Cohomological rigidity of 6-dimensional quasitoric manifolds

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The 43rd Symposium on Transformation Groups
November 17, 2016
Himeji Civic Hall

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Introduction

Problem

Given two closed smooth manifolds M and M', when does an isomorphism $H^*(M;\mathbb{Z}) \cong H^*(M';\mathbb{Z})$ imply that M and M' are diffeomorphic?

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Let γ be the tautological line bundle over $\mathbb{C}P^1$, and let Σ_n be the total space of the projective bundle $P(\underline{\mathbb{C}} \oplus \gamma^{\otimes n})$ for $n \in \mathbb{Z}$. Then, Σ_n is a closed smooth manifold.

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Note that $H^*(\Sigma_n; \mathbb{Z}) \cong \mathbb{Z}[x, y]/\langle x^2, y(nx + y)\rangle$, and

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Let $\mathbf{k}$ be a commutative ring with unit.

Definition

A family of closed manifolds is *cohomologically rigid* over $\mathbf{k}$ if manifolds in the family are distinguished up to homeomorphism by their cohomology rings with coefficients in $\mathbf{k}$.

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Simple polytopes

Definition

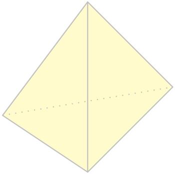
A *polytope* is a convex hull of finite points in $\mathbb{R}^n$.

e.g.) 2-dimensional polytope=polygon
3-dimensional polytope=polyhedron

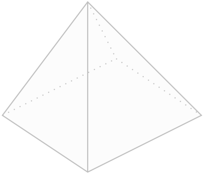
Definition

An n-dimensional polytope is *simple* if precisely n facets (codimension-1 faces) meet at each vertex.

simple



not simple



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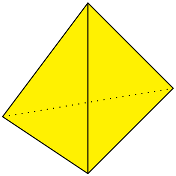

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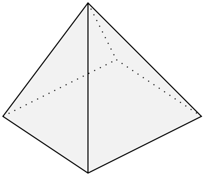

Quasitoric manifolds

The standard representation of T^n on $\mathbb{C}^n$ is

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n).$$

$\Rightarrow (0, \dots, 0)$ is a fixed point.
$\Rightarrow$ The orbit space $\mathbb{C}^n / T^n$ is a cone $\mathbb{R}^n_{\geq 0}$.

Davis-Januszkiewicz (1991)

A quasitoric manifold M is a closed smooth manifold of dimension $2n$ with a smooth action of T^n such that

1. the action of T^n is locally standard, and
2. there is a projection $\pi \colon M \to P$ such that the fibers of π are the T^n-orbits,

where P is a simple polytope of dimension n.

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Example and Non-example

Complex projective spaces

$$\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim,$$

where $(z_0, z_1, \ldots, z_n) \equiv (\lambda z_0, \lambda z_1, \ldots, \lambda z_n)$ for $\lambda \in \mathbb{C}^\times$.
Then $T^n \curvearrowright \mathbb{C}P^n$ as

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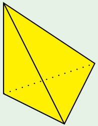

$\mathbb{C}P^3 / T^3$

Even dimensional spheres

Note that $S^{2n} = \{(z_1, \ldots, z_n, x) \in \mathbb{C}^n \oplus \mathbb{R} \mid |z_1|^2 + \cdots + |z_n|^2 + x^2 = 1\}$.
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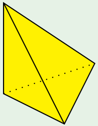

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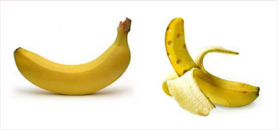

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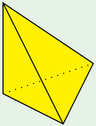

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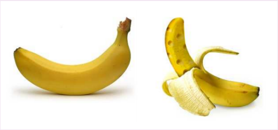

S^6/T^3

Construction

Let P be an n-dim'l simple polytope with facets $F_1, \dots, F_m$.
A function $\lambda\colon \{F_1, \dots, F_m\} \to \mathbb{Z}^n$ is a characteristic function on P if

$$\bigcap F_i : \text{ vertex } \implies \{\lambda(F_i)\} : \text{ a basis of } \mathbb{Z}^n.$$

For each $i = 1, \dots, m$, set $\lambda(F_i) = \boldsymbol{\lambda}_i$ and let T_i be the circle subgroup of T^n corresponding to $\boldsymbol{\lambda}_i$. For each point $x \in P$, define a torus $T(x) = \prod_{i\colon x \in F_i} T_i$.

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$$M(P, \lambda) = P \times T^n / \sim,$$

where $(x, t) \sim (x', t')$ whenever $x = x'$ and $t^{-1}t' \in T(x)$. Then $M(P, \lambda)$ is a quasitoric manifold over P.

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Equivalent quasitoric manifolds

Two quasitoric manifolds M and M' are *equivalent* if there exist a homeomorphism $f\colon M \to M'$ and an automorphism θ of T^n such that $f(t \cdot x) = \theta(t) \cdot f(x)$ for every $x \in M$ and every $t \in T^n$.

