PROJECTIVE BUNDLES OVER SMALL COVERS AND THE BUNDLE TRIVIALITY PROBLEM

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Abstract. This paper investigates the projective bundles over small covers. We first give a necessary and sufficient condition for the projectivization of a real vector bundle over a small cover to be a small cover. Then associated with moment-angle manifolds, we further study the structure of such a projectivization as a small cover by introducing a new characteristic function on simple convex polytopes. As an application, we characterize the real projective bundles over 2-dimensional small covers by interpreting the fibre sum operation to some combinatorial operation. We next study when the projectivization of Whitney sum of the tautological line bundle and the tangent bundle over real projective space is diffeomorphic to the product of two real projective spaces.

1. Introduction

A real projectivization $P(\xi)$ over a closed manifold M is defined by a vector bundle ξ over M via antipodal maps on fibers of associated sphere bundle $S(\xi)$ (we also call a *real projective bundle* over M in this paper). In [2], Borel and Hirzebruch computed the total Stiefel-Whitney class of $P(\xi)$, which has been applied to the study of the equivariant cobordism (see [9]). In his paper [28], Stong introduced a special kind of real projective bundles (i.e., *Stong manifolds*, also see Section 4), which can be used as generators in the Thom unoriented cobordism ring $\mathfrak{N}_*$.

As the topological version of real toric manifolds, Davis and Januszkiewicz introduced and studied a class of particularly nicely behaving manifolds M^n (called *small covers*), each of which admits a locally standard $\mathbb{Z}_2^n$-action such that its orbit space is a simple convex n-polytope P^n, where $\mathbb{Z}_2^n = \{-1, 1\}^n$ is a real torus. This establishes a direct connection among topology, combinatorics and commutative algebra etc. In this paper, we first consider the following natural questions:

Problem 1. Let $P(\xi)$ be a real projective bundle over a small cover. When is also $P(\xi)$ a small cover? If so, how can we characterize its topology?

With respect to Problem 1, we have

Theorem 1.1. *Let $P(\xi)$ be a real projective bundle over a small cover. $P(\xi)$ is a small cover if and only if the equivariant vector bundle ξ decomposes into the Whitney sum of equivariant line bundles, i.e., $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$.*

Associated with moment-angle manifolds, we further study the structure of a real projective bundle $P(\xi)$ as a small cover. As an application, we characterize the real projective bundles (as small covers) over 2-dimensional small covers. The 1st main result is stated as follows:

Theorem 1.2. *Let $P(\xi)$ be a real projective bundle over 2-dimensional small cover M^2 with its fibre $\mathbb{R}P^{k-1}$. If $P(\xi)$ is a small cover, then $P(\xi)$ can be constructed from real projective bundles $P(\kappa)$ over $\mathbb{R}P^2$ and $P(\zeta)$ over T^2 by using the fibre sum $\sharp^{\Delta^{k-1}}$.*

The 2nd main result is about the projectivization of a real vector bundle ξ over $\mathbb{R}P^n$ which might not be a small cover. Let $\tau_{\mathbb{R}P^n}$ be the tangent bundle over $\mathbb{R}P^n$. The following relation is well-known:

$$\epsilon \oplus \tau_{\mathbb{R}P^n} \equiv (n+1)\gamma, \tag{1.1}$$

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where ϵ is the trivial line bundle over $\mathbb{R}P^n$ and $(n+1)\gamma$ represents the $(n+1)$-times Whitney sum of the tautological line bundle γ. This relation (1.1) shows that

$$P(\epsilon \oplus \tau_{\mathbb{R}P^n}) \cong P((n+1)\gamma) \cong P((n+1)(\gamma \otimes \gamma)) \cong P((n+1)\epsilon) \cong \mathbb{R}P^n \times \mathbb{R}P^n.$$

Therefore, the projectivization of $\epsilon \oplus \tau_{\mathbb{R}P^n}$ always satisfies bundle triviality, so it also is a small cover. Since there are exactly two line bundles over $\mathbb{R}P^n$, i.e., the trivial bundle ϵ and the tautological line bundle γ, a question arises, saying how about the projectivization of $\gamma \oplus \tau_{\mathbb{R}P^n}$. (Note that this might not be a small cover). This is just the question asked by Richard Montgomery motivated from his interest of the study of singularities [5], which is stated as follows:

Problem 2 (R. Montgomery)**.** When does $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ have the trivial bundle structure? In other wards, when is $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ diffeomorphic (or homeomorphic) to $\mathbb{R}P^n \times \mathbb{R}P^n$?

