respectively. The set of all edges and vertices is called a *one-skeleton of Q^n* (or *graph of Q^n*), i.e., the set of all points with depth 0 or 1. By restricting the smooth structure on Q^n to faces, we may regard every codimension-k face as an $(n-k)$-dimensional smooth (sub)manifold with corners.

Definition 2.2 (Nice manifold with corners). An n-dimensional smooth manifold with corners Q is said to be a *nice manifold with corners* (or a *manifold with faces*) if Q satisfies the following two conditions:

(1) for every $k \in [n]$, there exists a codimension-k face;
(2) each codimension-k face of Q is a connected component of the intersection of exactly k facets.

If Q is a nice manifold with corners, then all faces of Q are also nice manifolds with corners.

For the given nice manifold with corners Q, we can define the *face poset* $\mathcal{P}(Q)$ by taking each lattice (i.e., element) of $\mathcal{P}(Q)$ as face, where the partial ordering $\preceq$ on $\mathcal{P}(Q)$ is given by inclusion of faces. We often denote Q with the face poset structure as $(\mathcal{P}(Q), \preceq)$.

Let Q_1 and Q_2 be n-dimensional smooth manifolds with corners. We call $(\mathcal{P}(Q_1), \preceq_1)$ and $(\mathcal{P}(Q_2), \preceq_2)$ are *combinatorially equivalent* if there is a bijective map $f : \mathcal{P}(Q_1) \to \mathcal{P}(Q_2)$ such that $f(a) \preceq_2 f(b)$ (resp. $f^{-1}(c) \preceq_1 f^{-1}(d)$) whenever $a \preceq_1 b$ (resp. $c \preceq_2 d$), i.e., $(\mathcal{P}(Q_1), \preceq_1)$ and $(\mathcal{P}(Q_2), \preceq_2)$ are isomorphic as a poset (we denote it by $(\mathcal{P}(Q_1), \preceq_1) \equiv (\mathcal{P}(Q_2), \preceq_2)$). We call such bijection a *combinatorially equivalent map* on manifolds with corners, and if there is a combinatorially equivalent map between Q_1 and Q_2, then we denote it by $Q_1 \sim_c Q_2$. Two manifolds with faces Q_1 and Q_2 are *diffeomorphic* if there is a diffeomorphism f between Q_1 and Q_2 (in the sense of smooth manifold with corners) such that the restriction map $f|_F$ for each faces $F \subset Q_1$ induces the diffeomorphism between faces $F \subset Q_1$ and $f(F) = F' \subset Q_2$ and f^{-1} also staisfies such conditions. We denote such manifold with faces by $Q_1 \cong Q_2$. We can easily check if $Q_1 \cong Q_2$ then $Q_1 \sim_c Q_2$.

2.2.2. *Orbit space.* Assume M is a locally standard $2n$-dimensional torus manifold. By the differentiable slice theorem, if the orbit $T(p)$ is codimension-$(n+k)$ for a point $p \in M$ $(0 \leq k \leq n)$, i.e., $(n-k)$-dimensional orbit, then there is an invariant normal bundle $N(T(p))$ which is equivariantly diffeomorphic to the open tubular neighborhood

$$T^n \times_K V^{n+k},$$

where K is the isotropy subgroup of p and K acts on $V^{n+k}(\cong \mathbb{R}^{n+k})$ linearly. Because the T^n-action on M is locally standard, K is isomorphic to T^k, i.e., connected. Moreover, there are exactly k characteristic submanifolds $M_{1,p}, \ldots, M_{k,p}$ such that

$$T(p) \subset \cap_{i=1}^{k} M_{i,p} (\subset M^K).$$

Hence, by using their omniorientation, we may regard

$$V^{n+k} = V(\alpha_{1,p}) \oplus \cdots \oplus V(\alpha_{k,p}) \oplus \mathbb{R}^{n-k}$$

for some 1-dimensional K-representations $\alpha_{i,p} : K \to T^1$ $(i = 1, \ldots, k)$, where $\mathbb{R}^{n-k}$ is the trivial (real) representation space of K (i.e., the normal bundle of $T(p)$ in M^K) and $\{\alpha_{1,p}, \ldots, \alpha_{k,p}\}$ spans the lattice of dual Lie algebra $\mathfrak{k}^*(\simeq (\mathfrak{t}^k)^*)$ of K. Furthermore, by locally standardness of T-action, we have that the invariant neighborhood $T(p)$ is diffeomorphic to an invariant neighborhood of compact torus embedding into $\mathbb{C}^n$. It follows from this that $T^n \times_K V^{n+k}$ is the trivial $\mathbb{R}^{n+k}$-bundle over T^n/K (in the sense of the ordinary non-equivariant vector bundle). This also implies that the tubular neighborhood satisfies

$$\begin{aligned} N(T(p)) &\cong T^n \times_K V^{n+k} \\ &\cong V(\alpha_{1,p}) \oplus \cdots \oplus V(\alpha_{k,p}) \oplus (\mathbb{R}^{n-k} \times T^n/K) \\ &\cong V(\alpha_{1,p}) \oplus \cdots \oplus V(\alpha_{k,p}) \oplus (\mathbb{C}^*)^{n-k} \\ &\cong \mathbb{C}^k \oplus (\mathbb{C}^*)^{n-k} \end{aligned}$$

where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and the 3rd isomorphism between $\mathbb{R}^{n-k} \times T^n/K$ and $(\mathbb{C}^*)^{n-k}$ is given by

$$\begin{array}{ccc} \mathbb{R}^{n-k} \times T^{n-k} & \longrightarrow & (\mathbb{C}^*)^{n-k} \\ \Psi & & \Psi \\ (r_1, \ldots, r_k, t_1, \ldots, t_{n-k}) & \longmapsto & (\exp(r_1)t_1, \ldots, \exp(r_{n-k})t_{n-k}) \end{array}$$

Let $U = N(T(p))$ and ψ_U be the above identification between U and $\mathbb{C}^k \oplus (\mathbb{C}^*)^{n-k}$. Then, $\{(U, \psi_U)\}$ may be regarded as a (locally standard) smooth atlas on M, i.e., for $U \cap V \neq \emptyset$, $\psi_V \circ \psi_U^{-1} : \psi_U(U \cap V) \to \psi_V(U \cap V)$ is a smooth function. Moreover, the orbit space of this normal bundle is

$$\begin{aligned} U/T &\cong (\mathbb{C}^k \oplus (\mathbb{C}^*)^{n-k})/T \\ &\cong \mathbb{R}_+^k \times \mathbb{R}_{>0}^{n-k} \subset \mathbb{R}_+^n, \end{aligned}$$

where $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ and the 2nd isomorphism is induced from

$$\begin{array}{ccc} \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} & \longrightarrow & \mathbb{R}_+^k \times \mathbb{R}_{>0}^{n-k} \\ \Psi & & \Psi \\ (z_1, \ldots, z_k, z_{k+1}, \ldots, z_n) & \longmapsto & (|z_1|, \ldots, |z_k|, |z_{k+1}|, \ldots, |z_n|) \end{array}$$

Let $\psi_{U/T}$ be the above identification between U/T and $\mathbb{R}_+^k \times \mathbb{R}_{>0}^{n-k}$. Because the orbit map $\pi : M \to M/T$ is an open map, the orbit space M/T of a locally standard torus manifold M is a manifold with boundary admitting the chart with corners $\{(U/T, \psi_{U/T})\}$. If $U/T \cap V/T \neq \emptyset$, then there is a homeomorphism in $\mathbb{R}_+^k \times \mathbb{R}_{>0}^{n-k}$

$$\psi_{V/T} \circ \psi_{U/T}^{-1} : W_1 = \psi_{U/T}(U/T \cap V/T) \to \psi_{V/T}(U/T \cap V/T) = W_2.$$

Now we can take the section of the orbit map $\pi_U : U = \mathbb{C}^k \oplus (\mathbb{C}^*)^{n-k} \to \mathbb{R}_+^k \times \mathbb{R}_{>0}^{n-k} = U/T$ by

$$\begin{array}{ccc} \mathbb{R}_+^k \times \mathbb{R}_{>0}^{n-k} & \xrightarrow{s_U} & \mathbb{C}^k \oplus (\mathbb{C}^*)^{n-k} \\ \Psi & & \Psi \\ (r_1 \ldots, r_k, r_{k+1}, \ldots, r_n) & \longmapsto & (r_1, \ldots, r_k, r_{k+1}, \ldots, r_n) \end{array} \tag{2.2}$$

This section s_U is obviously smooth in the sense of manifold with corners because we can take a smooth extension to $\mathbb{R}^k \times \mathbb{R}_{>0}^{n-k}$ as the same map. We call s_U a *smooth section on* U. Because $\psi_{V/T} \circ \psi_{U/T}^{-1} = \pi_V \circ (\psi_V \circ \psi_U^{-1}) \circ s_U$ and s_U, $\psi_V \circ \psi_U^{-1}$ and π_V are smooth, the chart with corners$\{(U/T, \psi_{U/T})\}$ is smoothly compatible. Namely, the orbit space $Q = M/T$ is a smooth manifold with corners whose chart with corners are induced from the locally standard atlas of M.

Moreover, it also follows from the locally standardness that Q is a nice manifold with corners (also see [**MaPa**]). So the following proposition is established.

PROPOSITION 2.3. *Let M be a $2n$-dimensional locally standard torus manifold and $\{(U, \psi_U)\}$ be the locally standard atlas. Then, its orbit space M/T admits the structure of the n-dimensional nice manifold with corners whose chart with corners is induced from $\{(U, \psi_U)\}$, it can be denoted by $\{(U/T, \psi_{U/T})\}$*

3. Equivariantly formal torus manifolds

In this section, we introduce the equivariantly formal torus manifolds.

3.1. Equivariant cohomology. We first recall the equivariant cohomology algebra in the general situation (see e.g. [**Hs, Ma08, Ku11-2**]). Let M be a space equipped with a Lie group G-action. By the classical Lie theory, there is a principal G-bundle $EG \to BG$, where EG is a contractible space with free G-action and BG is its orbit space EG/G, called a *classifying space* of G. Let $EG \times_G M$ be the G-orbit space of the Cartesian product $EG \times M$ by the action $g(e, x) \mapsto (eg^{-1}, gx)$ for $g \in G$ and $(e, x) \in EG \times M$. An *equivariant cohomology* $H^*_G(M)$ of the G-action on M is defined by the ordinary (singular) cohomology $H^*(EG \times_G M)$. One of the important properties of equivariant cohomology $H^*_G(M)$ is that $H^*_G(M)$ is not only a ring but also an $H^*(BG)$-algebra, i.e., $H^*_G(M)$ is an algebra over the ring $H^*(BG)$. This structure is defined as follows. Because G-action on EG is free, we have the following fibration:

$$M \longrightarrow EG \times_G M \xrightarrow{\pi} BG$$

Therefore, the induced homomorphism

$$H^*(BG) \xrightarrow{\pi^*} H^*_G(M)$$

defines the $H^*(BG)$-algebra structure on $H^*_G(M)$.

Let G be an n-dimensional compact torus T, i.e., $S^1 \times \cdots \times S^1$ (n times Cartesian product of circle group). As is well known, $H^*(BT) \simeq \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$, i.e., the polynomial ring generated by degree-2 elements. Therefore, the equivariant cohomology $H^*_T(M)$ of a torus manifold M has the $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$-algebra structure. We may also regard α_i, $i = 1, \ldots, n$, as a basis of $\mathfrak{t}^*_{\mathbb{Z}}$, where $\mathfrak{t}^*_{\mathbb{Z}}$ is the lattice of the dual of Lie algebra $\mathfrak{t}$ of T.

Now we may define the equivalence relation on two equivariant cohomology algebras. We call $H^*_T(M_1)$ and $H^*_T(M_2)$ are *weakly (or ρ^*-)isomorphic* as an $H^*(BT)$-algebra if there are a graded ring isomorphism $f^* : H^*_T(M_1) \to H^*_T(M_2)$ and an automorphism $\rho^* : H^*(BT) \to H^*(BT)$ such that the following diagram commutes

$$\begin{array}{ccc} H^*(BT) & \xrightarrow{\pi_1^*} & H^*_T(M_1) \\ \downarrow{\rho^*} & & \downarrow{f^*} \\ H^*(BT) & \xrightarrow{\pi_2^*} & H^*_T(M_2) \end{array}$$

and denote them by $H^*_T(M_1) \simeq_w H^*_T(M_2)$. If ρ^* is the identity, we call $H^*_T(M_1)$ is *isomorphic* to $H^*_T(M_2)$.

REMARK 3.1. Note that if M admits a G-action then there is the restricted K-action on M for any subgroup $K \subset G$. Let G be a connected Lie group and K be its maximal compact subgroup. By the classical Lie theory, G has the deformation retract to K. This implies that the equivariant cohomology $H^*_G(M)$ is isomorphic to $H^*_K(M)$ as an algebra over $H^*(BG) \simeq H^*(BK)$.

For example, the compact (topological) torus T^n is the maximal compact subgroup of $(\mathbb{C}^*)^n$ (called algebraic torus, where here we regard $(\mathbb{C}^*)^n$ as a Lie group). Now we may regard the non-singular (complete) toric variety V equipped with the Zariski topology as the complex analytic (compact) manifold equipped with the Hausdorff topology called a toric manifold M_V (see e.g. [**Od**, Chapter 2] (Serre's GAGA) about the functorial relation between V and M_V). By definition of the non-singular toric variety, there is the smooth $(\mathbb{C}^*)^n$-action on M_V with the dense orbit (in the sense of a Hausdorff topology) and this action has fixed points (also see [**IFM**]: they generalize the toric manifold from this point of view). Therefore, if we restrict the $(\mathbb{C}^*)^n$-action to the T^n-action, we may regard the toric manifold M_V as a torus manifold. Hence, by the fact stated as above, there is the $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$-algebra isomorphism:

$$H^*_{(\mathbb{C}^*)^n}(M_V) \simeq H^*_{T^n}(M_V).$$

3.2. Equivariantly formal torus manifold and its orbit space. Let M be a manifold with T-action such that M^T is finite. It is well known that two Euler characteristics of M and M^T are the same, i.e., $\chi(M) = \chi(M^T)$ (see e.g. [**Ka**]). Namely,

$$|M^T| = \chi(M) = \sum_{k=0}^{2n} (-1)^k \text{rank } H^k(M),$$

where the symbol $|M^T|$ represents the number of the fixed points M^T. If $H^{odd}(M) = 0$, the Euler characteristic is nothing but the sum of Betti numbers and in the above case, the sum of Betti numbers coincides with the number of fixed points. Moreover, by using the Serre spectral sequence for the fibration $ET \times_T M \to BT$, if $H^{odd}(M) = 0$ then $H^*_T(M) \simeq H^*(M) \otimes H^*(BT)$ as an $H^*(BT)$-module. This is one of the motivations to defined the following class of manifolds.