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Let Λ be an integer matrix whose ith column is $\boldsymbol{\lambda}_i$. Then $M(P, \lambda)$ and $M(P', \lambda')$ are equivalent if and only if

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Cohomology of quasitoric manifolds

The cohomology ring of a quasitoric manifold $M = M(P, \lambda)$ is

$$H^*(M(P,\lambda)) \cong \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_P + \mathcal{J}_\lambda,\ \deg(v_i) = 2,$$

where

$$\mathcal{I}_P = \langle v_{i_1} \cdots v_{i_k} \mid F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset \text{ in } P \rangle$$

and

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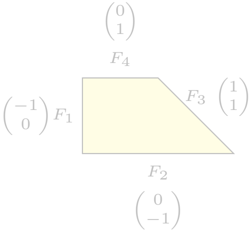

$$\begin{aligned} &H^*(M(P,\boldsymbol{\lambda})) \\ &\cong \mathbb{Z}[v_1, \ldots, v_4]/\langle v_1v_3, v_2v_4, v_1 - v_3, v_2 - v_3 - v_4 \rangle \\ &\cong \mathbb{Z}[v_3, v_4]/\langle v_3^2, v_4(v_3 + v_4) \rangle \end{aligned}$$

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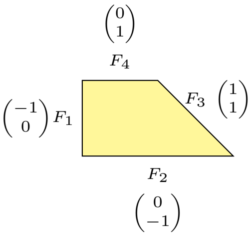

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Moment-angle manifold

Let P be an n-dim'l simple polytope with facets $F_1, \ldots, F_m$.
Let T_i be the 1-dim'l coordinate subgroup of T^m corresponding to F_i.
Then for each face $F = \cap_j F_j \neq \emptyset$ of P, we set $T_F = \prod_j T_j$.

Definition

The *moment-angle manifold* corresponding to P is

$$\mathcal{Z}_P = P \times T^m / \sim,$$

where $(x, t) \sim (x', t') \quad \Leftrightarrow \quad x = x' \ \& \ t^{-1}t' \in T_{F(x)}$. Here $F(x)$ is the face containing x in its interior.

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Example

$$\mathcal{Z}_{\Delta^n} = S^{2n+1} \text{ and } \mathcal{Z}_{\prod_{i=1}^k \Delta^{n_i}} = \prod_{i=1}^k S^{2n_i+1}.$$

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Relationship between $M(P,\lambda)$ and $\mathcal{Z}_P$

The matrix $\Lambda = \begin{pmatrix} \boldsymbol{\lambda}_1 & \cdots & \boldsymbol{\lambda}_m \end{pmatrix}$ corresponding to λ induces a surjective homomorphism $\overline{\lambda} : T^m \to T^n$.

$\Longrightarrow \ker(\overline{\lambda})$ is an $(m-n)$-dimensional subtorus of T^m.

Theorem [Davis-Januszkiewicz]

The subtorus $\ker(\overline{\lambda})$ acts freely on $\mathcal{Z}_P$, thereby defining a principal T^{m-n}-bundle $\mathcal{Z}_P \to M(P,\lambda)$.

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The following matrix defines a characteristic function on the standard simplex Δ^n

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}_{n \times (n+1)}.$$

Then $\ker(\overline{\lambda}) = \{(t, t, \ldots, t)\} \subset T^{n+1}$ and $S^{2n+1}/\ker(\overline{\lambda}) = \mathbb{C}P^n$.

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$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}_{n\times(n+1)}.$$

Then $\ker(\overline{\lambda}) = \{(t,t,\dots,t)\} \subset T^{n+1}$ and $S^{2n+1}/\ker(\overline{\lambda}) = \mathbb{C}P^n$.

Cohomology of moment-angle manifolds

Recall that $H^*(M(P,\lambda)) = \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_P + \mathcal{J}_\lambda$.
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Theorem [Buchstaber-Panov]

1. There are isomorphisms of (multi)graded commutative algebras

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Rigidity problems

In 2006, Masuda and Suh introduced the following problem.

Cohomological rigidity problems for quasitoric manifolds

If two quasitoric manifolds M and M' have the same cohomology ring with integral coefficients, are they homeomorphic? In other words, is the family of quasitoric manifolds cohomologically rigid?

This problem is still OPEN. There is no counter example, but there are many results which support the affirmative answer.

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Known results

1. Quasitoric manifolds of $\dim_{\mathbb{R}} \leq 4$ [Orlik-Raymond (1970)]
2. $\prod_{i=1}^{m} \mathbb{C}P^{n_i}$ [Masuda-Panov (2008), Choi-Masuda-Suh (2010)]
3. Projective smooth toric varieties with second Betti number 2 [Choi-Masuda-Suh (2010)]
4. Quasitoric manifolds with second Betti number 2 [Choi-P-Suh (2012)]
5. Quasitoric manifolds over the cube I^3 and dual cyclic polytopes [Hasui (2015)]
6. Projective bundles over smooth compact toric surfaces [Choi-P (2016)]

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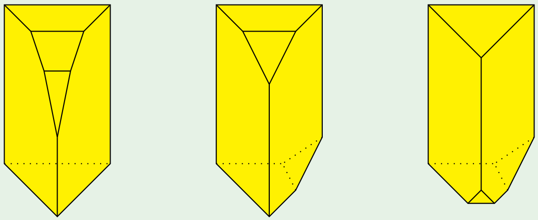
Example

The orbit space of $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ is a three times vertex-cut of Δ^3.