The answer is given as follows:

Theorem 1.3. *Let γ be the tautological line bundle and $\tau_{\mathbb{R}P^n}$ be the tangent bundle over $\mathbb{R}P^n$. Then, the following two statements are equivalent:*

(1) $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is diffeomorphic to $\mathbb{R}P^n \times \mathbb{R}P^n$;
(2) $n = 0, 2, 6$.

In particular, we also have that if $n = 0, 2, 6$ then $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is a small cover.

The organization of this paper is as follows. In Sections 2 and 3, we recall the basic facts about small covers and projective bundles. In Section 4, we give a proof of Theorem 1.1, and we also give the following two characterizations of projective bundles of small covers: (1) the characterization by the twisted product with real moment-angle manifolds; (2) the combinatorial characterization using simple convex polytopes and some function, like Davis-Januszkiewicz's small cover. In particular, to do (2), we introduce a new characteristic function on simple convex polytopes, called *projective characteristic functions*. In Section 5, we prove Theorem 1.2. To do this, we use the characterization (2) and introduce a new combinatorial operation which is the combinatorial analogue of the fibre sum, called a *projective fibre sum*. In Section 6, we classify all topological types of projective bundles over $\mathbb{R}P^2$ and T^2. In Section 7, we prove Theorem 1.3 and propose a question which we call *bundle triviality problem*. This problem is motivated by the question asked by Richard Montgomery. This paper gives the detailed proof for the results stated in [15] and also adds some results about the topological triviality problem.

Throughout this paper, all cohomologies and equivariant cohomologies will work on $\mathbb{Z}/2\mathbb{Z}$ coefficients. For a convenience, $H^*(-)$ means $H^*(-;\mathbb{Z}/2\mathbb{Z})$, and similarly for equivariant cohomologies.

2. Basic properties of small cover

In this section, we recall the notion of a small cover and the basic facts of its equivariant cohomology.

2.1. **Definition of small covers.** Let M^n be an n-dimensional, smooth, compact, connected manifold, and P^n a *simple* convex n-polytope, i.e., precisely n *facets* (codimension-1 faces) of P^n meet at each vertex. Put $\mathbb{Z}_2 = \{-1, 1\}$. A $\mathbb{Z}_2^\ell$-action on M (for some $1 \leq \ell \leq n$) is said to be *locally standard* if M is covered by $\mathbb{Z}_2^\ell$-invariant open charts $\{(U_i, \varphi_i)\}$ such that each chart $\varphi_i : U_i \to \mathbb{R}^n$ induces weakly equivariant homeomorphism from U_i to an open subset $\Omega \subset \mathbb{R}^n$ with the standard $\mathbb{Z}_2^\ell$-action, i.e., the action induced from an injection $\mathbb{Z}_2^\ell \to \mathbb{Z}_2^n$, where the $\mathbb{Z}_2^n$ action on $\mathbb{R}^n$ is standard. We call M^n a *small cover* if M admits a $\mathbb{Z}_2^n$-action such that

- **(a):** the $\mathbb{Z}_2^n$-action is *locally standard*, and
- **(b):** its orbit space has the structure of a simple convex polytope P^n, i.e., the corresponding orbit projection map $\pi : M^n \to P^n$ is constant on $\mathbb{Z}_2^n$-orbits and maps every rank k orbit (i.e., every orbit isomorphic to $\mathbb{Z}_2^k$) to an interior point of a k-dimensional face of the polytope P^n, $k = 0, 1, \ldots,\ n$.

