ON THE EQUIVALENCE OF SEVERAL DEFINITIONS OF COMPACT INFRA-SOLVMANIFOLDS

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Abstract. We show the equivalence of several definitions of compact infra-solvmanifolds that appear in various math literatures.

Infra-solvmanifolds are a speical class of aspherical manifolds studied by mathematicians for a long time. By the literal meaning, an infra-solvmanifold is a smooth manifold finitely covered by a solvmanifold (which is the quotient of a connected solvable Lie group by a closed subgroup). Infra-solvmanifolds include all flat Riemannian manifolds and infra-nilmanifolds.

It is known that compact Infra-solvmanifolds are *smoothly rigid*, i.e. any homotopy equivalence between two compact infra-solvmanifolds is homotopic to a diffeomorphism (see [3] and [8]). Geometrically, a compact infra-solvmanifold M can be described (see [7, Proposition 3.1]) as a manifold which admits a sequence of Riemannian metric $\{g_n\}$ with uniformly bounded sectional curvature so that (M, g_n) collapses in the *Gromov-Hausdorff* sense to a *flat orbifold*.

In the history of study on this topic, several definitions of compact infra-solvmanifolds were proposed in various math literatures. This is mainly because people look at these manifolds from very different angles (topologically, algebraically or geometrically). The following are five such definitions.

Def 1: Let G be a connected, simply connected solvable Lie group, K be a maximal compact subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G, and Γ be a discrete subgroup of $G \rtimes K$. If $[\Gamma : G \cap \Gamma] < \infty$ and the action of Γ on G is cocompact and free, the orbit space $\Gamma\backslash G$ is called an *infra-solvmanifold modeled on G*. See [7, Definition 1.1].

Def 2: A *compact infra-solvmanifold* is a manifold of the form $\Gamma\backslash G$, where G is a connected, simply connected solvable Lie group, and Γ is a torsion-free discrete subgroup of $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ which acts on G cocompactly

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and the closure of $hol(\Gamma)$ in $\mathrm{Aut}(G)$ is compact where $hol : \mathrm{Aff}(G) \to \mathrm{Aut}(G)$ is the holonomy projection.

Def 3: A *compact infra-solvmanifold* is a double coset space $\Gamma\backslash G/K$ where G is a virtually connected and virtually solvable Lie group, K is a maximal compact subgroup of G and Γ is a torsion-free, cocompact, discrete subgroup of G. See [2, Definition 2.10].

A *virtually connected* Lie group is a Lie group with finitely many connected components.

Def 4: A *compact infra-solvmanifold* M is an orbit space of the form $M = \Delta\backslash S$ where S is a connected, simply connected solvable Lie group acted upon cocompactly by a torsion-free closed subgroup $\Delta \subset \mathrm{Aff}(S)$ satisfying

- the identity component Δ_0 of Δ is contained in the nil-radical of S,
- the closure of $hol(\Delta)$ in $\mathrm{Aut}(S)$ is compact.

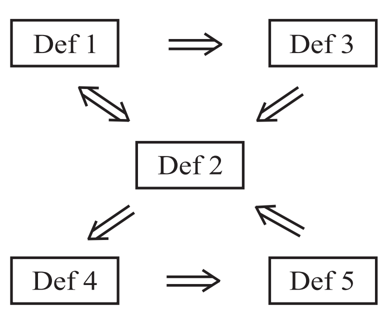
Def 5: A *compact infra-solvmanifold* is a manifold of the form $\Delta\backslash G$, where G is a connected, simply connected solvable Lie group, and Δ is a torsion-free subgroup of $\mathrm{Aff}(G)$ which acts cocompactly on G and the closure of $hol(\Delta)$ in $\mathrm{Aut}(G)$ is compact.

Note that the subgroup Δ in Def 4 and Def 5 is not necessarily discrete. The Def 4 is from [3, Definition 1.1] and Def 5 is from [1, Definition 1.1].

