Topological classification of some torus manifolds

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**Torus manifold** (Hattori-Masuda, 2003) $\overset{\text{def}}{\iff}$ a compact $2n$-dimensional manifold $M^{2n}$ with effective $T^n$-action and $M^T \neq \emptyset$.

**Example 1** (1) *Complex projective space* $\mathbb{C}P(n)$ *with the standard* $T^n$-*action,* $\left( S^{2n+1} \subset \mathbb{C}^{n+1}, \mathbb{C}P(n) \cong S^{2n+1}/S^1 \right)$.

(2) *Even dimensional sphere* $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ *with the* $T^n$-*action on the* $\mathbb{C}^n$-*factor.*
Motivation and Problem

2006 Masuda-Suh: Cohomological rigidity problem.
Let $M$ and $M'$ be (quasi)toric manifolds.

Suppose $H^*(M) \simeq H^*(M') \Rightarrow$ Are $M$ and $M'$ homeomorphic?

Let $M$ be a toric manifold.

Suppose $H^*(M) \simeq H^*(\prod CP(n_i)) \Rightarrow M \cong \prod CP(n_i)$.

In general $\Rightarrow$ Open problem.

Problem: How about torus manifolds?
Counter examples for torus manifolds I

2008 Kuroki: Classification of torus manifolds which have codimension one extended $G$-actions.

As a result, essentially, we have the following three types of fibre bundles:

$$\begin{align*}
CP(k_1 + k_2 - 1) & \hookrightarrow S^{2\ell+1} \times S^1 \text{ } P(C^{k_1}_\rho \oplus C^{k_2}) \rightarrow S^{2\ell+1}/S^1 \cong CP(\ell) \\
S^{2k} & \hookrightarrow S^{2\ell+1} \times S^1 \text{ } S(C^{k}_\rho \oplus R) \rightarrow S^{2\ell+1}/S^1 \cong CP(\ell) \\
S^{2k_1+2k_2} & \hookrightarrow S^{2\ell+1} \times S^1 \text{ } S(C^{k_1}_\rho \oplus R^{k_2+1}) \rightarrow S^{2\ell+1}/S^1 \cong CP(\ell)
\end{align*}$$

where we can identify the representation $\rho : S^1 \rightarrow S^1$ as an integer $\rho \in \mathbb{Z}$ by $t \mapsto t^\rho$.

There are counter examples in the above class.
Example 2 Assume $k \geq 2$ and $\ell = 1$. Consider the following manifolds:

$$S^{2k} \rightarrow M^{2k+2}(\rho) = S^3 \times S^1 S(C^k_\rho \oplus \mathbb{R}) \rightarrow S^2.$$ 

This manifold has the following topological invariants:

$$H^*(M^{2k+2}(\rho)) \cong H^*(S^{2k} \times S^2);$$
$$w(M^{2k+2}(\rho)) \equiv 1 + k\rho x \pmod{2};$$
$$p(M^{2k+2}(\rho)) = 1,$$

where $x \in H^2(M^{2k+2}(\rho))$. Therefore, our manifolds do not satisfy cohomological rigidity, by comparing $k\rho \equiv 0$ and $1$ (fix an odd $k$).

**Problem:** What is a complete invariant of topological types?
Main theorem

Let $M_1, M_2$ be two of the following manifolds $(k_1, k_2, k > 0)$:

\[
\begin{align*}
P(\ell, k_1, k_2) &= S^{2\ell+1} \times_{S^1} P(C_{\rho}^{k_1} \oplus C^{k_2}); \\
S(\ell, k) &= S^{2\ell+1} \times_{S^1} S(C_{\rho}^{k} \oplus \mathbb{R}); \\
S(\ell, k_1, k_2) &= S^{2\ell+1} \times_{S^1} S(C_{\rho}^{k_1} \oplus \mathbb{R}^{k_2+1}).
\end{align*}
\]

The following three statements hold.