It is easy to see that π sends $\mathbb{Z}_2^n$-fixed points in M^n to vertices of P^n by using the above condition (b). We often call P^n an *orbit polytope* of M.

Let M_1 and M_2 be two small covers and π_1 and π_2 be their orbit projections. If there is an orbit preserving equivariant homeomorphism between M_1 and M_2, i.e., there is a $\mathbb{Z}_2^n$-homeomorphism $f : M_1 \to M_2$ such that the following diagram commutes

$$\begin{tikzcd} M_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & M_2 \arrow[dl, "\pi_2"] \\ & P & \end{tikzcd}$$

then we call M_1 and M_2 are *Davis-Januszkiewicz equivalent* (or simply, *D-J equivalent*), see [6].

2.2. **Construction of small covers.** Conversely, for a given simple polytope P^n, the small cover M^n with orbit projection $\pi : M^n \to P^n$ can be reconstructed by using the *characteristic function* $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$, where $\mathcal{F}$ is the set of all facets in P and $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. In this subsection, we recall this construction (see [3], [10] for details).

Following the definition of a small cover $\pi : M \to P$, we have that $\pi^{-1}(\text{int}(F^{n-1}))$ consists of $(n-1)$-rank orbits, in other words, the isotropy subgroup at $x \in \pi^{-1}(\text{int}(F^{n-1}))$ is $K \subset \mathbb{Z}_2^n$ such that $K \simeq \mathbb{Z}_2$, where $\text{int}(F^{n-1})$ is the relative interior of the facet F^{n-1}. Hence, the isotropy subgroup at x is determined by a primitive vector $v \in (\mathbb{Z}/2\mathbb{Z})^n$ such that $(\mathbf{-1})^v$ generates the subgroup K, where $(\mathbf{-1})^v = ((-1)^{v_1}, \dots, (-1)^{v_n})$ for $v = (v_1, \dots, v_n) \in (\mathbb{Z}/2\mathbb{Z})^n$. In this way, we obtain a function λ from the set $\mathcal{F}$ to vectors in $(\mathbb{Z}/2\mathbb{Z})^n$, which is called a *characteristic function* or a *coloring* on P^n. We often describe λ as the $(m \times n)$-matrix

$$\Lambda = (\lambda(F_1) \cdots \lambda(F_m))$$

for $\mathcal{F} = \{F_1, \dots, F_m\}$ with a given ordering, and we call this matrix a *characteristic matrix.* Since the $\mathbb{Z}_2^n$-action is locally standard, a characteristic function has the following property:

($\star$): if $F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset$ for $F_{i_j} \in \mathcal{F}$ ($j = 1, \dots, n$), then $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$ spans $(\mathbb{Z}/2\mathbb{Z})^n$.

An interesting thing is that one can also construct small covers by using a given n-dimensional simple convex polytope P and a characteristic function λ with the property ($\star$). Let P be an n-dimensional simple convex polytope. Suppose that a characteristic function $\lambda : \mathcal{F} \to (\mathbb{Z}/2\mathbb{Z})^n$ with the above property ($\star$) is defined on P. Then a small cover can be constructed from P and λ as the quotient space $\mathbb{Z}_2^n \times P/\sim_\lambda$, where the equivalence relation $\sim_\lambda$ on $\mathbb{Z}_2^n \times P$ is defined as follows: $(t, x) \sim_\lambda (t', y)$ if and only if $x = y \in P$ and

$$\begin{array}{ll} t = t' & \text{if } x \in \text{int}(P); \\ t^{-1}t' \in \langle (\mathbf{-1})^{\lambda(F_{i_1})}, \cdots, (\mathbf{-1})^{\lambda(F_{i_r})} \rangle \simeq \mathbb{Z}_2^r & \text{if } x \in \text{int}(F_{i_1} \cap \cdots \cap F_{i_r}), \end{array}$$

where $\langle (\mathbf{-1})^{\lambda(F_{i_1})}, \cdots, (\mathbf{-1})^{\lambda(F_{i_r})} \rangle \subset \mathbb{Z}_2^n$ denotes the subgroup generated by $(\mathbf{-1})^{\lambda(F_{i_j})}$ for $j = 1, \dots, r$ with $r \leq n$. The small cover $\mathbb{Z}_2^n \times P/\sim_\lambda$ defined by this way is usually denoted by $M(P, \lambda)$.