Definition 3.2 (equivariantly formal space [**GKM**]). Let R be a ring. Let M be an even dimensional manifold with n-dimensional torus T-action (not necessarily maximum, i.e., $\dim M \geq 2n$). We say M is an *R-equivariantly formal manifold* if $H^*_T(M; R) \simeq H^*(M; R) \otimes H^*(BT; R)$, i.e., free as an $H^*(BT; R)$-module (as is well known $H^*(BT; R) \simeq R[\alpha_1, \ldots, \alpha_n]$, i.e., the polynomial ring). In this paper, the $\mathbb{Z}$-equivariantly formal torus manifold M is called an *equivariantly formal torus manifold*.

Remark 3.3. The original definition of equivariantly formal in [**GKM**] is the $\mathbb{C}$-equivariantly formal.

Because $\chi(M) = \chi(M^T)$ (see [**Ka**]), the fixed point set is non-empty $M^T \neq \emptyset$ when a $2n$-dimensional, compact, connected, oriented manifold M with an n-dimensional torus T-action satisfies that $H^{odd}(M) = 0$. Namely M is automatically an (equivariantly formal) torus manifold when $H^{odd}(M) = 0$. Together with this and [**MaPa**, Lemma 2.1], [**MaPa**, Theorem 2], we have the following theorem:

Theorem 3.4 (Masuda-Panov). *Let M be an even dimensional, compact, connected, oriented manifold with a half dimensional torus T-action. Then the following three conditions are equivalent:*

(1) $H^{odd}(M) = 0$;
(2) *M is an equivariantly formal torus manifold, i.e., there is a fixed point;*
(3) *the T-action on M is the locally standard and the orbit space M/T is an n-dimensional face acyclic,*

where we call an n-dimensional nice manifold with corners Q an n-dimensional face acyclic if all faces F (include Q) are acyclic, i.e., $H_*(F) \simeq H_0(F) \simeq \mathbb{Z}$.

Note that if M is $\mathbb{Z}$-equivariantly formal, then we have that M is $\mathbb{C}$-equivariantly formal (by the tensor $\otimes\mathbb{C}$), i.e., the original definition in [**GKM**].

4. Characteristic function and axial function

In this section, we introduce two combinatorial objects which have equivariant topological information of locally standard torus manifolds, called a characteristic function on the nice manifold with corners (Q, λ) and a torus graph $(\Gamma, \mathcal{A})$.

4.1. Characteristic functions. We first introduce the label (called a *characteristic function*) on the facets of the orbit space of a locally standard torus manifold.

Let M be an omnioriented $2n$-dimensional locally standard torus manifold. Then, by Proposition 2.3, the orbit space $Q = M/T$ is a nice manifold with corners. Let $\pi : M \to Q$ be the orbit projection and $\mathcal{F}(Q)(\subset \mathcal{P}(Q))$ be the set of facets. Then, for each $F_i \in \mathcal{F}(Q)$, the inverse image $\pi^{-1}(F_i) = M_i$ is a characteristic submanifold $M_i \subset M$. Because M_i is fixed by some circle subgroup $T_i \subset T^n$, this T_i acts on the normal (complex line) bundle ν_i on M_i and preserves its complex structure. Take the T_i-equivariant 1st Chern class of ν_i, say $c_1^{T_i}(\nu_i) \in H^2_{T_i}(M_i)$. Because T_i acts on M_i trivially, we may regard $c_1^{T_i}(\nu_i)$ is a generator of $H^2(BT_i) \simeq (\mathfrak{t}_i)^*_{\mathbb{Z}} \simeq \mathbb{Z}$, where $(\mathfrak{t}_i)^*_{\mathbb{Z}}$ is the lattice of the dual of $\mathrm{Lie}(T_i)$, i.e., the Lie algebra of T_i. This implies that we can determine the primitive element $\lambda(F_i)$ in $\mathfrak{t}_{\mathbb{Z}}$ such that

$$\langle c_1^{T_i}(\nu_i), \lambda(F_i)\rangle = +1,$$

where $\mathbb{Z}\lambda(F_i) = \mathrm{Lie}_{\mathbb{Z}}(T_i)(\simeq H_2(BT_i)) \subset \mathfrak{t}_{\mathbb{Z}}$ (where $\mathrm{Lie}_{\mathbb{Z}}(T_i)$ is the lattice of $\mathrm{Lie}(T_i)(\subset \mathrm{Lie}(T))$). This defines the map from the set of all facets of $Q = M/T$ to $\mathfrak{t}^n_{\mathbb{Z}}$, i.e.,

$$\lambda : \mathcal{F}(Q) \to \mathfrak{t}^n_{\mathbb{Z}}.$$

Because for each fixed point $p \in M^T$ there are exactly n characteristic submanifolds M_i $(i \in I_p)$ such that $\{p\} = \cap_{i\in I_p} M_i$. By (2.1), if we choose an omniorientation on M, there is the following decomposition:

$$T_pM \cong \oplus_{i\in I_p} V(\alpha_{i,p}),$$

where we may regard

$$\iota_p^*(c_1^{T_i}(\nu_i)) = \alpha_{i,p} \in \mathfrak{t}^*_{\mathbb{Z}}$$

by the restriction map

$$\iota_p^* : H^2_{T_i}(M_i) \to H^2_{T_i}(p) = H^2(BT_i) \simeq (\mathfrak{t}_i)^*_{\mathbb{Z}} \subset \oplus_{i\in I_p}(\mathfrak{t}_i)^*_{\mathbb{Z}} \simeq \mathfrak{t}^*_{\mathbb{Z}}$$

(where $\iota_p : \{p\} \to M_i$ and we often denote $\iota_p^*(c_1^{T_i}(\nu_i)) = c_1^{T_i}(\nu_i)|_p$). Because $\{\alpha_{i,p} \mid i \in I_p\}$ spans $\mathfrak{t}^*_{\mathbb{Z}}$, the determinant of the induced $(n \times n)$-matrix

$$(\lambda(F_{i_1}) \cdots \lambda(F_{i_n}))$$

satisfies that

$$\det(\lambda(F_{i_1}) \cdots \lambda(F_{i_n})) = \pm 1, \tag{4.1}$$

for n facets such that $\cap_{j=1}^n F_{i_j}$ is a vertex $\pi^{-1}(p)$ (called *the facets around a vertex*),

Remark 4.1. If we take the isotropy weight $w_i \in \mathfrak{t}_{\mathbb{Z}}$ of the characteristic submanifold $M_i = \pi^{-1}(F_i)$, i.e., a primitive vector corresponding to the circle subgroup $T_i \subset T$ which fixes M_i, then w_i coincides with $\lambda(F_i)$ up to sign.

Motivated by the above observation, we may define the characteristic function on a nice manifold with corners as follows (see [**BuPa, DaJa**] for simple polytopes and [**MaPa**] for nice manifold with corners):

Definition 4.2. Let Q be an n-dimensional nice manifold with corners and $\mathcal{F}(Q)$ be the set of its facets. Let $\mathfrak{t}_{\mathbb{Z}}$ be the lattice of Lie algebra of T^n. Then, a function $\lambda : \mathcal{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}$ is said to be a *characteristic function* if λ satisfies the relation (4.1) for the facets around all vertices.

Moreover, we can define the free $\mathbb{Z}$-module whose generators are $\mathcal{F}(Q)$, say $\mathbb{Z}\mathcal{F}(Q) = \oplus_{i=1}^m \mathbb{Z}F_i$ for $\mathcal{F}(Q) = \{F_1, \ldots, F_m\}$. Then, the characteristic function induces the linear surjection

$$\lambda_{\mathbb{Z}} : \mathbb{Z}\mathcal{F}(Q) \to \mathfrak{t}_{\mathbb{Z}} \tag{4.2}$$

by

$$\lambda_{\mathbb{Z}}(\sum_{i=1}^m k_i F_i) = \sum_{i=1}^m k_i \lambda(F_i).$$

Definition 4.3. Let (Q_1, λ_1) and (Q_2, λ_2) be n-dimensional nice manifolds with corners Q_1 and Q_2 with characteristic functions λ_1 and λ_2. Assume that there is a diffeomorphism $f_* : Q_1 \to Q_2$ in the sense of a manifold with corners. We say (Q_1, λ_1) and (Q_2, λ_2) are *weakly isomorphic* if there are an automorphism $\rho : T \to T$ and an induced linear isomorphism $f_{\mathbb{Z}} : \mathbb{Z}\mathcal{F}(Q_1) \to \mathbb{Z}\mathcal{F}(Q_2)$ defined by $f_{\mathbb{Z}}(F_i) = \epsilon_i f_*(F_i)$ for some $\epsilon_i = \pm 1$ (where F_i is a facet in Q_1; then $f_*(F_i)$ is a facet in Q_2) such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\mathcal{F}(Q_1) & \xrightarrow{(\lambda_1)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}} \\ \downarrow f_{\mathbb{Z}} & & \downarrow \rho_* \\ \mathbb{Z}\mathcal{F}(Q_2) & \xrightarrow{(\lambda_2)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}} \end{array}$$

where ρ_* is the induced isomorphism by ρ. We denote them by $(Q_1, \lambda_1) \cong_w (Q_2, \lambda_2)$. If the above ρ is the identity, we call (Q_1, λ_1) and (Q_2, λ_2) are *isomorphic* and denote them by $(Q_1, \lambda_1) \cong (Q_2, \lambda_2)$.

If $\epsilon_i = +1$ for all $i = 1, \ldots, m$ (where $m = |\mathcal{F}(Q_1)| = |\mathcal{F}(Q_2)|$), i.e., $f_{\mathbb{Z}}(F_i) = f_*(F_i)$ for all i, then we call (Q_1, λ_1) and (Q_2, λ_2) are *weakly omnioriented isomorphic* (if ρ is the identity then we call them are *omnioriented isomorphic*).

Now we have the following proposition:

Proposition 4.4. *Let (M_1, T, φ_1) and (M_2, T, φ_2) be locally standard (omnioriented) torus manifolds and (Q_1, λ_1) and (Q_2, λ_2) be their orbit spaces and induced characteristic functions. If there is an automorphism $\rho : T \to T$ and a weakly ρ-equivariant diffeomorphism $f : M_1 \to M_2$, then f induces the diffeomorphism $f_* : Q_1 \to Q_2$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z}\mathcal{F}(Q_1) & \xrightarrow{(\lambda_1)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}} \\ \downarrow f_{\mathbb{Z}} & & \downarrow \rho_* \\ \mathbb{Z}\mathcal{F}(Q_2) & \xrightarrow{(\lambda_2)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}} \end{array}$$

where $f_{\mathbb{Z}}(F_i) = \pm f_*(F_i)$ for all facets $F_i \in \mathcal{F}(Q_1)$, and ρ_* is the induced isomorphism by ρ. Namely, if $(M_1, T, \varphi_1) \cong_w (M_2, T, \varphi_2)$ then $(Q_1, \lambda_1) \cong_w (Q_2, \lambda_2)$.

Furthermore, if f preserves the omniorientations of M_1 and M_2, then (Q_1, λ_1) and (Q_2, λ_2) are weakly omnioriented isomorphic.

Proof. Because f is a weakly ρ-equivariant diffeomorphism, f preserves the orbits. Namely, the induced map $f_* : Q_1 \to Q_2$ is homeomorphism preserving the faces, i.e., $f_*(F) \subset Q_2$ is a face. Furthermore, by using the section s_U defined in (2.2), we can easily verify that the restriction of f_* to the smooth chart with corners U/T extends to the smooth function on an open subset in $\mathbb{R}^n$. Similarly we can check that f_*^{-1} has a smooth extension on the smooth chart with corners. Therefore, f_* is a diffeomorphism in the sense of a manifold with corners. Now it is easy to check that to change the orientation of a characteristic submanifold M_i corresponds to change the sign of its characteristic function $\lambda(F_i)$, where F_i is the facet whose pull back image by the orbit projection is $\pi^{-1}(F_i) = M_i$. Therefore, the induced linear isomorphism $f_{\mathbb{Z}} : \mathbb{Z}\mathcal{F}(Q_1) \to \mathbb{Z}\mathcal{F}(Q_2)$ can be defined by $f_{\mathbb{Z}}(F_i) = \epsilon_i f_*(F_i)$ for some $\epsilon_i = \pm 1$. Finally, because f commutes with the T-action up to the automorphism ρ and, we have $\lambda_2 \circ f_{\mathbb{Z}} = \lambda_1 \circ \rho_*$. This establishes the first statement.

If f preserves the omniorientation then we can take $\epsilon_i = +1$ for all $i = 1, \ldots, m$ ($m = |\mathcal{F}(Q_1)| = |\mathcal{F}(Q_2)|$). This also establishes the 2nd statement. □

4.2. Torus graph. The notion of *torus graph* is defined by Maeda-Masuda-Panov [**MMP**] motivated by the GKM graph induced by Guillemin-Zara [**GuZa**]. We first recall *torus graph* abstractly (combinatorially).

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an abstract graph, where $V(\Gamma)$ is the set of vertices and $E(\Gamma)$ is the set of edges of Γ. We denote the set of all outgoing edges from the vertex p by $E_p(\Gamma)$. We say Γ is an *n-valent graph* if $|E_p(\Gamma)| = n$ for all $p \in V(\Gamma)$.

Assume that Γ is an n-valent graph (not necessarily connected). Put

$$\mathcal{A} : E(\Gamma) = \cup_p E_p(\Gamma) \to H^2(BT).$$

If $\mathcal{A}$ satisfies the following two conditions:

(1) $\mathcal{A}(pq) = \pm\mathcal{A}(qp)$;
(2) the set $\{\mathcal{A}(E_p(\Gamma))\}$ spans $\mathfrak{t}^*_{\mathbb{Z}} \simeq H^2(BT)$,

then we call $\mathcal{A}$ is an *axial function* on Γ. We denote the n-valent graph labeled by the axial function by the pair $(\Gamma, \mathcal{A})$.

Let $\nabla_{pq} : E_p(\Gamma) \to E_q(\Gamma)$ be a bijective map (recall $|E_p(\Gamma)| = |E_q(\Gamma)| = n$). We call the collection $\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ a *connection* of $(\Gamma, \mathcal{A})$ if ∇ satisfies the following conditions:

(1) $\nabla_{qp} = \nabla_{pq}^{-1}$;
(2) $\mathcal{A}(e) - \mathcal{A}(\nabla_{pq}(e)) \equiv 0 \mod \mathcal{A}(pq)$ for all $e \in E_p(\Gamma)$ (we call this relation a *congruence relation*).

Definition 4.5 ([**MMP**]). Let Γ be a connected n-valent graph and $\mathcal{A}$ be an axial function on Γ. Then, a labeled graph $(\Gamma, \mathcal{A})$ is said to be a *torus graph* if there is a connection on $(\Gamma, \mathcal{A})$.