**Theorem 1 (Cohomology)** If $\exists f : H^*(M_1) \xrightarrow{\sim} H^*(M_2)$ then $M_1 \cong M_2 \iff M_1$ and $M_2$ are two of the following manifolds:

<table>
<thead>
<tr>
<th>$P(\ell, k_1, k_2)$</th>
<th>$\forall \ell, k_1, k_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(\ell, k)$</td>
<td>$\ell &gt; 1$ and $k \leq \ell$</td>
</tr>
<tr>
<td>$S(1, k)$</td>
<td>$k \equiv_2 0$</td>
</tr>
<tr>
<td>$S(1, k_1, k_2)$</td>
<td>$k_1 \equiv_2 0$</td>
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</table>
Theorem 2 (Pontrjagin class) If \( \exists f : H^*(M_1) \xrightarrow{\sim} H^*(M_2) \) such that \( f(p(M_1)) = p(M_2) \) then \( M_1 \cong M_2 \) iff \( M_1 \) and \( M_2 \) are two of the the following manifolds:

<table>
<thead>
<tr>
<th>( S(\ell, k) )</th>
<th>( k &gt; \ell &gt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(\ell, k_1, k_2) )</td>
<td>( \ell &gt; 1 )</td>
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</table>

Theorem 3 (Stiefel-Whitney class) If \( \exists f : H^*(M_1) \xrightarrow{\sim} H^*(M_2) \) such that \( f(w(M_1)) = w(M_2) \) then \( M_1 \cong M_2 \) iff \( M_1 \) and \( M_2 \) are two of the the following manifolds:

<table>
<thead>
<tr>
<th>( S(1, k) )</th>
<th>( k \equiv_2 1 \ (k &gt; 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(1, k_1, k_2) )</td>
<td>( k_1 \equiv_2 1 )</td>
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Outline of proof

(1) Compute the topological invariants: $H^*(M)$, $p(M)$ and $w(M)$.

(2) Divide the following two cases by $H^*(M) \simeq \mathbb{Z}[x, y]/ < x^{\ell+1}, f(x, y) > $ (deg $x = 2$, deg $y = 2$, $2k$ or $2k_1 + 2k_2$):

1. deg $y = 2 \Rightarrow M = P(\ell, k_1, k_2) \Rightarrow$ By Choi-Mausda-Suh, this case satisfies the cohomological rigidity (Theorem 1).

2. deg $y > 2 \Rightarrow$ Divide into the three cases
   (i) $\ell \geq 4 \Rightarrow$ Analyzing $f : H^*(M_1) \simeq H^*(M_2)$, we have Theorem 1, 2.
   (ii) $\ell = 2, 3 \Rightarrow$ Analyzing $f : H^*(M_1) \simeq H^*(M_2)$ and $\widetilde{KO}(CP(\ell))$, we have Theorem 1, 2.
   (iii) $\ell = 1 \Rightarrow$ Analyzing $f : H^*(M_1) \simeq H^*(M_2)$ and using $\pi_2(BO(m)) \simeq \mathbb{Z}_2$ for $m > 2$, we have Theorem 1, 3.
Observation (non-trivial examples)

From our proof, a **homeomorphic type of** $M$ is determined by $\ell, k$ (or $k_1, k_2$) and $|\rho|$, except $\ell = 1, 2, 3$ cases. In fact there are the following example.

**Example 3** Homeomorphism types of $\ell = 1$ and $k > 1$ ($k_1 > 1$) are determined by $\rho \mod 2$. The following two manifolds are homeomorphic:

$$S(1, 3) = S^3 \times S^1 S(C_2^3 \oplus R); \quad S(1, 2, 1) = S^3 \times S^1 S(C_4^2 \oplus R^3).$$

Homeomorphism types of $\ell = 2, 3$ are determined by $k\rho^2$ ($k_1\rho^2$). The following two manifolds are homeomorphic:

$$S(2, 1, 4) = S^5 \times S^1 S(C_2 \oplus R^9); \quad S(2, 4, 1) = S^5 \times S^1 S(C_1^4 \oplus R^3).$$