Summing up, there is the following bijective correspondence:

$$\boxed{\begin{array}{c}\text{Small covers} \\ \text{with } \mathbb{Z}_2^n\text{-actions}\end{array}} \begin{array}{c}\longrightarrow \\ \longleftarrow\end{array} \boxed{\begin{array}{c}\text{Simple convex polytopes} \\ \text{with characteristic functions}\end{array}}$$

2.3. **Equivariant cohomology and ordinary cohomology of small cover.** In this subsection, we recall the equivariant cohomology and ordinary cohomology of the small covers (see [3], [10] for details). Let $M = M(P, \lambda)$ be an n-dimensional small cover. We denote an ordered set of facets of P by $\mathcal{F} = \{F_1, \dots, F_m\}$ such that $\cap_{i=1}^n F_i \neq \emptyset$. Due to [10], $M(P, \lambda_1)$ is D-J equivalent to $M(P, \lambda_2)$ if and only if there is an automorphism $X \in GL(n; \mathbb{Z}/2\mathbb{Z})$ such that $\lambda_1 = X \circ \lambda_2$ (such (P, λ_1) and (P, λ_2) (or their characteristic matrix Λ_1 and Λ_2) are said to be *equivalent*). Therefore, up to D-J equivalence (or equivalently, equivalence of (P, λ)), we may assume the values of the characteristic function λ on $F_1, \dots, F_n$ as

$$\lambda(F_i) = e_i$$

where $e_1, \ldots, e_n$ are the standard basis vectors of $(\mathbb{Z}/2\mathbb{Z})^n$. Then we can write the characteristic matrix as

$$\Lambda = (I_n \mid \Lambda'),$$

where I_n is the $(n \times n)$-identity matrix and Λ' is an $(l \times n)$-matrix, where $l = m - n$.

The *equivariant cohomology* of a G-manifold X is defined by the ordinary cohomology of the Borel construction $EG \times_G X$, where EG is the total space of a universal G-bundle, and denoted by $H^*_G(X)$. In this paper, we have assumed the coefficient group of cohomology is $\mathbb{Z}/2\mathbb{Z}$. Due to [10], the equivariant cohomology of a small cover M is given by

$$H^*_{\mathbb{Z}_2^n}(M) \simeq \mathbb{Z}/2\mathbb{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I},$$

where the symbol $\mathbb{Z}/2\mathbb{Z}[\tau_1, \ldots, \tau_m]$ represents the polynomial ring generated by the degree 1 elements τ_i $(i = 1, \ldots, m)$, and the ideal $\mathcal{I}$ is generated by the following monomial elements:

$$\prod_{i \in I} \tau_i$$

where I runs through every subset of $\{1, \ldots, m\}$ such that $\cap_{i \in I} F_i = \emptyset$. On the other hand, the ordinary cohomology ring of M is given by

$$H^*(M) \simeq H^*_{\mathbb{Z}_2^n}(M)/\mathcal{J},$$

where the ideal $\mathcal{J}$ is generated by the following degree 1 homogeneous elements:

$$\tau_i + \lambda_{i1}x_1 + \cdots \lambda_{il}x_l,$$

for $i = 1, \ldots, n$. Here, $(\lambda_{i1} \cdots \lambda_{il})$ is the ith row vector of Λ' $(i = 1, \ldots, n)$, and $x_j = \tau_{n+j}$ $(j = 1, \ldots, l)$.