Remark 4.6. If a torus graph $(\Gamma, \mathcal{A})$ satisfies $\mathcal{A}(pq) = -\mathcal{A}(qp)$ for all edges pq, this is nothing but a *GKM graph* defined in [**GuZa**] with maximal rank axial function, i.e., Γ is an n-valent graph and $\dim T = n$. If there is a connection ∇ on a 3-linearly independent GKM graph, then ∇ is unique by [**GuZa**]. With the method similar to this proof, we also have this property for a torus manifold, i.e., if there is a connection ∇ on a n-valent torus graph for $n \geq 3$, then ∇ is unique. It is easy to check the connection of the case when $n = 1, 2$ is also unique. Therefore, we may write a torus graph as $(\Gamma, \mathcal{A})$ by omitting the connection.

Remark 4.7. Let $\mathcal{P}_k(\Gamma, \mathcal{A})$ be the set of k-valent torus subgraphs in $(\Gamma, \mathcal{A})$ $(0 \leq k \leq n)$, i.e., k-valent graphs in Γ closed under the connection ∇. Then, the set

$$\mathcal{P}(\Gamma, \mathcal{A}) = \cup_{k=0}^{n} \mathcal{P}_k(\Gamma, \mathcal{A})$$

admits the structure of a simplicial poset by inclusion (see [**MMP**]). We denote this structure by $(\mathcal{P}(\Gamma, \mathcal{A}), \preceq)$.

In order to define the equivalence relations among torus graphs, we define the combinatorially equivalent map $f_* : \Gamma_1 \to \Gamma_2$ for n-valent graphs Γ_1 and Γ_2. We call the map $f_* : \Gamma_1 = (V(\Gamma_1), E(\Gamma_1)) \to \Gamma_2 = (V(\Gamma_2), E(\Gamma_2))$ a *combinatorially equivalent map* if the restricted map $f_*|_V : V(\Gamma_1) \to V(\Gamma_2)$ and $f_*|_E : E(\Gamma_1) \to E(\Gamma_2)$ are bijective and the following map commutes:

$$\begin{array}{ccc} E(\Gamma_1) & \xrightarrow{f_*|_E} & E(\Gamma_2) \\ \downarrow \pi_{V_1} & & \downarrow \pi_{V_2} \\ V(\Gamma_1) & \xrightarrow{f_*|_V} & V(\Gamma_2) \end{array}$$

where $\pi_V : E(\Gamma) \to V(\Gamma)$ is the map projecting to the initial vertex, i.e., $\pi_V(pq) = p$. Namely, the bijection $f_*|_V$ preserves the edges. Now we may define the equivalence relation.

Definition 4.8. Let $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ be n-valent torus graphs. We say $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ are *weakly isomorphic* if there is a combinatorially equivalent map $f_* : \Gamma_1 \to \Gamma_2$ and an automorphism $\rho^* : \mathfrak{t}^* \to \mathfrak{t}^*$ such that

$$\begin{array}{ccc} E(\Gamma_1) & \xrightarrow{\mathcal{A}_1} & \mathfrak{t}^*_{\mathbb{Z}} \\ \downarrow f_*|_E & & \downarrow \rho^* \\ E(\Gamma_2) & \xrightarrow{\mathcal{A}_2} & \mathfrak{t}^*_{\mathbb{Z}} \end{array}$$

If ρ^* is the identity, $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ are said to be *isomorphic*

For the given torus manifold M with an omniorientation $\mathcal{O}$, we can define the torus graph $(\Gamma_M, \mathcal{A}_M)$. If we fix an omniorientation $\mathcal{O}$ in the torus manifold M, then by (2.1) we have the following decomposition around the fixed point p:

$$T_pM \simeq \oplus_{i\in I_p} V(\alpha_{i,p}).$$

Now we may defined the labeled graph $(\Gamma_M, \mathcal{A}_M)$ as follows:

- fixed points M^T as vertices $V(\Gamma_M)$, i.e., vertices in the nice manifold with corners M/T;
- invariant S^2's as edges $E(\Gamma_M)$ whose 2 fixed points corresponding to the 2 vertices connected by this edge, i.e., edges in the nice manifold with corners M/T (S^2 may also be regarded as the fixed pointwise component of the intersection of some $(n-1)$ characteristic submanifolds);
- the function $\mathcal{A}_M : E(\Gamma_M) \to \mathfrak{t}^*_{\mathbb{Z}} \simeq H^2(BT)$ is defined by
$$\mathcal{A}_M(e_{j,p}) = \alpha_{j,p}$$
for $e_{j,p} \in E_p(\Gamma)$ which corresponds to $(p \in)S^2 \subset \cap_{i\in I_p\setminus\{j\}} M_i$.

Note that $e_{j,p}$ may also be regarded as the normal bundle of M_j at p, i.e., $\iota_p^*\nu_j = V(\alpha_{j,p})$. Therefore, we can easily check that the set of bijective maps $\nabla_{pq}(e_{i,p}) = e_{i,q}$ for all $pq \in E(\Gamma_M)$ defines the connection on $(\Gamma_M, \mathcal{A}_M)$. Hence, $(\Gamma_M, \mathcal{A}_M)$ is a torus graph, and we call it *a torus graph induced from the torus manifold M*. We sometimes denote $(\Gamma_M, \mathcal{A}_M)$ by $(\Gamma_{(M,\mathcal{O})}, \mathcal{A}_{(M,\mathcal{O})})$ if we emphasis the omniorientation $\mathcal{O}$.

REMARK 4.9. An omniorientation also determines the sign of representations $T_pM \simeq T_p(M,\mathcal{O}) = \oplus_{i=1}^n V(\alpha_{i,p})$, i.e., the sign is plus if the orientation of T_pM and $T_p(M,\mathcal{O})$ are the same; otherwise the sign is minus. This notion will be used when we define the equivariant connected sum operation in Section 6.4. If the omniorientation is determined from the invariant almost complex structure, then all the signs are plus and $\mathcal{A}(pq) = -\mathcal{A}(qp)$ (see [**MMP, MaPa**]).

REMARK 4.10. The set of k-valent torus subgraph $\mathcal{P}_k(\Gamma_M, \mathcal{A}_M)$ of a torus graph induced from M is nothing but the set of k-dimensional faces in a orbit space M/T. Moreover, we have that the face poset $(\mathcal{P}(M/T), \preceq)$ and the simplicial poset $(\mathcal{P}(\Gamma_M, \mathcal{A}_M), \preceq)$ induced from the torus graph $(\Gamma_M, \mathcal{A}_M)$ are the same, i.e.,

$$(\mathcal{P}(\Gamma_M, \mathcal{A}_M), \preceq) \equiv (\mathcal{P}(M/T), \preceq).$$

We also have the following Proposition.

PROPOSITION 4.11. *Let (M_1, T, φ_1) and (M_2, T, φ_2) be (omnioriented) torus manifolds, and $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ be their induced torus graphs. If there is an automorphism $\rho : T \to T$ and a weakly ρ-equivariant diffeomorphism $f : M_1 \to M_2$ which preserves omniorientations, then f induces the combinatorially equivalent map f_* between two n-valent graphs Γ_{M_1} and Γ_{M_2} such that the following diagram commutes:*

$$\begin{array}{ccc} E(\Gamma_1) & \xrightarrow{\mathcal{A}_1} & \mathfrak{t}^*_{\mathbb{Z}} \\ \downarrow{\scriptstyle f_*|_E} & & \downarrow{\scriptstyle \rho^*} \\ E(\Gamma_2) & \xrightarrow{\mathcal{A}_2} & \mathfrak{t}^*_{\mathbb{Z}} \end{array}$$

where ρ^* is the induced isomorphism by ρ. Namely, a weakly equivariant diffeomorphism $f : M_1 \to M_2$ preserves the omniorientations, then their induced torus graphs $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ are weakly isomorphic.

PROOF. Under the assumption of this Proposition, we have that the differential $df_p : T_pM_1 \to T_{f(p)}M_2$ (on $p \in M_1^T$) preserves the decomposition of the invariant complex one-dimensional representations in (2.1) up to the automorphism ρ, i.e., df_p induces the isomorphism

$$\oplus_{i\in I_{f(p)}} V(\alpha_{i,f(p)}) \simeq \oplus_{i\in I_p} V(\rho^*(\alpha_{i,p})).$$

Therefore, we can easily check the statements. □

4.3. Torus graph and nice manifold with corners with characteristic function. In this section, we recall the relation between torus graphs and nice manifolds with corners (see [**MMP**]). Let (Q, λ) be a nice manifold with corners Q with the characteristic function $\lambda : \mathcal{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}$. Then, we can define the torus graph $(\Gamma_Q, \mathcal{A}_\lambda)$ as follows. Let Γ_Q be the one-skeleton of Q and p be a vertex which is the connected component of $\cap_{i\in I_p} F_i$ for some $I_p \subset [m]$ such that $|I_p| = n$, where $m = |\mathcal{F}(Q)|$ and $F_i \in \mathcal{F}(Q)$. Then, because Q is nice, for each edge e_j in $E_p(\Gamma_Q)$ there are exactly $(n-1)$ facets such that $e_j \subset \cap_{i\in I_p\setminus\{j\}} F_i$. Here, we call F_j the *normal facet* of e_j at p and e_j a *normal edge* of F_j at p. Because $\{\lambda(F_i) \mid i \in I_p\}$ spans $\mathfrak{t}_{\mathbb{Z}}$, we can define $\mathcal{A}_\lambda(e_j) \in \mathfrak{t}^*_{\mathbb{Z}}$ as the dual of $\lambda(F_j)$, where e_j is a normal edge of F_j at p, i.e.,

$$\langle \mathcal{A}_\lambda(e_j), \lambda(F_i)\rangle = 0$$

for $i \in I_p \setminus \{j\}$ and

$$\langle \mathcal{A}_\lambda(e_j), \lambda(F_j)\rangle = 1,$$

where $\langle \alpha, x\rangle$ is a paring of $\alpha \in \mathfrak{t}^*$ and $x \in \mathfrak{t}$. We now define the set of bijections $\nabla_{pq} : E_p(\Gamma_Q) \to E_q(\Gamma_Q)$ by the correspondence between normal edges on p and q of a facet F_i, where $pq = e_j \subset \cap_{i\in I_p\setminus\{j\}} F_i$. Then, we can easily check that $\{\nabla_e \mid e \in E(\Gamma_Q)\}$ is a connection. This implies that $(\Gamma_Q, \mathcal{A}_\lambda)$ is a torus graph.

Conversely, let $(\Gamma, \mathcal{A})$ be an n-valent (abstract) torus graph. Assume that there is a nice manifold with corners Q_Γ whose face poset $(\mathcal{P}(Q_\Gamma), \preceq)$ is combinatorially equivalent to $(\mathcal{P}(\Gamma, \mathcal{A}), \preceq)$. Then, we can define the characteristic function $\lambda_{\mathcal{A}}$ on Q_Γ as follows. Let F be a facet in Q_Γ. Then, because $(\mathcal{P}(\Gamma, \mathcal{A}), \preceq) \equiv (\mathcal{P}(Q_\Gamma), \preceq)$, there is a corresponding $(n-1)$-valent torus subgraph $(\Gamma_F, \mathcal{A}_F)$. Because $\{\mathcal{A}_F(e) \mid e \in E(\Gamma_F)\}$ spans $(n-1)$-dimensional subspace of $(\mathfrak{t}_{\mathbb{Z}})^*$, we can define the element $\lambda_{\mathcal{A}}(F) \in \mathfrak{t}_{\mathbb{Z}}$ as follows:

$$\langle \mathcal{A}(e), \lambda_{\mathcal{A}}(F)\rangle = 0$$

for all $e \in E(\Gamma_F)$ and

$$\langle \mathcal{A}(e_{p,F}), \lambda_{\mathcal{A}}(F)\rangle = 1$$

for the edge $e_{p,F} \in E_p(\Gamma)\setminus E_p(\Gamma_F)$ for all $p \in V(\Gamma_F)$. Then, we can easily check $\lambda_{\mathcal{A}} : \mathcal{F}(Q_\Gamma) \to \mathfrak{t}_{\mathbb{Z}}$ is the axial function.

It is easy to check that the above two construction is *dual*, i.e., the following proposition holds.

PROPOSITION 4.12. *Let Q be an n-dimensional nice manifold with corners and Γ be its one-skeleton (n-valent graph). Then the following two statements hold:*

(1) *if there is a characteristic function λ on Q then there is an induced axial function $\mathcal{A}_\lambda$ on Γ;*
(2) *if there is an axial function $\mathcal{A}$ on Γ then there is an induced characteristic function $\lambda_{\mathcal{A}}$ on Q.*

Furthermore, these two functions are dual, i.e.,

$$\lambda_{\mathcal{A}_\lambda} = \lambda, \quad \mathcal{A}_{\lambda_{\mathcal{A}}} = \mathcal{A}.$$

By this proposition and the definitions of equivalence relations on (Q, λ) and $(\Gamma, \mathcal{A})$, we also have the following proposition:

PROPOSITION 4.13. *Let (Q_1, λ_1) and (Q_2, λ_2) be nice manifolds with corners with characteristic functions. Then the following two statements are equivalent:*

(1) *(Q_1, λ_1) and (Q_2, λ_2) are (resp. weakly) omnioriented isomorphic;*
(2) *the induced torus graphs $(\Gamma_{Q_1}, \mathcal{A}_{\lambda_1})$ and $(\Gamma_{Q_2}, \mathcal{A}_{\lambda_2})$ are (resp. weakly) isomorphic.*

5. Construction of 6-dimensional simply connected equivariantly formal torus manifolds

From now on, let M be a simply connected, equivariantly formal 6-dimensional (omnioriented) torus manifold. In this section, we recall the basic facts about such M.

5.1. Canonical model. Let (Q_M, λ_M) be the pair of the orbit space $Q_M = M/T$ of M and its induced characteristic function $\lambda_M : \mathcal{F}(Q) \to \mathfrak{t}^3_{\mathbb{Z}}$ (see Section 4.1). Then, it follows from Theorem 3.4 that Q_M is a face acyclic. Because the 3-dimensional manifold with corers Q_M (which is homeomorphic to 3-dimensional manifold with boundary) itself is acyclic, its boundary ∂Q_M may be regarded as the 2-dimensional compact, oriented manifold which becomes a boundary of an acyclic manifold. This implies that ∂Q_M is homeomorphic to the 2-sphere and Q_M is a homology 3-dimensional disk, i.e., $D^3 \# hS^3$ for some homology 3-sphere. Because M is simply connected, we also have Q_M is simply connected (see e.g. [**Wi**, Lemma 2.7]). Therefore, Q_M is homeomorphic to the standard 3-dimensional disk D^3.