The main purpose of this paper is to explain why the above five definitions of compact infra-solvmanifolds are equivalent. The authors are fully aware that the reason is probably known to many people. However, we do not find any formal proof of this equivalence and feel this phenomenon a little confusing. So we want to write a proof here for the convenience of future reference. However, we do not intend to give a completely new treatment of this subject. Our proof will directly quote some results on infra-solvmanifolds from [1], [3] and [8] and use many subtle facts in Lie theory. This paper can be treated as an elementary exposition of compact infra-solvmanifolds and some related concepts. Now let us start to prove the following main result of the paper.

Proposition 1: The five definitions Def 1 – Def 5 of compact infra-solvmanifolds are all equivalent.

Proof. We divide our proof into several parts. The framework of the proof is shown below.



Def 1 $\iff$ Def 2

Let M be a compact infra-solvmanifold in the sense of Def 1. First of all, any $g \in \Gamma$ can be decomposed as $g = k_g u_g$ where $k_g \in K \subset \mathrm{Aut}(G)$ and $u_g \in G$. The holonomy projection $hol : G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ sends g to k_g. Since hol is a group homomorphism, its image $hol(\Gamma)$ is a subgroup of K. By assumption, $|hol(\Gamma)| = [\Gamma : G \cap \Gamma]$ is finite, so $hol(\Gamma)$ is compact. In addition, since G is a connected simply-connected solvable Lie group, G is diffeomorphic to an Euclidean space. Then by Smith fixed point theorem ([6, Theorem I]), Γ acting on G freely implies that Γ is torsion-free. So M satisfies Def 2.

Conversely, if M is a compact infra-solvmanifold in Def 2, then [8, Theorem 3 (a) $\Rightarrow$ (f)] tells us that there exists a connected, simply-connected solvable Lie group G' so that the Γ (which defines M) can be thought of as discrete subgroup of $G' \rtimes F$ where F is a finite subgroup of $\mathrm{Aut}(G')$. Moreover, there exists an equivariant diffeomorphism from G to G' with respect to the action of Γ (by [8, Theorem 1 and Theorem 2]). Hence

$$M = \Gamma \backslash G \underset{\text{diff.}}{\cong} \Gamma \backslash G'.$$

Then $[\Gamma : \Gamma \cap G'] = |hol_{G'}(\Gamma)| \le |F|$ is finite.

It remains to show that the action of Γ on G' is free. Since F is finite, we can choose an F-invariant Riemannian metric on G' (in fact we only need F to be compact). If the action of an element $g \in \Gamma$ on G' has a fixed point, say h_0. Let L_{h_0} be the left translation of G' by h_0. Then $L_{h_0}^{-1} g L_{h_0}$ fixes the identity element, which implies $L_{h_0}^{-1} g L_{h_0} \in F$. So there exists a compact neighborhood U of h_0 so that $g \cdot U = U$ and g acts isometrically on U with respect to the Riemannian metric just as the element $L_{h_0}^{-1} g L_{h_0} \in F$ acts around e. So $A = Dg : T_{h_0} U \to T_{h_0} U$ is an orthogonal transformation. Then A is conjugate in $O(n)$ to the block diagonal matrix of the form

$$\begin{pmatrix} B(\theta_1) & & 0 \\ & \ddots & \\ 0 & & B(\theta_m) \end{pmatrix},$$

where $B(0) = 1$, $B(\pi) = -1$, and $B(\theta_j) = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}$, $1 \le j \le m$.

Since Γ is torsion-free, g is an infinite order element. This implies that at least one θ_j is irrational. Then for any $v \neq 0 \in T_{h_0} G'$, $|\{A^n v\}_{n \in \mathbb{Z}}| = \infty$. So there exists $h \in U$ so that $|\{g^n \cdot h\}_{n \in \mathbb{Z}}| = \infty$. Then the set $\{g^n \cdot h\}_{n \in \mathbb{Z}} \subset U$ has at least one accumulation point. This contradicts the fact that the orbit space $\Gamma \backslash G'$ is Hausdorff (since $\Gamma \backslash G' = M$ is a manifold).