Note that the above ideal $\mathcal{J}$ coincides with the ideal generated by $\pi^*(H^+(B\mathbb{Z}_2^n)) = \mathrm{Im}\ \pi^+$, i.e.,

$$\mathcal{J} = \langle \mathrm{Im}\ \pi^+ \rangle,$$

where $H^+(B\mathbb{Z}_2^n) = H^*(B\mathbb{Z}_2^n) \setminus H^0(B\mathbb{Z}_2^n)$ and $\pi^* : H^*(B\mathbb{Z}_2^n) \to H^*_{\mathbb{Z}_2^n}(M)$ is the induced homomorphism from the natural projection $E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M \to B\mathbb{Z}_2^n$, where $B\mathbb{Z}_2^n = (\mathbb{R}P^\infty)^n$.

3. General facts of projective bundles

In this section, we recall some general notations and basic facts for projective bundles (see e.g. [9], [23] for details). We first recall the definition of the projective bundle. Let ξ be a k-dimensional, real vector bundle over M. We will denote the total space of ξ by $E(\xi)$, the projection from $E(\xi)$ onto M by $\widetilde{\rho}$, and the fibre on $x \in M$ by $F_x(\xi)$, i.e., $F_x(\xi) = \widetilde{\rho}^{-1}(x)$. Put ξ_0 the bundle induced by ξ removing the 0-section. Then each fibre of ξ_0 has the multiplicative action of $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Taking its orbit space, we have the fibre bundle $\rho : P(\xi) \to M$ whose fibre is the $(k-1)$-dimensional real projective space $\mathbb{R}P^{k-1}$. We call $P(\xi)$ the real *projective bundle* of ξ. We often denote the fibre of $P(\xi)$ on $x \in M$ by $P_x(\xi)$, i.e., $P_x(\xi) = \rho^{-1}(x) \simeq \mathbb{R}P^{k-1}$.

We next recall the properties of cohomology of projective bundles. Let $\iota : \mathbb{R}P^{k-1} \simeq P_x(\xi) \to P(\xi)$ be the natural embedding. As is well known, the induced ring homomorphism

$$H^*(P(\xi)) \xrightarrow{\iota^*} H^*(\mathbb{R}P^{k-1}) \tag{3.1}$$

is surjective. On the other hand, the induced ring homomorphism

$$H^*(M) \xrightarrow{\rho^*} H^*(P(\xi)) \tag{3.2}$$

is injective. Moreover, we have the kernel of ι^* is the ideal generated by $\mathrm{Im}\ \rho^+$, where $\mathrm{Im}\ \rho^+ = \rho^*(H^+(M))$, where $H^+(M) = H^*(M) \setminus \{0\}$. Now we want to consider the ring structure of the cohomology $H^*(P(\xi))$. In order to do this, we define the following line bundle over $P(\xi)$ associated from ξ:

$$\gamma_\xi = \sqcup_{x \in M}\{(L, r) \in P_x(\xi) \times F_x(\xi) \mid r \in L\}, \tag{3.3}$$

where we regard $L \in P_x(\xi)$ as the line in the fibre $F_x(\xi)$ of ξ. We call γ_ξ the *tautological (real) line bundle* of $P(\xi)$. Note that we have the following diagram:

$$\begin{tikzcd} & E(\gamma_\xi) \arrow[r] \arrow[dr, "\mathbb{R}"'] & E(\rho^*\xi) \arrow[r] \arrow[d, "\mathbb{R}^k"'] & E(\xi) \arrow[d, "\mathbb{R}^k"', "\widetilde{\rho}"] \\ & \mathbb{R}P^{k-1} \arrow[r, "\iota"] & P(\xi) \arrow[r, "\rho"] & M \end{tikzcd} \tag{3.4}$$

where $\rho^*\xi$ is the pull-back of ξ by ρ. Let $w_i(\xi) \in H^i(M)$ be the i^{th} Stiefel-Whitney class of the k-dimensional vector bundle ξ for $i = 1, \ldots, k$, and $w_1(\gamma_\xi) \in H^1(P(\xi))$ be the 1^{st} Stiefel-Whitney class of γ_ξ. Then $\iota^*(w_1(\gamma_\xi))$ is the ring generator of $H^*(\mathbb{R}P^{k-1})$. Because

$$H^*(\mathbb{R}P^{k-1}) \simeq \mathbb{Z}/2\mathbb{Z}[a]/\langle a^k \rangle \tag{3.5}$$