Conversely, we can recover the simply connected, equivariantly formal 6-dimensional torus manifold M from the pair of the 3-dimensional disk Q admitting the structure of face acyclic with characteristic function λ, say (Q, λ) (see [**BuPa, DaJa, MaPa**]). Let $M(Q, \lambda)$ be the set of the following identifying space:

$$M(Q, \lambda) = Q \times T^3 / \sim_\lambda,$$

where the equivalence relation $(p, t_1) \sim_\lambda (q, t_2)$ is defined by $p = q \in Q$ and $t_1 t_2^{-1} \in T(p)$. Here, $T(p) \subset T$ is a subtorus generated by all $\lambda(F)$ such that $p \in F \in \mathcal{F}(Q)$, where if $p \in \mathrm{int}(Q)$ then we define $t_1 = t_2$. The topological manifold $M(Q, \lambda)$ is called the *canonical model* and it is known that $M(Q, \lambda)$ is equivariantly homeomorphic to a torus manifold M whose induced pair (Q_M, λ_M) is isomorphic to (Q, λ) (see [**MaPa**, Lemma 4.5]). Moreover, we have that $M(Q, \lambda)$ and M are equivariantly diffeomorphic by using the following one of the key fact in this paper (see [**Wi**, Theorem 1.3]).

THEOREM 5.1 (Wiemeler). *Let M_1 and M_2 be simply connected equivariantly formal 6-dimensional torus manifolds. Then, M_1 and M_2 are equivariantly homeomorphic if and only if they are equivariantly diffeomorphic.*

By using Theorem 5.1 and the result of [**Ju**], the canonical model $M(Q, \lambda)$ has the unique smooth structure. In addition, we have the following lemma:

LEMMA 5.2. *Let (Q_1, λ_1) and (Q_2, λ_2) be 3-dimensional disks admitting face acyclic structures and characteristic functions. Then, the following two statements are equivalent:*

(1) *two canonical models $M(Q_1, \lambda_1)$ and $M(Q_2, \lambda_2)$ are (weakly) equivariantly diffeomorphic;*
(2) $(Q_1, \lambda_1) \cong_{(w)} (Q_2, \lambda_2)$.

PROOF. The statement without "weakly" is the corollary of the two facts: Theorem 5.1; and the canonical model is determined up to equivariant homeomorphism. We shall prove the case when we put "weakly" in the statement. By Proposition 4.4, the statement from (1) to (2) is trivial. Assume $(Q_1, \lambda_1) \cong_w (Q_2, \lambda_2)$. Then, there is a diffeomorphism $f_* : Q_1 \to Q_2$ (in the sense of a manifold with corners) and an automorphism $\rho : T \to T$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\mathcal{F}(Q_1) & \xrightarrow{(\lambda_1)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}} \\ \downarrow f_{\mathbb{Z}} & & \downarrow \rho_* \\ \mathbb{Z}\mathcal{F}(Q_2) & \xrightarrow{(\lambda_2)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}} \end{array}$$

where $f_{\mathbb{Z}}$ and ρ_* are induced isomorphisms.

Now the product map $\rho \times f_* : T \times Q_1 \to T \times Q_2$ is a diffeomorphism (in the sense of manifold with corners); in particular, $\rho \times f_*$ is a homeomorphism between manifolds with boundary. Moreover, there is the surjective (continuous) map $T \times Q_i \to M(Q_i, \lambda_i) = M_i$ by the identifying quotient $/ \sim_{\lambda_i}$ $(i = 1, 2)$. Therefore, the following commutative diagram:

$$\begin{array}{ccc} T \times Q_1 & \xrightarrow{\rho \times f_*} & T \times Q_2 \\ \downarrow /\sim_{\lambda_1} & & \downarrow /\sim_{\lambda_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

induces a weakly (ρ-)equivariant homeomorphism $f : M_1 \to M_2$ (defined by $f[t,p] = [\rho(t), f_*(p)]$).

We claim that M_1 and M_2 are weakly (ρ-)equivariantly diffeomorphic. At first, we can easily verify that

$$\rho_*^{-1} \circ \lambda_2 : \mathcal{F}(Q_2) \xrightarrow{\lambda_2} \mathfrak{t}_{\mathbb{Z}} \xrightarrow{\rho_*^{-1}} \mathfrak{t}_{\mathbb{Z}}$$

defines the characteristic function on Q_2. We next prove that $M(Q_2, \lambda_2) = M_2$ and $M(Q_2, \rho_*^{-1} \circ \lambda_2)$ are weakly ρ^{-1}-equivariantly diffeomorphic. Because the product map $\rho^{-1} \times Id_* : T \times Q_2 \to T \times Q_2$ is a diffeomorphism (in the sense of a manifold with corners), the following commutative diagram:

$$\begin{array}{ccc} T \times Q_2 & \xrightarrow{\rho^{-1} \times \mathrm{Id}_*} & T \times Q_2 \\ \big\downarrow /\sim_{\lambda_2} & & \big\downarrow /\sim_{\rho_*^{-1} \circ \lambda_2} \\ M_2 & \xrightarrow{g} & M(Q_2, \rho_*^{-1} \circ \lambda_2) \end{array}$$

induces a weakly (ρ^{-1}-)equivariant homeomorphism $g : M_2 \to M(Q_2, \rho_*^{-1} \circ \lambda_2)$ by $g([t,p]) = [\rho^{-1}(t), p]$, where $[t,p] \in T \times Q_2/\sim_{\lambda_2}$. We may only prove that this homeomorphism g is the weakly (ρ-)equivariant diffeomorphism. In order to prove that, we note that for the atlas with corners $\{(U, \psi_U)\}$ of Q_2 induces the locally standard (smooth) atlas $\{(T \times U/\sim_{\lambda_2}, \psi_{T\times U/\sim_{\lambda_2}})\}$ of M_2. Let $\psi_{T\times U/\sim_{\lambda_2}}(T \times U/\sim_{\lambda_2}) = W$ be the equivariant open subset in $\mathbb{C}^3$. Note that we may regard $W = TU \subset \mathbb{C}^3$ for the relative open subset $U \subset \mathbb{R}^3_+$, i.e., $W \subset \mathbb{C}^3$ is the set of T-orbits of the elements in U. Similarly, $M(Q_2, \rho_*^{-1} \circ \lambda_2)$ also has this atlas. Because g is induced from $\rho^{-1} \times \mathrm{Id}_*$, the composition map

$$g_U = \psi_{T\times U/\sim_{\rho_*^{-1}\circ\lambda_2}} \circ g \circ \psi^{-1}_{T\times U/\sim_{\lambda_2}} : W(=TU) \to W(=TU)$$

is defined by

$$g_U(r_1t_1, r_2t_2, r_3t_3) = (r_1\rho_1(t), r_2\rho_2(t), r_3\rho_3(t))$$

where $(r_1, r_2, r_3) \in U \subset \mathbb{R}^3_+ \subset \mathbb{C}^3$, $(t_1, t_2, t_3) \in T(= S^1 \times S^1 \times S^1) \subset \mathbb{C}^3$ and $\rho(t_1, t_2, t_3) = (\rho_1(t), \rho_2(t), \rho_3(t)) \in T \subset \mathbb{C}^3$. Because the automorphism ρ is also a diffeomorphism on T, the composition maps g_U and g_U^{-1} are smooth isomorphisms. This implies that g is a weakly (ρ^{-1}-)equivariant diffeomorphism.

Together with the above commutative diagrams, the composition map $g \circ f$ is an equivariant homeomorphism between $M(Q_1, \lambda_1) = M_1$ and $M(Q_2, \rho_*^{-1} \circ \lambda_2)$. Therefore, by Theorem 5.1, M_1 and $M(Q_2, \rho_*^{-1} \circ \lambda_2)$ are equivariantly diffeomorphic. By the above claim, we have $M(Q_2, \rho_*^{-1} \circ \lambda_2)$ and M_2 are weakly ρ-equivariantly diffeomorphic. This establishes that M_1 and M_2 are weakly ρ-equivariantly diffeomorphic. □

5.2. Construction from torus graphs. Let $(\Gamma_M, \mathcal{A}_M)$ be a torus graph induced from the simply connected equivariantly formal 6-dimensional (omnioriented) torus manifold M. Then, by the argument in Section 4.3 and 5.1, we may regard the 3-valent graph Γ_M as the one-skeleton of the 3-dimensional disk $Q_M(= M/T)$ with the structure of face acyclic. Moreover, two induced objects (Q_M, λ_M) and $(\Gamma_M, \mathcal{A}_M)$ have the dual relation which stated in Proposition 4.12. Therefore, if there is a simply connected equivariantly formal 6-dimensional torus manifold M whose torus graph is $(\Gamma, \mathcal{A})$, we can construct the canonical model $M(\Gamma, \mathcal{A})$ of a torus graph $(\Gamma, \mathcal{A})$ by

$$M(\Gamma, \mathcal{A}) = M(Q_\Gamma, \lambda_{\mathcal{A}}),$$

where $(Q_\Gamma, \lambda_{\mathcal{A}})$ is the 3-dimensional disk Q_Γ whose one-skeleton is Γ and $\lambda_{\mathcal{A}}$ is its characteristic function whose dual is $\mathcal{A}$. Together with Proposition 4.13 and Lemma 5.2, we can easily show the following key lemma:

LEMMA 5.3. *Let $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ be torus graphs induced from simply connected equivariantly formal 6-dimensional torus manifolds M_1 and M_2, respectively. If $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ are (weakly) isomorphic, then M_1 and M_2 are (weakly) equivariantly diffeomorphic.*

In particular, we have that the canonical model $M(\Gamma_M, \mathcal{A}_M)$ of induced torus graph $(\Gamma_M, \mathcal{A}_M)$ from M is equivariantly diffeomorphic to the original torus manifold M (if M is a simply connected equivariantly formal 6-dimensional torus manifold).

6. Equivariantly formal six-dimensional torus manifolds

In this section, we prove the 1st main theorem (Theorem 6.10). Before we prove Theorem 6.10, we will exhibit some examples of simply connected, equivariantly formal 6-dimensional torus manifolds.

6.1. 6-sphere. Let $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$ be the unit sphere with the induced (positive) orientation from $\mathbb{C}^3 \oplus \mathbb{R}(\simeq \mathbb{R}^7)$. Then, T^3 acts on the first three complex coordinates by

$$(t_1, t_2, t_3)(z_1, z_2, z_3, r) \mapsto (\rho_1(t)z_1, \rho_2(t)z_2, \rho_3(t)z_3, r)$$

where $z_i \in \mathbb{C}$, $r \in \mathbb{R}$ such that $|z_1|^2 + |z_2|^2 + |z_3|^2 + r^2 = 1$, $t = (t_1, t_2, t_3) \in T$ and $\rho : T \to T$ is an automorphism. This action is effective and there are two fixed points (the north pole $p = (0, 0, 0, 1)$ and the south pole $q = (0, 0, 0, -1)$). Therefore, (S^6, T^3) is a (simply connected) equivariantly formal torus manifold. We can easily check that there are three characteristic submanifolds in (S^6, T^3) and we may choose their orientations by the orientation induced from $\mathbb{C}^2 \oplus \mathbb{R}(\simeq \mathbb{R}^5)$. We denote this omniorientation as $\mathcal{O}_{S^6}$. Then, the tangential representation around the fixed points my be regarded as follows:

$$T_p(S^6, \mathcal{O}_{S^6}) = \mathbb{C}^3 \times \{+1\} \subset \mathbb{C}^3 \oplus \mathbb{R}, \quad T_q(S^6, \mathcal{O}_{S^6}) = \mathbb{C}^3 \times \{-1\} \subset \mathbb{C}^3 \oplus \mathbb{R},$$

i.e., both of them are the natural torus representations by ρ. In particular, if ρ is the identity then the irreducible decomposition of $T_p(S^6, \mathcal{O}_{S^6})$ and $T_q(S^6, \mathcal{O}_{S^6})$ are obtained by

$$\mathbb{C}^3 = V(\alpha) \oplus V(\beta) \oplus V(\gamma),$$

where α, β, γ are the generators of $\mathfrak{t}^*_{\mathbb{Z}}$. Hence, the torus graph of $(S^6, \mathcal{O}_{S^6})$ with the standard T-action is the labeled graph in Figure 1.

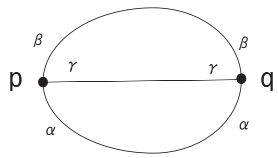

Figure 1. Torus graph induced from $(S^6, \mathcal{O}_{S^6})$ with the standard T^3-action. We call this torus graph the *standard torus graph with two vertices.*

If the 3-valent graph has only two vertices, then its combinatorial type is just as above graph. Moreover, due to the definition of axial functions of torus graphs, all of the axial functions on this graph is determined by the automorphism on $\mathfrak{t}^*_{\mathbb{Z}}$ of the above axial function. More precisely, we have the following lemma:

Lemma 6.1. *Let $(\Gamma, \mathcal{A})$ be a 3-valent torus graph such that $V(\Gamma) = \{p, q\}$, i.e., $|V(\Gamma)| = 2$. Then, the set of (oriented) edges is*

$$E(\Gamma) = \{e_1, e_2, e_3, \overline{e}_1, \overline{e}_2, \overline{e}_3\}$$

such that e_i connects two vertices p and q ($\overline{e}_i(= e_i)$ connects q and p), and the axial function $\mathcal{A}$ is

$$\mathcal{A}(e_i) = \mathcal{A}(\overline{e}_i) = k_{i1}\alpha + k_{i2}\beta + k_{i3}\gamma \in (\mathfrak{t}^3_{\mathbb{Z}})^* = H^2(BT^3; \mathbb{Z})$$

for the standard generator $\alpha, \beta, \gamma \in (\mathfrak{t}^3_{\mathbb{Z}})^*$ such that

$$\det \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} = \pm 1.$$

Furthermore, all of such torus graphs are weakly isomorphic to the standard torus graph with two vertices (Figure 1).

It is easy to see that the axial function in Lemma 6.1 can be realized by the following T^3-action on $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$:

$$(t_1, t_2, t_3)(z_1, z_2, z_3, r) \mapsto (t_1^{k_{11}} t_2^{k_{12}} t_3^{k_{13}} z_1, t_1^{k_{21}} t_2^{k_{22}} t_3^{k_{23}} z_2, t_1^{k_{31}} t_2^{k_{32}} t_3^{k_{33}} z_3, r).$$

Therefore, together with Lemma 5.3, the following lemma holds:

LEMMA 6.2. *Let $(\Gamma, \mathcal{A})$ be a 3-valent torus graph with $|V(\Gamma)| = 2$. Then, there is the canonical model $M(\Gamma, \mathcal{A})$ of $(\Gamma, \mathcal{A})$, and $M(\Gamma, \mathcal{A})$ is equivariantly diffeomorphic to a T^3-action on $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$.*

Furthermore, $M(\Gamma, \mathcal{A})$ is weakly equivariantly diffeomorphic to the standard T^3-action on $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$.

6.2. 6-dimensional quasitoric manifolds. The quasitoric manifold ([**BuPa, DaJa**]) is one of the examples of simply connected equivariantly formal torus manifolds. In this section, we introduce the quasitoric manifold.

If a torus manifold X satisfies the following two conditions, then we call X a *quasitoric manifold*:

(1) the T^n-action is locally standard;
(2) the orbit space M/T is a nice manifold with corners admitting the combinatorial structure of an n-dimensional convex polytope, i.e., the convex hull of finite points located in the general position in $\mathbb{R}^n$.

For example, the (complex) n-dimensional complex projective space $\mathbb{C}P^n$ with the standard T^n-action is the quasitoric manifold whose orbit space is n-dimensional simplex.