Combing the above arguments, M is an infra-solvmanifold modeled on G' in the sense of Def 1. □

Def 1 $\Longrightarrow$ Def 3

Let M be an infra-solvmanifold in the sense of Def 1. Let $hol(\Gamma)$ be the image of the the holonomy projection $hol : \Gamma \to \mathrm{Aut}(G)$. Then $hol(\Gamma)$ is a finite subgroup of $\mathrm{Aut}(G)$.

Define $\widetilde{G} = G \rtimes hol(\Gamma)$ which is virtually solvable. It is easy to see that $hol(\Gamma)$ is a maximal compact subgroup of $\widetilde{G}$ and, Γ is a cocompact, discrete subgroup of $\widetilde{G}$. Then $M \cong \Gamma\backslash\widetilde{G}/hol(\Gamma)$. So M satisfies Def 3. □

Def 3 $\Longrightarrow$ Def 2

Let $M = \Gamma\backslash G/K$ be a compact infra-solvmanifold in the sense of Def 3. Since K is a maximal compact subgroup of G, G/K is contractible. Note that K is not necessarily a normal subgroup of G. Hence G/K may not directly inherit a group structure from G.

Let G_0 be the connected component of G containing the identity element. Then by [4, Theorem 14.1.3 (ii)], $K_0 = K \cap G_0$ is connected and K_0 is a maximal compact subgroup of G_0. Moreover, K intersects each connected component of G and $K/K_0 \cong G/G_0$. By the classical Lie theory, the Lie algebra of a compact Lie group is a direct product of an abelian Lie algebra and some simple Lie algebras. Then since the Lie algebra $\mathrm{Lie}(G)$ of G is solvable and K is compact, the Lie algebra $\mathrm{Lie}(K) \subset \mathrm{Lie}(G)$ must be abelian. This implies that K_0 is a torus and hence a maximal torus in G_0.

In addition, since G is virtually solvable, G_0 is actually solvable. This is because the radical R of G is a normal subgroup of G_0 and $\dim(R) = \dim(G_0)$ (since G is virtually solvable). So G_0/R is discrete. Then since G_0 is connected, G_0 must be equal to R.

Let $Z(G)$ be the center of G and define $C = Z(G) \cap K$. Then C is clearly a normal subgroup of G. Let $G' = G/C$ and $K' = K/C$ and let $\rho : G \to G'$ be the quotient map. Since $\Gamma \cap K = \{1\}$, $\Gamma \cong \rho(\Gamma) \subset G'$, we can think of Γ as a subgroup of G'. So we have

$$M = \Gamma\backslash G/K \cong \Gamma\backslash G'/K'. \tag{1}$$

Let $G'_0 = \rho(G_0)$ be the identity component of G'. Then G'_0 is a finite index solvable normal subgroup of G'.

$$\mathrm{Lie}(G'_0) = \mathrm{Lie}(G') = \mathrm{Lie}(G)/\mathrm{Lie}(C) = \mathrm{Lie}(G_0)/\mathrm{Lie}(C). \tag{2}$$

Besides, let $K'_0 = K' \cap G'_0$ which is a maximal torus of G'_0. We have

$$\mathrm{Lie}(K'_0) = \mathrm{Lie}(K') = \mathrm{Lie}(K)/\mathrm{Lie}(C) = \mathrm{Lie}(K_0)/\mathrm{Lie}(C). \tag{3}$$

Claim-1: G'_0 is linear and so G' is linear.

A group is called *linear* if it admits a faithful finite-dimensional representation. By [4, Theorem 16.2.9 (b)], a connected solvable Lie group S is linear if and only if $\mathfrak{t} \cap [\mathfrak{s}, \mathfrak{s}] = \{0\}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are Lie algebras of S and its maximal torus T_S, respectively. And for a general connected solvable group S, the Lie subalgebra $\mathfrak{t} \cap [\mathfrak{s}, \mathfrak{s}]$ is always central in $\mathfrak{s}$. So for our G_0 and its maximal torus K_0, the Lie subalgebra $\mathrm{Lie}(K_0) \cap [\mathrm{Lie}(G_0), \mathrm{Lie}(G_0)]$ is central in $\mathrm{Lie}(G_0) = \mathrm{Lie}(G)$. So $\mathrm{Lie}(K_0) \cap [\mathrm{Lie}(G_0), \mathrm{Lie}(G_0)] \subset \mathrm{Lie}(Z(G))$ and