with $\deg a = 1$, we have $\iota^*(w_1(\gamma_\xi)^k) = 0$ in $H^*(\mathbb{R}P^{k-1})$. However, $w_1(\gamma_\xi)^k$ might not be zero in $H^k(P(\xi))$. The following formula, called the *Borel-Hirzebruch formula*, gives us the explicit expression of this element (see [2] or [9, (23.3)]):

$$w_1(\gamma_\xi)^k = \sum_{i=1}^{k} \rho^*(w_i(\xi)) w_1(\gamma_\xi)^{k-i}. \tag{3.6}$$

Therefore, together with (3.2), $H^*(P(\xi))$ is isomorphic to

$$H^*(M)[x]/\langle \sum_{i=0}^{k} \rho^*(w_i(\xi)) x^{k-i} \rangle \tag{3.7}$$

as the $H^*(M)$-algebra, where $x = w_1(\gamma_\xi)$ and $\rho^*(w_i(\xi))$ is regarded as the element in $H^*(M)$ (because of the injectivity of ρ^*). Moreover, by using the Borel-Hirzebruch formula, we have the following proposition.

Proposition 3.1. *Let M be a compact, connected manifold, and ξ a k-dimensional real vector bundle, where $k > 1$. Then, the following two statements are equivalent:*

(1) $H^*(P(\xi)) \simeq H^*(M \times \mathbb{R}P^{k-1})$;
(2) $w(\xi) = (1 + X)^k$ for some $X \in H^1(M)$.

Proof. Because M is a compact, connected manifold, its cohomology ring $H^*(M)$ is finitely generated. Therefore, we may assume that

$$H^*(M) = \mathbb{Z}/2\mathbb{Z}[\eta_1, \cdots, \eta_m]/\langle f_j \mid j = 1, \ldots, l \rangle \tag{3.8}$$

for some polynomial $f_j = f_j(\eta_1, \cdots, \eta_m)$ and generators $\eta_1, \ldots, \eta_m$. Because $\rho^* : H^*(M) \to H^*(P(\xi))$ is injective, we may regard $\eta_1, \ldots, \eta_m$ as the generators in $H^*(P(\xi))$. Moreover, since $\iota^*(w_1(\gamma_\xi))$ is the generator of $H^*(\mathbb{R}P^{k-1})$, we may denote the cohomology ring of $H^*(P(\xi))$ as follows:

$$\mathbb{Z}/2\mathbb{Z}[\eta_1, \ldots, \eta_m,\ w_1(\gamma_\xi)]/\langle f_j,\ w_1(\gamma_\xi)^k - \sum_{i=1}^{k} \rho^*(w_i(\xi)) w_1(\gamma_\xi)^{k-i} \mid j = 1, \ldots, l \rangle \tag{3.9}$$

by using the Borel-Hirzebruch formula (3.6).

Assume that the statement (1) holds, that is, $H^*(P(\xi)) \simeq H^*(M \times \mathbb{R}P^{k-1})$. Then, we may put

$$H^*(P(\xi)) \simeq \mathbb{Z}/2\mathbb{Z}[\eta_1, \ldots, \eta_m,\ A]/\langle f_j,\ A^k \mid j = 1, \ldots, l \rangle, \tag{3.10}$$

for some $A \in H^1(P(\xi))$. Comparing (3.9) and (3.10), we may write

$$\begin{aligned} A &= w_1(\gamma_\xi) + \epsilon_1 \eta_1 + \cdots + \epsilon_m \eta_m \\ &= w_1(\gamma_\xi) + X \end{aligned} \tag{3.11}$$

for some $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, \ldots, m)$. Therefore, we have

$$
\begin{aligned}
A^k &\overset{(3.11)}{=} (w_1(\gamma_\xi) + X)^k \equiv_2 w_1(\gamma_\xi)^k + \sum_{i=1}^{k} \binom{k}{i} X^i w_1(\gamma_\xi)^{k-i} \\
&\overset{(3.6)}{\equiv_2} \sum_{i=1}^{k} \rho^*(w_i(\xi)) w_1(\gamma_\xi)^{k-i} + \sum_{i=1}^{k} \binom{k}{i} X^i w_1(\gamma_\xi)^{k-i} \\
&\overset{(3.10)}{\equiv_2} 0.
\end{aligned} \tag{3.12}
$$