Let X be a 6-dimensional quasitoric manifold. Then, by [**DaJa**], this is a 6-dimensional (simply connected) equivariantly formal torus manifold. The Figure 2 shows the torus graph induced from $(\mathbb{C}P^3, \mathcal{O}_{\mathbb{C}})$, i.e., the omniorientation $\mathcal{O}_{\mathbb{C}}$ induced from the standard complex structure on $\mathbb{C}P^3$ and the standard T-action on $\mathbb{C}P^3$.

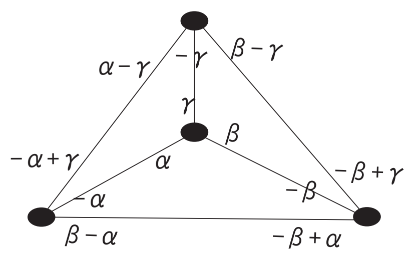

FIGURE 2. Torus graph induced from $(\mathbb{C}P^3, \mathcal{O}_{\mathbb{C}})$.

REMARK 6.3. Here, we note the relations between 6-dimensional quasitoric manifolds and (complex) 3-dimensional (complete) non-singular toric varieties, say 6-dimensional compact toric manifolds. By Remark 3.1, we may regard a 6-dimensional compact toric manifold as a 6-dimensional torus manifold. If X is a 6-dimensional compact toric manifold, then the orbit space of $0, 1$-dimensional orbits in X has the structure of a graph such that 3-valent, simple (i.e., there are no multiple edges and no loops) and planner. Using Steinitz's theorem in [**Zi**, Chapter 4], such graph can be always realized as the 1-skeleton of a 3-dimensional simple convex polytope. Therefore, Lemma 5.3 shows that a 6-dimensional compact toric manifold is equivariantly diffeomorphic to a 6-dimensional quasitoric manifold.

On the other hand, quasitoric manifolds do not always admit the structure of a toric manifold. For example, $(\mathbb{C}P^2 \# \mathbb{C}P^2) \times S^2$ is one of the 6-dimensional quasitoric manifolds, but its cohomology ring is never isomorphic to that of a toric manifold (also see [**BuPa**]). This shows that $(\mathbb{C}P^2 \# \mathbb{C}P^2) \times S^2$ does not admit the structure of a toric manifold.

By Steinitz's theorem and Lemma 5.3, we also have the following lemma:

LEMMA 6.4. *If a 3-valent torus graph $(\Gamma, \mathcal{A})$ satisfies that simple and planner, then there is a unique 6-dimensional quasitoric manifold $M(\Gamma, \mathcal{A})$ up to equivariant diffeomorphism.*

6.3. S^4-bundles over S^2. In this section, we introduce the structure of a torus manifold on the S^4-bundle over S^2. There are two free T^1-actions on $S^3 \subset \mathbb{C}^2$ by

$$(z_1, z_2) \mapsto (t^{-1}z_1, t^{\epsilon}z_2),$$

where $\epsilon = \pm 1$. We denote S^3 with the above T^1-action by S^3_ϵ. Note that S^3_ϵ / T^1 is diffeomorphic to the 2-sphere S^2. Because the complex line bundle over S^2 can be denoted by

$$S^3_\epsilon \times_{T^1} \mathbb{C}_a,$$

where $\mathbb{C}_a$ is the complex 1-dimensional vector space with the T^1-representation by a-times rotation for $a \in \mathbb{Z}$. Then, by taking the unit sphere bundle of the 5-dimensional real vector bundle

$$S^3_\epsilon \times_{T^1} (\mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{R})$$

where $S^3_\epsilon \times_{T^1} \mathbb{R}$ is the trivial real line bundle over S^2, we define S^4-bundle over S^2 denoted by

$$M(\epsilon, a, b) = S^3_\epsilon \times_{S^1} S(\mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{R}),$$

for $\epsilon = \pm 1$, $a, b \in \mathbb{Z}$. Then, we define the following T^3-action on $M(\epsilon, a, b)$:

$$[(w, z), (x, y, r)] \mapsto [(t_1 w, z), (t_2 x, t_3 y, r)],$$

where $(t_1, t_2, t_3) \in T^3$ and $[(w, z), (x, y, r)] \in M(\epsilon, a, b)$ such that $|w|^2 + |z|^2 = 1$ and $|x|^2 + |y|^2 + r^2 = 1$. It is easy to check that there are 4 fixed points and its orbit space admits the face acyclic structure induced from $D^2 \times I$, i.e., the product of the 2-dimensional disk and the interval. Therefore, $M(\epsilon, a, b)$ is a simply connected equivariantly formal 6-dimensional torus manifold.

Moreover, we can define the omniorientation on $M(\epsilon, a, b)$ standardly, and by using this omniorientation we have the torus graph drawn in Figure 3:

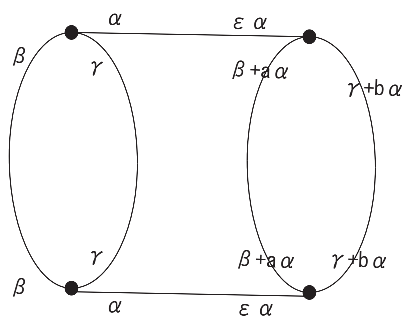

FIGURE 3. Torus graph induced from $M(\epsilon, a, b)$ with the standard omniorientation, where $\alpha, \beta, \gamma \in \mathfrak{t}^*_{\mathbb{Z}}$ are the standard generators.

By the easy argument, if the 3-valent graph Γ which is not the one-skeleton of 3-dimensional simplex (i.e., the graph in Figure 2) has 4 vertices, then Γ is combinatorially equivalent to the one-skeleton of $D^2 \times I$, i.e., the graph in Figure 3. Moreover, all of the axial functions on the one-skeleton of $D^2 \times I$ are the axial functions in Figure 3 up to automorphism $\rho : T \to T$. Therefore, by Lemma 5.3, we have the following lemma:

LEMMA 6.5. *Let $(\Gamma, \mathcal{A})$ be a 3-valent torus graph with $|V(\Gamma)| = 4$. Assume Γ is not the one-skeleton of 3-dimensional simplex. Then, there is the canonical model $M(\Gamma, \mathcal{A})$ of $(\Gamma, \mathcal{A})$, and $M(\Gamma, \mathcal{A})$ is weakly equivariantly diffeomorphic to a T^3-action on $M(\epsilon, a, b)$.*

REMARK 6.6. In Lemma 6.5, if Γ is a one-skeleton of 3-dimensional simplex, then $M(\Gamma, \mathcal{A})$ is weakly equivariantly diffeomorphic to the standard T^3-action on $\mathbb{C}P^3$.

6.4. Connected sum. The equivariant connected sum of two simply connected equivariantly formal 6-dimensional torus manifolds M_1 and M_2 constructs many examples of such torus manifolds. There seems to be several ways to define such an equivariant connected sum operations. In this paper, we define the equivariant connected sum around the fixed points of two 6-dimensional torus manifolds M_1 and M_2 as follows. Assume that two fixed points $p \in M_1$ and $q \in M_2$ have the same tangential representation

$$T_pM_1 = V(\alpha) \oplus V(\beta) \oplus V(\gamma) = T_qM_2,$$

where α, β, γ are the generators of $\mathfrak{t}^*_{\mathbb{Z}}$. Then, there are three characteristic submanifolds $M_i(\alpha)$, $M_i(\beta)$ and $M_i(\gamma)$ for $i = 1, 2$ such that $M_1(\alpha) \cap M_1(\beta) \cap M_1(\gamma) = \{p\}$ ($M_2(\alpha) \cap M_2(\beta) \cap M_2(\gamma) = \{q\}$) and $T_pM_1(\delta)(= T_qM_2(\delta)) = (V(\alpha) \oplus V(\beta) \oplus V(\gamma))/V(\delta)$, where $\delta = \alpha,\ \beta$ or γ. Moreover we assume signs of T_pM_1 and T_qM_2 are different and those of $T_pM_1(\delta)$ and $T_qM_2(\delta)$ are also different for all $\delta = \alpha, \beta, \gamma$ (see Remark 4.9). Then, we glue around the invariant tubular neighborhoods of p and q equivariantly and get another torus manifold $M_1 \#_{(p,q)} M_2$ (or we also denote $M_1 \# M_2$). This operation is called the *equivariant omnioriented connected sum*, or just *equivariant connected sum* in this paper. Because we assume M_1 and M_2 are simply connected equivariantly formal torus manifolds, the connected sum $M_1 \#_{(p,q)} M_2$ is also a simply connected equivariantly formal torus manifold. Because $Q_1 = M_1/T$ and $Q_2 = M_2/T$ are 3-dimensional disks with the structure of a face acyclic, $(M_1 \#_{(p,q)} M_2)/T$ is the connected sum $Q_1 \# Q_2$ along two vertices corresponding to fixed points p, q. Note that $Q_1 \# Q_2$ is also a 3-dimensional disks with the structure of a face acyclic.

Let $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ be (abstract) 3-valent torus graphs. If $p \in V(\Gamma_1)$ and $q \in V(\Gamma_2)$ have the same out-going axial functions, i.e.,

$$\{\mathcal{A}_1(e_i) \mid e_i \in E_p(\Gamma_1)\} = \{\mathcal{A}_2(f_i) \mid f_i \in E_q(\Gamma_2)\},$$

then we can do the *connected sum of torus graphs* between $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$, say $(\Gamma, \mathcal{A}) = (\Gamma_1, \mathcal{A}_1) \#_{(p,q)} (\Gamma_2, \mathcal{A}_2)$, as follows (also see Figure 4).

(1) $V(\Gamma) = V(\Gamma_1) \setminus \{p\} \sqcup V(\Gamma_2) \setminus \{q\}$;
(2) $E(\Gamma) = E(\Gamma_1) \setminus \{pp_1, pp_2, pp_3\} \sqcup E(\Gamma_2) \setminus \{qq_1, qq_2, qq_3\} \sqcup \{p_1q_1, p_2q_2, p_3q_3\}$, where $\mathcal{A}_1(pp_i) = \mathcal{A}_2(qq_i)$ for $i = 1, 2, 3$;
(3) $\mathcal{A} : E(\Gamma) \to (\mathfrak{t}^3_{\mathbb{Z}})^*$ such that $\mathcal{A}(e) = \mathcal{A}_1(e)$ and $\mathcal{A}(f) = \mathcal{A}_2(f)$ for $e \in E(\Gamma_1) \setminus \{pp_1, pp_2, pp_3\}$ and $f \in E(\Gamma_2) \setminus \{qq_1, qq_2, qq_3\}$, and $\mathcal{A}(p_iq_i) = \mathcal{A}_1(p_ip)$ and $\mathcal{A}(q_ip_i) = \mathcal{A}_2(q_iq)$.

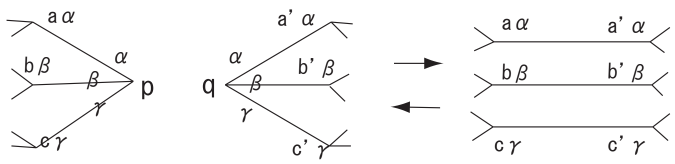

FIGURE 4. The equivariant connected sum $\sharp$ (from left to right) and its inverse $\sharp^{-1}$ (from right to left). Here, α, β, γ are generators of $(\mathfrak{t}^3_{\mathbb{Z}})^*$ and $a, a', b, b', c, c' = \pm 1$.

Because it is easy to check that the connection is well-defined on the above connected sum operation of the torus graphs, the obtained labeled graph $(\Gamma, \mathcal{A})$ is a torus graph.

By the conditions of signs of characteristic submanifolds in the definition of equivariant connected sums and using Corollary 5.3, we have the following lemma:

LEMMA 6.7. *Let $(\Gamma_1, \mathcal{A}_1)$, $(\Gamma_2, \mathcal{A}_2)$ be torus graphs and $(\Gamma, \mathcal{A}) = (\Gamma_1, \mathcal{A}_1) \#_{(p,q)} (\Gamma_2, \mathcal{A}_2)$ be their connected sum. Let $M(\Gamma_1, \mathcal{A}_1)$, $M(\Gamma_2, \mathcal{A}_2)$ and $M(\Gamma, \mathcal{A})$ be their canonical models, respectively. Assume that we can define connected sum around two fixed points p, q and we obtain the torus manifold $M = M(\Gamma_1, \mathcal{A}_1) \#_{(p,q)} M(\Gamma_2, \mathcal{A}_2)$. Then, there exists the following equivariant diffeomorphism:*

$$M(\Gamma, \mathcal{A}) \cong M = M(\Gamma_1, \mathcal{A}_1) \#_{(p,q)} M(\Gamma_2, \mathcal{A}_2).$$

REMARK 6.8. In the situation in Lemma 6.7, the omniorientation induced from the collapsing map $M(\Gamma, \mathcal{A}) \to M(\Gamma_1, \mathcal{A}_1) \cup \{q\}$ (resp. $M(\Gamma, \mathcal{A}) \to \{p\} \cup M(\Gamma_2, \mathcal{A}_2)$) defines the same torus graph $(\Gamma_1, \mathcal{A}_1)$ (resp. $(\Gamma_2, \mathcal{A}_2)$).

REMARK 6.9. Similarly, we can define the notion of the equivariant connected sum on (Q, λ), i.e., the nice manifold with corners with characteristic functions

6.5. The 1st main theorem. Now we may prove the 1st main theorem.

THEOREM 6.10. *Let M be a simply connected, equivariantly formal 6-dimensional torus manifold. Then, M is equivariantly diffeomorphic to one of the following manifolds:*

(1) $S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$ *with the torus action induced from a T^3-action on $\mathbb{C}^3$.;*
(2) 6*-dimensional quasitoric manifold X;*
(3) S^4*-bundle over S^2 equipped with the structure of a torus manifold which is weakly equivariantly diffeomorphic to $M(\epsilon, a, b)$,*

or otherwise, there is a 6-dimensional quasitoric manifold X and S^4-bundles over S^2, say S_i for some $i = 1, \dots, \ell$, such that M is equivariantly diffeomorphic to

$$X\#S_1\#\cdots\#S_\ell.$$

Here, S_i is weakly equivariantly diffeomorphic to $M(\epsilon_i, a_i, b_i)$ for some $\epsilon_i = \pm 1$ and $a_i, b_i \in \mathbb{Z}$.

PROOF. Let M be a simply connected, equivariantly formal 6-dimensional torus manifold and $(\Gamma_M, \mathcal{A}_M)$ be its induced torus graph. By Lemma 6.2, the number of vertices $|V(\Gamma_M)| = 2$ if and only if M is diffeomorphic to S^6. Therefore, if $|V(\Gamma_M)| = 2$ then the statement (1) occurs. So we may only prove the case when $|V(\Gamma_M)| > 2$, i.e., M is not diffeomorphic to S^6.