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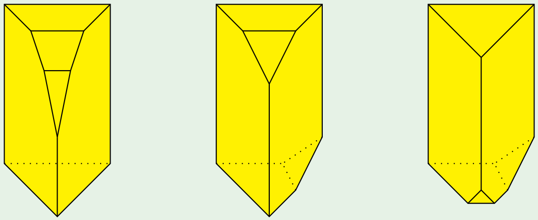

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Rigidity problems for polytopes

Definition [Masuda-Suh]

A simple polytope P is said to be *C-rigid* if it satisfies any of the following

- there is no quasitoric mnifold whose orbit space is P; or
- there exists a quasitoric manifold whose orbit space is P, and whenever there exists a quasitoric manifold N over another polytope Q with a graded ring isomorphism $H^*(M) \cong H^*(N)$, there is combinatorial equivalence $P \approx Q$.

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Pogorelov class

Definition

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Over the Pogorelov class

Theorem [Fan-Ma-Wang]

Flag 3-polytopes without 4-belts are B-rigid.

Note that

- Every simple polytope of dimension 3 admits a characteristic function by the Four Color Theorem. $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \sum_{i=1}^{3} \mathbf{e}_i)$

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Corollary

The polytopes in $\mathcal{P}$ are C-rigid.

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Cohomological rigidity

Lemma [Fan-Ma-Wang]

Consider the cohomology classes

$$\mathcal{T}(P) = \{\pm[u_i v_j] \in H^3(\mathcal{Z}_P) \mid F_i \cap F_j = \emptyset\}.$$

If $P \in \mathcal{P}$, then for any cohomology ring isomorphism $\psi\colon H^*(\mathcal{Z}_P) \to H^*(\mathcal{Z}_{P'})$, we have $\psi(\mathcal{T}(P)) = \mathcal{T}(P')$.

Lemma

Consider the set of cohomology classes

$$\mathcal{D}(M) = \{\pm[v_i] \in H^2(M) \mid i = 1, \dots, m\}.$$

If $P \in \mathcal{P}$ and M' is a quasitoric manifold over P', ten for any cohomology ring isomorphism $\varphi\colon H^*(M) \to H^*(M')$ we have $\varphi(\mathcal{D}(M)) = \mathcal{D}(M')$. Moreover, the set $\mathcal{D}(M)$ is mapped bijectively under φ.

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Let $M = M(P, \lambda)$ and $M' = M(P', \lambda')$. Assume that P belongs to the Pogorelov class $\mathcal{P}$. Then the following are equivalent.

1. $H^*(M)$ and $H^*(M')$ are isomorphic;
2. M and M' are diffeomorphic; and
3. M and M' are equivalent.

Remark

Let Σ_n and Σ_m be Hirzebruch surfaces. Then

- Σ_n and Σ_m are diffeomorphic if and only if $n \equiv m \mod 2$, and
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Definition

A *small cover* M is a closed smooth manifold of dimension n with a locally standard action of $\mathbb{Z}_2^n$ such that there is a projection $\pi\colon M \to P$ such that the fibers of π are the $\mathbb{Z}_2^n$-orbits.

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Let N and N' be small covers of P and P', respectively. Assume that P belongs to the Pogorelov class $\mathcal{P}$. Then the following are equivalent.

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Cohomological rigidity for 6-dim'l quasitoric manifolds

Theorem [Wall, Jupp]

Let $\varphi\colon H^*(N) \to H^*(N')$ be an isomorphism of the cohomology rings of smooth closed simply connected 6-dimensional manifolds N and N' with $H^3(N) = H^3(N') = 0$. Suppose that

1. $\varphi(w_2(N)) = w_2(N')$, where $w_2(N) \in H^2(N;\mathbb{Z}_2)$ is the second Stiefel-Whiteny class; and
2. $\varphi(p_1(N)) = \varphi(p_1(N'))$, where $p_1(N) \in H^4(N)$ is the first Pontryagin class.

Then the manifolds N and N' are diffeomorphic.

Cohomological rigidity for 6-dim'l quasitoric manifolds

Lemma [Choi-Masuda-Suh]

Suppose that the ring $H^*(N;\mathbb{Z}_2)$ is generated by $H^k(N;\mathbb{Z}_2)$ for some $k > 0$. Then any cohomology ring isomorphism $\varphi\colon H^*(N;\mathbb{Z}_2) \to H^*(N';\mathbb{Z}_2))$ preserves the total Stiefel-Whitney class, i.e., $\varphi(w(N)) = w(N')$.

Corollary

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Thank you very much!