$$\mathrm{Lie}(K_0) \cap [\mathrm{Lie}(G_0), \mathrm{Lie}(G_0)] \subset \mathrm{Lie}(K) \cap \mathrm{Lie}(Z(G)) = \mathrm{Lie}(C) \tag{4}$$

Then for the Lie group G_0' and its maximal torus K_0', we have

$$\mathrm{Lie}(K_0') \cap [\mathrm{Lie}(G_0'), \mathrm{Lie}(G_0')] = \frac{\mathrm{Lie}(K_0)}{\mathrm{Lie}(C)} \cap \left[\frac{\mathrm{Lie}(G_0)}{\mathrm{Lie}(C)}, \frac{\mathrm{Lie}(G_0)}{\mathrm{Lie}(C)} \right] = 0. \tag{5}$$

So by [4, Theorem 16.2.9 (b)], G_0' is linear. Moreover, suppose V is a faithful finite-dimensional representation of G_0'. Then $\mathbb{R}[G'] \otimes_{\mathbb{R}[G_0']} V$ is a faithful finite-dimensional representation of G' where $\mathbb{R}[G']$ and $\mathbb{R}[G_0']$ are the group rings of G' and G_0' over $\mathbb{R}$, respectively. So the Claim-1 is proved.

From the Claim-1 and (1), we can just assume that our group G is linear at the beginning. Under this assumption, G_0 is a connected linear solvable group. So there exists a simply connected solvable normal Lie subgroup S of G_0 so that $G_0 = S \rtimes K_0$ and $[G_0, G_0] \subset S$ (see [4, Lemma 16.2.3]). So

$$\mathrm{Lie}(G_0) = \mathrm{Lie}(K_0) \oplus \mathrm{Lie}(S).$$

More specifically, we can take $S = p^{-1}(V)$ where $p : G_0 \to G_0/[G_0, G_0]$ is the quotient map and V is a vector subgroup of the abelian group $G_0/[G_0, G_0]$ so that $G_0/[G_0, G_0] \cong p(K_0) \times V$ (see the proof of [4, Theorem 16.2.3]). Note that the vector subgroup V is not unique, nor is S.

Claim-2: We can choose S to be normal in G and so $G \cong S \rtimes K$.

Indeed since K is compact, we can choose a metric on $\mathrm{Lie}(G_0)$ which is invariant under the adjoint action of K. Then we can choose V so that $\mathrm{Lie}(S)$ is orthogonal to $\mathrm{Lie}(K_0)$ in $\mathrm{Lie}(G_0)$. Then because K_0 is normal in K, the adjoint action of K on $\mathrm{Lie}(G_0)$ preserves $\mathrm{Lie}(K_0)$, so it also preserves the orthogonal complement $\mathrm{Lie}(S)$ of $\mathrm{Lie}(K_0)$. This implies that S is preserved under the adjoint action of K.

Let G_0, $h_1 G_0, \cdots, h_m G_0$ be all the connected components of G. Since K intersects each connected component of G, we can assume that $h_i \in K$ for all $1 \leq i \leq m$. Then any element $g \in G$ can be written as $g = g_0 h_i$ for some $g_0 \in G_0$ and $h_i \in K$. So $gSg^{-1} = g_0 h_i S h_i^{-1} g_0^{-1} \subset g_0 S g_0^{-1} \subset S$. The Claim-2 is proved.

From the semidirect product $G = S \rtimes K$, we can define an injective group homomorphism $\alpha : G \to \mathrm{Aff}(S) = S \rtimes \mathrm{Aut}(S)$ as follows. For any $g \in G$, we can write $g = s_g k_g$ for a unique $s_g \in S$ and $k_g \in K$ since $S \cap K = S \cap K_0 = \{1\}$. Then $\alpha(g) : S \to S$ is the composition of the adjoint action of k_g on S and the left translation on S by s_g, i.e. $\alpha(g) = L_{s_g} \circ \mathrm{Ad}_{k_g}$.