Assume that M is not diffeomorphic to S^6. Then, by the combinatorial argument for 3-valent graphs, we have $|V(\Gamma_M)| \geq 4$. If $|V(\Gamma_M)| = 4$, it follows from Lemma 6.5 and Remark 6.6 that M is equivariantly diffeomorphic to a quasitoric manifold $\mathbb{C}P^3$ or an S^4-bundle over S^2 which is weakly equivariantly diffeomorphic to $M(\epsilon, a, b)$ for some $\epsilon = \pm 1$, $a, b \in \mathbb{Z}$, discussed in Section 6.3. Therefore, if $|V(\Gamma_M)| = 4$ then the statement (2) or (3) occur. So we may assume $|V(\Gamma_M)| \geq 5$. Recall that Γ_M is the one-skeleton of the orbit space M/T. Because now M/T is homeomorphic to the 3-dimensional disk, its one-skeleton Γ_M can be realized as a planner graph (by using the stereographic projection from the 2-sphere to the plane). Therefore, if there is no multiple edges, i.e., two vertices p, q connected by more than 2 edges, then Γ_M can be realized as the one-skeleton of a 3-dimensional simple polytope by Steinitz's theorem. This implies that if there is no multiple edges in Γ_M then there is a quasitoric manifold X which is equivariantly diffeomorphic to M by Lemma 6.4, i.e., the statement (2).

Therefore, we may only discuss the case when $|V_\Gamma| \geq 5$ and there are (at least) two vertices connected two edges in Γ_M. In this case, by Lemma 6.7, it is enough to show that $(\Gamma_M, \mathcal{A}_M)$ can be decomposed into the following torus graphs:

$$(\Gamma_M, \mathcal{A}_M) = (\Gamma_X, \mathcal{A}_X)\#(\Gamma_{S_1}, \mathcal{A}_{S_1})\#\cdots\#(\Gamma_{S_\ell}, \mathcal{A}_{S_\ell}).$$

Assume two vertices p, q are connected by two edges (see the bottom graph in Figure 5). Then, by following the connection in the definition of the torus graph, it is easy to check that the axial functions around the multiple edges connecting p, q satisfy the axial functions expressed in Figure 5, where we can take α, β, γ as any choices of generators in $(\mathfrak{t}^3_{\mathbb{Z}})^*$. In this case, we can do (inverse) connected sum operation around the vertex enclosed with a circle in Figure 5 (from the bottom to the top in Figure 5). Then, the induced torus graph $(\Gamma_M, \mathcal{A}_M)$ is decomposed into two induced torus graphs $(\Gamma_{S_1}, \mathcal{A}_{S_1})$ and $(\Gamma_{M'}, \mathcal{A}_{M'})$, where M' is some simply connected equivariantly formal 6-dimensional torus manifold and S_1 is a torus manifold which is weakly equivariantly diffeomorphic to $M(\epsilon, a, b)$. Namely, we have

$$(\Gamma_M, \mathcal{A}_M) \cong (\Gamma_{M'}, \mathcal{A}_{M'})\#(\Gamma_{S_1}, \mathcal{A}_{S_1}).$$

If there is no multiple edges in $\Gamma_{M'}$ then M' is a quasitoric manifold. If there is a multiple edge, then we iterate this argument and finally we have the following decomposition:

$$(\Gamma_M, \mathcal{A}_M) = (\Gamma_X, \mathcal{A}_X)\#(\Gamma_{S_1}, \mathcal{A}_{S_1})\#\cdots\#(\Gamma_{S_\ell}, \mathcal{A}_{S_\ell}),$$

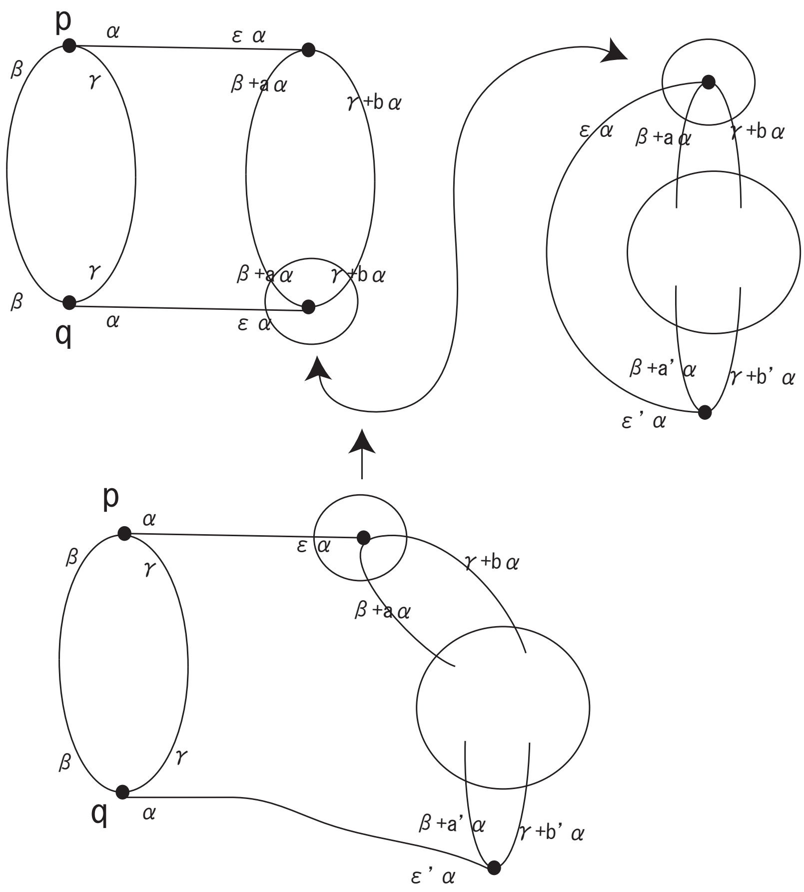

FIGURE 5. We may regard α, β, γ as any generators in $(\mathfrak{t}^3_{\mathbb{Z}})^*$ and $a, a', b, b' \in \mathbb{Z}$ and $\epsilon, \epsilon' = \pm 1$.

where Γ_X is a one-skeleton of some simple polytope. This establishes the statement of the theorem. □

By Theorem 6.10, we also have the following well-known result.

COROLLARY 6.11. *Let M be a simply connected 6-dimensional torus manifold whose cohomology ring is generated by the 2nd degree cohomology. Then, M is a quasitoric manifold.*

REMARK 6.12. Corollary 6.11 does not hold for eight-dimensional torus manifold. (There is such a torus manifold over a *Barnette sphere* which is non-polytopal 3-sphere, see [**IFM**])

7. Equivariant cohomological rigidity

In this section, we study the classification of equivariantly formal torus manifolds by using their equivariant cohomology. The goal of this section is to prove the following theorem:

THEOREM 7.1. *Let M_1 and M_2 be simply connected, equivariantly formal 6-dimensional torus manifolds. Then, the following two statements are equivalent:*

(1) $(M_1, T^3) \cong_w (M_2, T^3)$, *i.e., weakly equivariantly diffeomorphic;*
(2) $H^*_T(M_1) \simeq_w H^*_T(M_2)$, *i.e., weakly isomorphic as the $H^*(BT)$-algebra.*

Because the statement from (1) to (2) is trivial, we shall prove the statement from (2) to (1).

7.1. Preliminary to prove. We first recall the structure of equivariant cohomology $H^*(BT)$-algebras of equivariantly formal torus manifolds proved by Maeda-Masuda-Panov. In order to state the result, we define the equivariant cohomology of torus graphs.

A *(graph) equivariant cohomology of the torus graph* $(\Gamma, \mathcal{A})$ is defined as the following set of functions:

$$H_T^*(\Gamma, \mathcal{A}) = \{f : V(\Gamma) \to H^*(BT) \mid f(p) - f(q) \equiv 0 \mod \mathcal{A}(pq)\}.$$

It is easy to show that this set is a graded ring which is induced from the graded ring structure of $H^*(BT)$. Moreover, we can define the function $i_\alpha : V(\Gamma) \to H^*(BT)$ by $i_\alpha(p) = \alpha$. Obviously, $i_\alpha \in H_T^*(\Gamma, \mathcal{A})$. This defines the homomorphism $i : H^*(BT) \to H_T^*(\Gamma, \mathcal{A})$ by

$$i(\alpha) = i_\alpha.$$

Therefore, by this homomorphism, we may regard $H_T^*(\Gamma, \mathcal{A})$ as the $H^*(BT)$-algebra. By using these structures, we may define the following graded ring:

$$H^*(\Gamma, \mathcal{A}) = H_T^*(\Gamma, \mathcal{A})/i(H^{>0}(BT)),$$

where $i(H^{>0}(BT))$ represents the image of $H^*(BT)$ $(* \neq 0)$ by i. We call $H^*(\Gamma, \mathcal{A})$ a *(graph) cohomology ring of the torus graph* $(\Gamma, \mathcal{A})$.

Now we can define an element in $H_T^*(\Gamma, \mathcal{A})$ which represents the subgraph K. Let K be a $(n-k)$-valent torus subgraph in $(\Gamma, \mathcal{A})$, where Γ is an n-valent. Then, we can define the element $\tau_K \in H_T^{2k}(\Gamma, \mathcal{A})$ as follows:

$$\tau_K(p) = \begin{cases} \prod_{e \in E_p(\Gamma) \setminus E_p(K)} \mathcal{A}(e) & \text{if} \quad p \in V(K) \\ 0 & \text{if} \quad p \notin V(K) \end{cases}$$

The element τ_K is called the *Thom class* of K. Because $|E_p(\Gamma) \setminus E_p(K)| = k$, the degree of τ_K is $2k$, i.e., $\tau_K \in H_T^{2k}(\Gamma, \mathcal{A})$. We formally define $\tau_\Gamma = 1 \in H_T^0(\Gamma, \mathcal{A}) \simeq \mathbb{Z}$ and $\tau_\emptyset = 0 \in H_T^{-1}(\Gamma, \mathcal{A}) = \{0\}$. Let $\mathcal{P}_*(\Gamma, \mathcal{A})$ be the simplicial poset induced from torus subgraphs in $(\Gamma, \mathcal{A})$. Now we may define the *face ring* $\mathbb{Z}[\mathcal{P}_*(\Gamma, \mathcal{A})]$ (*generalized Stanley-Reisner ring*) as follows:

$$\mathbb{Z}[\mathcal{P}_*(\Gamma, \mathcal{A})] = \mathbb{Z}[K \mid K \in \mathcal{P}_*(\Gamma, \mathcal{A})]/\mathcal{I}$$

where $\mathbb{Z}[K \mid K \in \mathcal{P}_*(\Gamma, \mathcal{A})]$ is the polynomial ring generated by all torus subgraphs K (where we define $\deg K = 2(n - |E_p(K)|)$, for $p \in V(K)$), the ideal $\mathcal{I}$ is generated by the following homogeneous polynomials for $G, H \in \mathcal{P}_*(\Gamma, \mathcal{A})$;

$$GH = G \vee H \sum_{E \in G \cap H} E$$

where $G \vee H$ represents a minimal face containing both G and H, $\sum_{E \in G \cap H} E$ runs through all of the connected component of $G \cap H$ and we formally define $\Gamma = 1$ and $\emptyset = 0$. In general, such a least upper bound $G \vee H$ may fail to exist or be non-unique; however it exists and is unique provided that the intersection $G \cap H$ is non-empty. (If $G \cap H$ is empty, the right hand side is zero, i.e., $GH = 0$.) Then, the following theorem holds:

THEOREM 7.2 ([**MaPa**] and [**MMP**]). *Suppose* $(\Gamma, \mathcal{A})$ *be an (abstract) torus graph. Then, the correspondence between a torus subgraph and its Thom class,* $K \mapsto \tau_K$, *induces the following ring isomorphism:*

$$\mathbb{Z}[\mathcal{P}_*(\Gamma, \mathcal{A})] \simeq H_T^*(\Gamma, \mathcal{A}).$$

Furthermore, if M *is an equivariantly formal, omnioriented torus manifold, then there is the following ring isomorphism:*

$$H_T^*(M) \simeq H_T^*(\Gamma_M, \mathcal{A}_M).$$

Recall that $H^*(BT) \simeq \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ for $\deg \alpha_i = 2$. Theorem 7.2 also implies that the $H^*(BT)$-algebraic structure of $H_T^*(\Gamma, \mathcal{A})$ can be determined by Thom classes of $(n-1)$-valent torus subgraphs, say $\tau_1, \dots, \tau_m$. More precisely, for any element $\alpha \in H^2(BT)(= \mathfrak{t}^*_{\mathbb{Z}})$, there exists the element $\lambda_i \in H_2(BT)(= \mathfrak{t}_{\mathbb{Z}})$ which satisfies

$$\alpha \mapsto \sum_{i=1}^{m} \langle \alpha, \lambda_i \rangle \tau_i \in H^2_T(\Gamma, \mathcal{A}), \tag{7.1}$$

and

$$\sum_{i=1}^{m} \langle \alpha, \lambda_i \rangle \tau_i|_p = \sum_{i \in I_p} \langle \alpha, \lambda_i \rangle \tau_i|_p = \alpha \quad \text{for all vertices } p \in V(\Gamma).$$

Remark 7.3. If there is a face acyclic Q_Γ whose one-skeleton is Γ, then the above $\lambda_i \in \mathfrak{t}_{\mathbb{Z}}$ defines the characteristic function $\lambda_{\mathcal{A}}$ on Q_Γ which is the dual of the axial function $\mathcal{A}$.

If two torus graphs $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ induce the same simplicial poset (see Section 2.2), then it follows from Theorem 7.2 that

$$H^*_T(\Gamma_1, \mathcal{A}_1) \simeq H^*_T(\Gamma_2, \mathcal{A}_2) \quad \text{as a ring.}$$

This implies that if we take two distinct omniorientations $\mathcal{O}_1$ and $\mathcal{O}_2$ on the equivariantly formal torus manifold M, then

$$H^*_T(\Gamma_{(M,\mathcal{O}_1)}, \mathcal{A}_{(M,\mathcal{O}_1)}) \simeq H^*_T(\Gamma_{(M,\mathcal{O}_2)}, \mathcal{A}_{(M,\mathcal{O}_2)}) \quad \text{as a ring,}$$

where $(\Gamma_{(M,\mathcal{O}_i)}, \mathcal{A}_{(M,\mathcal{O}_i)})$ represents the torus graph induced from omnioriented torus manifold $(M, \mathcal{O}_i)$, for $i = 1, 2$. Moreover, the choice of omniorientation $\mathcal{O}$ (with the fixed orientation M) is just to choose the signs of Thom classes in $H^2_T(\Gamma, \mathcal{A})$, i.e., two orientations of a characteristic submanifold M_j corresponding with $+\tau_{K_j}$ and $-\tau_{K_j}$, where K_j is a $(n-1)$-valent torus subgraph corresponding with M_j. This implies that the ring isomorphism as above can be given by the map $\tau_i \mapsto \pm\tau_i$ for $i = 1, \dots, m$. Therefore, we have the following proposition:

Proposition 7.4. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be omniorientations on the equivariantly formal torus manifold M. Then, for the graph equivariant cohomology of their induced torus graphs, the following holds:*

$$H^*_T(\Gamma_{(M,\mathcal{O}_1)}, \mathcal{A}_{(M,\mathcal{O}_1)}) \simeq H^*_T(\Gamma_{(M,\mathcal{O}_2)}, \mathcal{A}_{(M,\mathcal{O}_2)}) \quad \textit{as } H^*(BT)\textit{-algebra.}$$

Furthermore, if M is an equivariantly formal, omnioriented torus manifold, then there is the following $H^*(BT)$-algebra isomorphism:

$$H^*_T(M) \simeq H^*_T(\Gamma_M, \mathcal{A}_M).$$

Here, in Proposition 7.4, $H^*_T(\Gamma_1, \mathcal{A}_1) \simeq H^*_T(\Gamma_2, \mathcal{A}_2)$ represents that there is a graded ring isomorphism $f^* : H^*_T(\Gamma_1, \mathcal{A}_1) \to H^*_T(\Gamma_2, \mathcal{A}_2)$ such that the following diagram commutes

$$\begin{array}{ccc} H^*(BT) & \xrightarrow{i_1} & H^*_T(\Gamma_1, \mathcal{A}_1) \\ \big\downarrow \mathrm{Id} & & \big\downarrow f^* \\ H^*(BT) & \xrightarrow{i_2} & H^*_T(\Gamma_2, \mathcal{A}_2) \end{array}$$

where i_1 and i_2 are the homomorphism which defines the $H^*(BT)$-algebra structure. If we use an isomorphism $\rho^* : H^*(BT) \to H^*(BT)$ instead of Id in the above diagram, then we denote $H^*_T(\Gamma_1, \mathcal{A}_1) \simeq_w H^*_T(\Gamma_2, \mathcal{A}_2)$

Let $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ be torus graphs induced from equivariantly formal torus manifolds M_1 and M_2 respectively. By Theorem 7.2, it is easy to check that $H^*_T(M_1) \simeq_w H^*_T(M_2)$ if and only if $H^*_T(\Gamma_1, \mathcal{A}_1) \simeq_w H^*_T(\Gamma_2, \mathcal{A}_2)$. Therefore, in order to show the statement from (2) to (1) in Theorem 7.1, by using the arguments in Section 4.3 and Lemma 5.2, it is enough to show the following theorem

Theorem 7.5. *Let M_1, M_2 be simply connected, equivariantly formal 6-dimensional torus manifolds, and let $(\Gamma_1, \mathcal{A}_1)$, $(\Gamma_2, \mathcal{A}_2)$ be their induced torus graphs and (Q_1, λ_1), (Q_2, λ_2) be their induced 3-dimensional disks with characteristic functions, respectively. If $H^*_T(\Gamma_1, \mathcal{A}_1) \simeq_w H^*_T(\Gamma_2, \mathcal{A}_2)$, then $(Q_1, \lambda_1) \cong_w (Q_2, \lambda_2)$.*

7.2. Proof of Theorem 7.5. We shall prove Theorem 7.5. In this section, we assume that all torus graphs $(\Gamma, \mathcal{A})$ are induced from a simply connected, equivariantly formal 6-dimensional torus manifold, i.e., for torus graph $(\Gamma, \mathcal{A})$, there is an omnioriented torus manifold $(M, \mathcal{O})$ such that

$$(\Gamma, \mathcal{A}) = (\Gamma_{(M,\mathcal{O})}, \mathcal{A}_{(M,\mathcal{O})}).$$

Due to Proposition 7.4, we may also denote this by $(\Gamma_M, \mathcal{A}_M)$. Moreover, by using Theorem 6.10, the combinatorial type of such a torus graph is one of the following cases:

CASE 1: $|V(\Gamma)| = 2$ and $E(\Gamma) = 3$, say Γ_{S^6} (see Figure 1);

CASE 2: the connected sum of torus graphs induced from $M(\epsilon, a, b)$, say $\Gamma_{S_1}\#\cdots\#\Gamma_{S_\ell}$ (see Figure 3);

CASE 3: Otherwise, i.e., the connected sum of a torus graph obtained from a one-skeleton of 3-dimensional simple convex polytope and a torus graph in CASE 2, say $\Gamma_X\#\Gamma_{S_1}\#\cdots\#\Gamma_{S_\ell}$ where Γ_X is the one-skeleton of a polytope.

7.2.1. *CASE 1.* Let $H_T^*(\Gamma_{S^6}, \mathcal{A}_{S^6})$ be the equivariant cohomology of the torus graph induced from a T^3-action on S^6. By Lemma 6.2 and Proposition 7.4, we may regard $H_T^*(\Gamma_{S^6}, \mathcal{A}_{S^6})$ as the equivariant cohomology induced from the standard T^3-action on S^6 (see Figure 1).

Assume that for some induced torus graph $(\Gamma_M, \mathcal{A}_M)$ there is a weak $H^*(BT)$-algebra isomorphism

$$f^* : H_T^*(\Gamma_{S^6}, \mathcal{A}_{S^6}) \to H_T^*(\Gamma_M, \mathcal{A}_M).$$

Because f^* induces the isomorphism between $H^*(\Gamma_{S^6}, \mathcal{A}_{S^6}) \simeq H^*(\Gamma_M, \mathcal{A}_M)$, we have that there are just two vertices in $(\Gamma_M, \mathcal{A}_M)$ (because $\#V(\Gamma_M) = \chi(M)$). It follows from Theorem 6.10 that we have that $(\Gamma_{S^6}, \mathcal{A}_{S^6})$ and $(\Gamma_M, \mathcal{A}_M)$ are weakly isomorphic. In particular, by Definition 4.3 and Proposition 4.13, we have

Proposition 7.6. *Let $(\Gamma_M, \mathcal{A}_M)$ be a torus graph induced from a simply connected equivariantly formal 6-dimensional torus manifold. Assume that $H_T^*(\Gamma_{S^6}, \mathcal{A}_{S^6}) \simeq_w H_T^*(\Gamma_M, \mathcal{A}_M)$. Then,*

$$(Q_M, \lambda_M) \cong_w (Q_{S^6}, \lambda_{S^6}).$$

So, in the rest of this section, we may assume $|V(\Gamma)| > 2$; moreover, (by the combinatorial argument) we may assume

$$|V(\Gamma)| \geq 4$$

for the induced torus graph $(\Gamma, \mathcal{A})$ from a simply connected, equivariantly formal 6-dimensional torus manifold.

7.2.2. *Zero-length.* In order to prove Theorem 7.5 for CASE 2 and CASE 3, we introduce the *zero-length* of the element $\xi \in H_T^*(\Gamma, \mathcal{A})$ in [**Ma08**]. Set

$$Z(\xi) = \{p \in V(\Gamma) \mid \xi(p) = 0\}. \tag{7.2}$$

We call the cardinality of $Z(\xi)$ the *zero-length of* $\xi \in H_T^*(\Gamma, \mathcal{A})$ and denote it by $|Z(\xi)|$. Because in our case $H_T^*(\Gamma, \mathcal{A}) \simeq H_T^*(M)$ for some torus manifold (M, T), the localization theorem holds (see [**Hs**, p. 40]). Hence, with the method similar to that demonstrated in [**Ma08**, Section 3], we have that

$$|Z(\xi)| = |Z(f(\xi))|$$

for a weak $H^*(BT)$-algebra isomorphism $f : H_T^*(\Gamma_1, \mathcal{A}_1) \to H_T^*(\Gamma_2, \mathcal{A}_2)$.

We first show the following lemma:

Lemma 7.7. *Let $\xi \in H_T^*(\Gamma, \mathcal{A})$ $(* = 2, 4)$ be a non-zero element such that $|Z(\xi)| = |V(\Gamma)| - 2$, i.e., $V(\Gamma) \setminus Z(\xi) = \{p, q\}$. Then, if $* = 2$ (resp. $* = 4$), there exists a 2-valent multiple edge K (resp. an edge e) connecting p, q and a non-zero integer k such that*

$$\xi = k\tau_K \quad (\textit{resp. } k\tau_e),$$

where $\tau_K \in H_T^2(\Gamma, \mathcal{A})$ (resp. $\tau_e \in H_T^4(\Gamma, \mathcal{A})$) is the Thom class of K (resp. e).

Moreover, let $\xi \in H_T^6(\Gamma, \mathcal{A})$ be a non-zero element such that $|Z(\xi)| = |V(\Gamma)| - 1$, i.e., $V(\Gamma) \setminus Z(\xi) = \{p\}$ Then, there is a non-zero integer k such that

$$\xi = k\tau_p,$$

where $\tau_p \in H^6_T(\Gamma, \mathcal{A})$ *is the Thom class of* p.

Proof. Because the latter statement (for the case when $* = 6$) is straightforward by definition, we only prove the former statement.

Let $V(\Gamma) \setminus Z(\xi) = \{p,\ q\}$. Assume p and q are not connected by edges. Then, by the definition of $H^*_T(\Gamma, \mathcal{A})$, there are integers k_p and k_q such that

$$\xi(p) = k_p \alpha_{1,p}\alpha_{2,p}\alpha_{3,p}, \quad \xi(q) = k_q \alpha_{1,q}\alpha_{2,q}\alpha_{3,q}$$

where $\alpha_{i,p} = \mathcal{A}(e_{i,p}) \in H^2(BT^3)$ for $e_{i,p} \in E_p(\Gamma)$. Because the degree of $\xi \in H^*_T(\Gamma, \mathcal{A})$ is 2 or 4, we have $k_p = k_q = 0$. This gives a contradiction to $\xi \neq 0$. Therefore, because we assume $|V(\Gamma)| > 2$, there are 1 or 2 edges connecting p and q.

If there is the only one such edge, say e, similarly we have

$$\xi(p) = k_p \alpha_{1,p}\alpha_{2,p}, \quad \xi(q) = k_q \alpha_{1,q}\alpha_{2,q},$$

where $\alpha_{i,p} - \alpha_{i,q} \equiv 0 \pmod{\mathcal{A}(e)}$ for $i = 1, 2$. Because $\xi(p) - \xi(q) \equiv 0 \pmod{\mathcal{A}(e)}$, we have that $k_p = k_q = k$. Therefore, ξ can be written by

$$\xi = k\tau_e \in H^4_T(\Gamma, \mathcal{A}).$$

If there are two edges connecting p and q, say e_1, e_2, i.e., p and q are connected by the multiple edge. Put this multiple edge as the 2-valent subgraph $K \subset \Gamma$ such that $V(K) = \{p, q\}$ and $E(K) = \{e_1, e_2\}$. By the axial functions around the multiple edge (see the bottom graph in Figure 5), K is a 2-valent torus subgraph. Because $\xi \in H^2_T(\Gamma, \mathcal{A})$ and $V(\Gamma) \setminus Z(\xi) = \{p,\ q\}$, we have

$$\xi(p) = k\alpha_p, \quad \xi(q) = k\alpha_q$$

for some integer k and the normal axial function $\alpha_p = \tau_K(p)$ and $\alpha_q = \tau_K(q)$. This establishes the former statement. □

Let $f : H^*_T(\Gamma_1, \mathcal{A}_1) \to H^*_T(\Gamma_2, \mathcal{A}_2)$ be a weak $H^*(BT)$-algebra isomorphism. Let $\mathcal{T}^{(i)}_k$ be the set of all Thom classes of 2-valent torus subgraphs K in $(\Gamma_i, \mathcal{A}_i)$ $(i = 1, 2)$ such that $|Z(\tau)| = k$ for $\tau \in \mathcal{T}^{(i)}_k$ $(k = 0, \ldots, |V(\Gamma_i)| - 2)$. By Lemma 7.7, we have the following corollary:

Corollary 7.8. *Let* $\tau \in \mathcal{T}^{(1)}_{|V(\Gamma_1)|-2}$. *Then, there is an element* $\tau' \in \mathcal{T}^{(2)}_{|V(\Gamma_2)|-2}$ *such that*

$$f(\tau) = \pm\tau'.$$

Namely, by a weak algebra isomorphism, the Thom class of a multiple edge in Γ_1 *maps to the Thom class of a multiple edge in* Γ_2 *up to sign.*

Next, we generalize the statement in Corollary 7.8 to the Thom classes of $\mathcal{T}^{(1)}_k$ for all $k \geq 0$ in Lemma 7.11. To do that, we prepare two lemmas:

Lemma 7.9. *Let* $\xi \in H^2_T(\Gamma, \mathcal{A})$ *be an element with* $0 < |Z(\xi)| < |V(\Gamma)| - 2$. *Express* $\xi = \sum_{K \subset \Gamma} a_K \tau_K$ *for some integers* a_K, *where* K *runs through all* 2*-valent torus subgraphs in* $(\Gamma, \mathcal{A})$. *If* $a_K \neq 0$ *for some* K, *then* $Z(\xi) \subset Z(\tau_K)$. *Furthermore, if* $a_K \neq 0$ *and* $a_H \neq 0$ *for some distinct* K *and* H, *then* K *or* H *is a multiple edge, i.e.,* K *or* H *consists of* 2 *vertices and* 2 *edges, or otherwise* $Z(\xi) \subsetneq Z(\tau_K)$.

Proof. Let $p \in V(\Gamma)$ and $p \in Z(\xi)$. Because $0 = \xi|_p = \sum_K a_K \tau_K|_p$, if $a_K \neq 0$ then $\tau_K|_p = 0$. This implies that if $a_K \neq 0$ then $Z(\xi) \subset Z(\tau_K)$. Moreover, if both a_K and a_H are non-zero, then $Z(\xi) \subset Z(\tau_K) \cap Z(\tau_H)$. Therefore, we may only prove that K or H is a multiple edge, or otherwise $Z(\tau_K) \cap Z(\tau_H)$ is properly contained in $Z(\tau_K)$.

Suppose that $Z(\tau_K) \cap Z(\tau_H) = Z(\tau_K)$. Then $Z(\tau_H) \supset Z(\tau_K)$. By definition of the Thom class, we have that

$$\tau_K|_q = 0 \quad \text{if and only if} \quad q \notin V(K). \tag{7.3}$$

Therefore, $V(H) \subset V(K)$. By Theorem 6.10, the combinatorial type of all Γ (when $|V(\Gamma)| > 2$) is determined by one of the torus graphs stated in CASE 2 or CASE 3 (in the beginning of Section 7.2). Therefore, it is easy to check that if there are 2-valent torus subgraphs H and K such that $V(H) \subset V(K)$ then H must be a multiple edge, i.e., there are just 2 vertices. Otherwise, $Z(\tau_K) \cap Z(\tau_H) \neq Z(\tau_K)$, i.e., $Z(\xi) \subsetneq Z(\tau_K)$. This establishes the statement. □

LEMMA 7.10. *Assume that there is a* 2*-valent torus subgraph* $K \subset \Gamma$ *such that its Thom class* $\tau_K \in H^2_T(\Gamma, \mathcal{A})$ *satisfies* $|Z(\tau_K)| = 0$. *Then,* $|V(\Gamma)|$ *is an even number, say* $2(\ell+1)$ *(for some* $\ell \geq 1$*). Furthermore, there exist exactly* $\ell+1$ *multiple edges, and* 2 *torus subgraphs* K_1 *and* K_2 *such that* $V(K_1) = V(K_2) = V(\Gamma)$.