Claim-3: $\alpha(\Gamma)\backslash S$ is diffeomorphic to the double coset space $\Gamma\backslash G/K$.

Notice that each left coset in G/K contains a unique element of S, so we have

$$G/K = \{sK \,;\, s \in S\}.$$

For any $\gamma \in \Gamma$, let $\gamma = s_\gamma k_\gamma$ where $s_\gamma \in S$ and $k_\gamma \in K$. We have

$$\gamma s K = s_\gamma k_\gamma s K = s_\gamma k_\gamma s k_\gamma^{-1} K = \alpha(\gamma)(s) K, \ \forall \, s \in S. \tag{6}$$

So the natural action of Γ on the left coset space G/K can be identified with the action of $\alpha(\Gamma) \subset \mathrm{Aff}(S)$ on S. The Claim-3 is proved.

Let $\mathrm{Ad} : K \to \mathrm{Aut}(S)$ denote the adjoint action of K on S. Since K is compact and Ad is continuous, $\mathrm{Ad}(K) \subset \mathrm{Aut}(S)$ is also compact. Notice that $\alpha(\Gamma)$ is a subgroup of $S \rtimes \mathrm{Ad}(K) \subset S \rtimes \mathrm{Aut}(S)$. So the closure $\overline{hol(\alpha(\Gamma))}$ of the holonomy group $hol(\alpha(\Gamma))$ in $\mathrm{Aut}(S)$ is contained in $\mathrm{Ad}(K)$. So $\overline{hol(\alpha(\Gamma))}$ is compact. This implies that $\Gamma\backslash G/K \cong \alpha(\Gamma)\backslash S$ is an infra-solvmanifold in the sense of Def 2. □

Remark 1: A simply-connected solvable Lie group is always linear, but for non-simply-connected solvable Lie groups, this is not always so. A counterexample is the quotient group of the Heisenberg group by an infinite cyclic group (see [5, p.169 Example 5.67]).

It is obviously true that Def 2 $\Longrightarrow$ Def 4 $\Longrightarrow$ Def 5. So it remains to show Def 5 $\Longrightarrow$ Def 2.

Def 5 $\Longrightarrow$ Def 2

This follows from a nontrivial result in [1]. Indeed, the fundamental group of the manifold $\Delta\backslash G$ in Def 5 is $\Gamma = \Delta/\Delta_0$, which is a torision-free virtually poly-cyclic group (see [2]). It is shown in [1] that such a group Γ determines a virtually solvable real linear algebraic group H_Γ which contains Γ as a discrete and Zariski-dense subgroup. H_Γ is called the *real algebraic hull* of Γ in [1]. In addition, we have $H_\Gamma = U \rtimes T$ where T is a maximal reductive subgroup of H_Γ and U is the unipotent radical of H_Γ. The splitting of H_Γ gives an injective group homomorphism $\alpha : H_\Gamma \to \mathrm{Aff}(U)$ and a corresponding affine action of $\Gamma < H_\Gamma$ on U. It is shown in [1] that $M_\Gamma = \alpha(\Gamma)\backslash U$ is a compact infra-solvmanifold whose fundamental group is Γ. M_Γ is called the *standard* Γ*-manifold*. Now since $\alpha(\Gamma)$ is a discrete subgroup of $\mathrm{Aff}(U)$, M_Γ is a compact infra-solvmanifold in the sense of Def 2. Moreover $\Delta\backslash G$ is diffeomorphic to M_Γ by [1, Theorem 1.4].

So we finish the proof of Proposition 1. □

Remark 2: If we remove the "cocompact" condition in Def 1 − Def 5, we may get noncompact infra-solvmanifolds, which are vector bundles over some compact infra-solvmanifolds (see [8, Theorem 6]).

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