PROOF. If there is a 2-valent torus subgraph $K \subset \Gamma$ such that its Thom class $\tau_K \in H^2_T(\Gamma, \mathcal{A})$ satisfies $|Z(\tau_K)| = 0$, then it follows from the combinatorial types of CASE 2 and CASE 3 that Γ is one of the torus graphs in CASE 2. Therefore, we have $\Gamma = S_1 \# \cdots \# S_\ell$. By considering the axial function on it (e.g. see Figure 5), it is easy to check the statement. □

Now we may prove Lemma 7.11.

LEMMA 7.11. *Every weak* $H^*(BT)$*-algebra isomorphism* $f : H^*_T(\Gamma_1, \mathcal{A}_1) \to H^*_T(\Gamma_2, \mathcal{A}_2)$ *preserves* $\mathcal{T}^{(1)}_k$ *to* $\mathcal{T}^{(2)}_k$ *up to sign.*

PROOF. Because f induces the isomorphism $H^*(\Gamma_1, \mathcal{A}_1) \to H^*(\Gamma_2, \mathcal{A}_2)$, with the method similar to that demonstrated in Section 7.2.1, we may put

$$V(\Gamma_1) = V = V(\Gamma_2).$$

By Corollary 7.8, f preserves $\mathcal{T}^{(1)}_{|V|-2}$ and $\mathcal{T}^{(2)}_{|V|-2}$ up to sign. Therefore, we may only think the elements of $\mathcal{T}^{(1)}_k$ and $\mathcal{T}^{(2)}_k$ $(k < |V|-2)$.

Let $\tau^{(i)}_K$ be the Thom class of $K \subset \Gamma_i$ $(i = 1, 2)$. If we put

$$\xi = \sum_{\tau_K \notin \mathcal{T}^{(1)}_{|V|-2}} a_K \tau^{(1)}_K, \tag{7.4}$$

then we have

$$f(\xi) \quad = \sum_{\tau_K \notin \mathcal{T}^{(1)}_{|V|-2}} a_K f(\tau^{(1)}_K) = \sum_{\tau_H \notin \mathcal{T}^{(2)}_{|V|-2}} b_H \tau^{(2)}_H.$$

Because there are no multiple edges in $\mathcal{T}^{(i)}_k$ for $k < |V|-2$ and $i = 1, 2$, it follows from Lemma 7.9 that if $a_K, a_{K'} \neq 0$ in (7.4) then $|Z(\xi)| < |Z(\tau^{(1)}_K)|$. Therefore, if h_1 is the highest zero-length (except $|V|-2$) in $H^2_T(\Gamma_1, \mathcal{A}_1)$ and if $|Z(\xi)| = h_1$, then $\xi = a_K \tau^{(1)}_K$ for some non-zero integer a_K and $\tau^{(1)}_K \in \mathcal{T}^{(1)}_{h_1}$. Then, because f preserves the zero-length, it is easy to check that the highest zero-length in $H^2_T(\Gamma_2, \mathcal{A}_2)$ coincides with h_1 and if $|Z(\xi')| = h_1$ then $\xi' = b_H \tau^{(2)}_H$ for some non-zero integer b_H and $\tau^{(2)}_H \in \mathcal{T}^{(2)}_{h_1}$. This implies that for any $\tau^{(1)}_K \in \mathcal{T}^{(1)}_{h_1}$ there are an element $\tau^{(2)}_H \in \mathcal{T}^{(2)}_h$ and a non-zero integer b such that

$$f(\tau^{(1)}_K) = b\tau^{(2)}_H.$$

Because f is an isomorphism and $\tau^{(1)}_K$, $\tau^{(2)}_H$ are generators, we also have that $b = \pm 1$, i.e., f maps $\mathcal{T}^{(1)}_{h_1}$ to $\mathcal{T}^{(2)}_{h_1}$ bijectively up to sign.

Take an element $\tau^{(1)}_K \in \mathcal{T}^{(1)}_{h_2}$ for the second largest zero-length (except $|V|-2$) h_2. Because $\mathcal{T}^{(1)}_{h_1}$ and $\mathcal{T}^{(2)}_{h_1}$ are preserved under f and f^{-1}, $f(\tau^{(1)}_K)$ does not have a term described by a linear combination of elements in $\mathcal{T}^{(2)}_{h_1}$ and $\mathcal{T}^{(2)}_{|V|-2}$. Therefore, with the method similar to that demonstrated as above, we also have that f maps $\mathcal{T}^{(1)}_{h_2}$ to $\mathcal{T}^{(2)}_{h_2}$ bijectively up to sign. By repeating this argument, we have that f preserves $\mathcal{T}^{(1)}_k$ to $\mathcal{T}^{(2)}_k$ up to sign for $k \neq 0$.

If there is an element in $\mathcal{T}^{(i)}_0$ $(i = 1, 2)$, then this is CASE 2 and we may put $\mathcal{T}^{(i)}_0 = \{\nu^{(i)}_1, \nu^{(i)}_2\}$ by Lemma 7.10. Because f preserves $\mathcal{T}^{(1)}_k$ to $\mathcal{T}^{(2)}_k$ up to sign for $k \neq 0$, we may also put

$$f(\nu^{(1)}_1) = a\nu^{(2)}_1 + b\nu^{(2)}_2;$$
$$f(\nu^{(1)}_2) = c\nu^{(2)}_1 + d\nu^{(2)}_2,$$

for some integers a, b, c, d such that $ad-bc = \pm 1$. It is enough to show that the equations $b = c = 0$ or $a = d = 0$ holds. Let $e_{1,j}^{(i)}, e_{2,j}^{(i)}$ be the Thom classes of two edges in a multiple edge $K_j \subset \Gamma_i$ ($i = 1, 2$ and $j = 1, \ldots, \ell$). Then, by Theorem 7.2 and the combinatorial structure of CASE 2, we my put

$$e_{1,j}^{(i)} = \nu_1^{(i)} \tau_{K_j}^{(i)};$$
$$e_{2,j}^{(i)} = \nu_2^{(i)} \tau_{K_j}^{(i)}.$$

Therefore, together with Lemma 7.7 and Corollary 7.8, we have:

$$f(e_{1,j}^{(1)}) = f(\nu_1^{(1)} \tau_{K_j}^{(1)}) = (a\nu_1^{(2)} + b\nu_2^{(2)})\epsilon_j \tau_{K_{\sigma(j)}}^{(2)} = \epsilon_j a e_{1,\sigma(j)}^{(2)} + \epsilon_j b e_{2,\sigma(j)}^{(2)} = \epsilon_j' e_{1,\sigma(j)}^{(2)} \text{ (or } \epsilon_j' e_{2,\sigma(j)}^{(2)});$$
$$f(e_{2,j}^{(1)}) = f(\nu_2^{(1)} \tau_{K_j}^{(1)}) = (c\nu_1^{(2)} + d\nu_2^{(2)})\epsilon_j \tau_{K_{\sigma(j)}}^{(2)} = \epsilon_j c e_{1,\sigma(j)}^{(2)} + \epsilon_j d e_{2,\sigma(j)}^{(2)} = \epsilon_j'' e_{2,\sigma(j)}^{(2)} \text{ (or } \epsilon_j'' e_{1,\sigma(j)}^{(2)}),$$

for some $\epsilon_j, \epsilon_j', \epsilon_j'' = \pm 1$ and the permutation $\sigma : [\ell] \to [\ell]$. Because $e_{1,\sigma(j)}^{(2)}$ and $e_{2,\sigma(j)}^{(2)}$ are linearly independent, we have that $b = c = 0$ or $a = d = 0$. This establishes the statement. □

By using Lemma 7.7 and 7.11, we have the following key fact:

PROPOSITION 7.12. *Let f be a weak $H^*(BT)$-algebra isomorphism $H_T^*(\Gamma_1, \mathcal{A}_1) \to H_T^*(\Gamma_2, \mathcal{A}_2)$. Then, f induces the combinatorial equivalence f^C between the simplicial posets $\mathcal{P}_*(\Gamma_1, \mathcal{A}_1)$ and $\mathcal{P}_*(\Gamma_2, \mathcal{A}_2)$.*

PROOF. Because of Lemma 7.7 and 7.11, f preserves all generators of $H_T^*(\Gamma_1, \mathcal{A}_1)$ and $H_T^*(\Gamma_2, \mathcal{A}_2)$ up to sign. This induces the bijective map f^C between $\mathcal{P}_*(\Gamma_1, \mathcal{A}_1)$ and $\mathcal{P}_*(\Gamma_2, \mathcal{A}_2)$. Let H be a 2-valent torus subgraph, e be an edge and p be a vertex in $\mathcal{P}_*(\Gamma_1, \mathcal{A}_1)$. We put H', e' and p' corresponding torus subgraphs in $\mathcal{P}_*(\Gamma_2, \mathcal{A}_2)$ by f^C, respectively. It is easy to show that $e \subset H$ if and only if $|Z(\tau_e \tau_H)| = |V(\Gamma_1)| - 2$, where $\tau_e \in H_T^4(\Gamma_1, \mathcal{A}_1)$ and $\tau_H \in H_T^2(\Gamma_1, \mathcal{A}_1)$ are the Thom classes of e and H, respectively. Therefore, because $f^C(e) = e'$ and $f^C(H) = H'$ and f preserves the zero-length, if $e \subset H$ then $f^C(e) \subset f^C(H)$. Similarly, we also have that $p \subset e$ then $f^C(p) \subset f^C(e)$. This implies that f^C preserves the orders of simplicial posets, i.e., combinatorial equivalence. □

Now we may prove Theorem 7.5:

PROOF OF THEOREM 7.5. Let $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ be torus graphs induced from simply connected equivariantly formal 6-dimensional torus manifolds, and let (Q_1, λ_1) and (Q_2, λ_2) be their induced 3-dimensional disks with face acyclic structures and characteristic functions respectively. Assume there is a weak $H^*(BT)$-algebra isomorphism $f : H_T^*(\Gamma_1, \mathcal{A}_1) \to H_T^*(\Gamma_2, \mathcal{A}_2)$ which is defined by the following commutative diagram:

$$\begin{array}{ccc} H^2(BT) & \xrightarrow{\pi_1^*} & H_T^2(\Gamma_1, \mathcal{A}_1) \\ \downarrow \rho^* & & \downarrow f \\ H^2(BT) & \xrightarrow{\pi_2^*} & H_T^2(\Gamma_2, \mathcal{A}_2) \end{array}$$

such that

$$\begin{array}{ccc} \alpha & \longrightarrow & \sum_{i=1}^m \langle \alpha, \lambda_i \rangle \tau_i^{(1)} \\ \downarrow \rho^* & & \downarrow f \\ \rho^*(\alpha) & \longrightarrow & \sum_{i=1}^m \langle \rho^*(\alpha), \lambda_i' \rangle \tau_i^{(2)} \end{array}$$

for some $\lambda_i, \lambda_i' \in H_2(BT)$, where ρ^* is the induced homomorphism from an isomorphism $\rho : T \to T$. If $|V(\Gamma_1)| = |V(\Gamma_2)| = 2$, then the statement of Theorem 7.5 holds because of Proposition 7.6. So, we may only prove for the case when $|V(\Gamma_1)|,\ |V(\Gamma_2)| > 2$. In this case, by Lemma 7.11, we have the following equation:

$$\begin{aligned}\sum_{i=1}^{m}\langle\rho^*(\alpha),\lambda_i'\rangle\tau_i^{(2)} &= \sum_{i=1}^{m}\langle\alpha,\lambda_i\rangle f(\tau_i^{(1)}) \\ &= \sum_{i=1}^{m}\langle\alpha,\lambda_i\rangle\epsilon_i\tau_{\sigma(i)}^{(2)} \\ &= \sum_{i=1}^{m}\langle\alpha,\epsilon_{\sigma(i)}\lambda_{\sigma(i)}\rangle\tau_i^{(2)},\end{aligned}$$

where $\sigma : [m] \to [m]$ is a permutation and $\epsilon_i = \pm 1$ for $i = 1, \ldots, m$. Therefore, we have

$$\rho_*(\lambda_i') = \epsilon_{\sigma(i)}\lambda_{\sigma(i)}, \tag{7.5}$$

for all $i = 1, \ldots, m$. On the other hand, by Proposition 7.12, f induces the combinatorial equivalence between Q_1 and Q_2; moreover, because Q_1 and Q_2 are 3-dimensional standard disks admitting face acyclic structures, f also induces the diffeomorphism $f_* : Q_1 \to Q_2$ in the sense of manifold with corners (also see [**Wi**, Lemma 6.2]). Moreover, it easily follows from the relations between the characteristic function and the axial function (also see Remark 7.3) that we have

$$\lambda_i = \lambda_1(H_i^{(1)}) \quad \text{and} \quad \lambda_i' = \lambda_2(H_i^{(2)}),$$

where $H_i^{(1)}$ (resp. $H_i^{(2)}$) is the facet in Q_1 (resp. Q_2) which corresponds with $\tau_i^{(1)}$ (resp. $\tau_i^{(2)}$). (Note that because the 2nd degree Thom class τ_K are defined by the 2-valent torus subgraphs K, there is the corresponding facet whose one-skeleton is K.) It follows from (7.5) that the following equation holds:

$$\rho_*(\lambda_2(H_i^{(2)})) = \epsilon_{\sigma(i)}\lambda_1(H_{\sigma(i)}^{(1)}).$$

Moreover, we can define the induced isomorphism $f_{\mathbb{Z}} : \mathbb{Z}\mathcal{F}(Q_1) \to \mathbb{Z}\mathcal{F}(Q_2)$ from the diffeomorphism $f_* : Q_1 \to Q_2$ by

$$f_{\mathbb{Z}}(H_i^{(1)}) = \epsilon_i H_{\sigma(i)}^{(2)}.$$

This implies that the following diagram is commutative:

$$\begin{array}{ccc}\mathbb{Z}\mathcal{F}(Q_1) & \xrightarrow{(\lambda_1)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}} \\ \big\downarrow{\scriptstyle f_{\mathbb{Z}}} & & \big\downarrow{\scriptstyle \rho_*^{-1}} \\ \mathbb{Z}\mathcal{F}(Q_2) & \xrightarrow{(\lambda_2)_{\mathbb{Z}}} & \mathfrak{t}_{\mathbb{Z}}\end{array}$$

where $(\lambda_i)_{\mathbb{Z}}$ is the induced homomorphism from the characteristic function λ_i ($i = 1, 2$). Now we may regard ρ_*^{-1} is induced from the automorphism $\rho^{-1} : T \to T$. Therefore, $(Q_1, \lambda_1) \cong_w (Q_2, \lambda_2)$. This establishes Theorem 7.5. □

Consequently, we have Theorem 7.1.

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