Based upon the $H^*(M)$-algebraic structure of $H^*(P(\xi))$ in (3.7), we have that $w_1(\gamma_\xi)^0, \ldots, w_1(\gamma_\xi)^{k-1}$ are the $H^*(M)$-module generators of $H^*(P(\xi))$. Therefore, the equation (3.12) implies that

$$\rho^*(w_i(\xi)) \equiv_2 \binom{k}{i} X^i.$$

Furthermore, because ρ^* is injective, we may write

$$w(\xi) = (1 + X)^k,$$

where $k \equiv_2 0$ or 1, and $w(\xi)$ is the total Stiefel-Whitney class of ξ. This establishes the statement (2).

Assume that the statement (2) holds, that is, $w(\xi) = (1+X)^k$ for some $X \in H^1(M)$. By using (3.6) and the injectivity of ρ^*, one can easily show that $(w_1(\gamma_\xi)+X)^k = 0$. Using (3.5) and (3.8), we may put

$$H^*(M \times \mathbb{R}P^{k-1}) = \mathbb{Z}/2\mathbb{Z}[\eta_1, \ldots, \eta_m,\ a]/\langle f_j,\ a^k \mid j = 1, \ldots, l\rangle, \tag{3.13}$$

for some $a \in H^1(M \times \mathbb{R}P^{k-1})$. Therefore, using (3.9) and the above (3.13), there is the following isomorphism from $H^*(M \times \mathbb{R}P^{k-1})$ to $H^*(P(\xi))$:

$$
\begin{aligned}
&\varphi : \eta_i \mapsto \eta_i \quad (i = 1, \ldots, m); \\
&\varphi : a \mapsto w_1(\gamma_\xi) + X.
\end{aligned}
$$

This establishes the statement (1). □

4. Projective bundles over small covers

In this section, we introduce some notations and basic facts for projective bundles over small covers. We first recall the definition of a G-equivariant vector bundle over G-space M (also see the notations in Section 3). A *G-equivariant vector bundle* is a vector bundle ξ over G-space M together with a lift of the G-action to $E(\xi)$ by fibrewise linear transformations, i.e., $E(\xi)$ is also a G-space, the projection $E(\xi) \to M$ is G-equivariant and the induced fibre isomorphism between $F_x(\xi)$ and $F_{gx}(\xi)$ is linear, for all $x \in M$ and $g \in G$.

Example 4.1 (Generalized real Bott manifold)**.** A *generalized real Bott manifold* of height m is an iterated real projective fibration defined as a sequence of real projective fibrations

$$\mathbb{R}B_m \xrightarrow{\pi_m} \mathbb{R}B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} \mathbb{R}B_1 \xrightarrow{\pi_1} \mathbb{R}B_0 = \{a\ point\}$$

where $\mathbb{R}B_i = P(\gamma_{i_1} \oplus \cdots \oplus \gamma_{i_l})$ is the projectivization of a Whitney sum of line bundles over $\mathbb{R}B_{i-1}$. Note that $\mathbb{R}B_1$ is just the real projective space. If the dimension of each fibre is exactly 1, then this is called a *real Bott manifold*. See [19] for details.

Example 4.2 (Stong manifold ([28]))**.** Let $\pi_i : B = \mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_l} \to \mathbb{R}P^{n_i}$ be the natural projection, for $i = 1, \ldots, l$. We define the line bundle γ_i over B by the pull-back of the tautological line bundle over $\mathbb{R}P^{n_i}$ along π_i. Then, a *Stong manifold* S is defined by the following projectivization over B:

$$S = P(\gamma_1 \oplus \cdots \oplus \gamma_l) \to B.$$

It is easy to check that this is a generalized real Bott manifold.

4.1. **Necessary and sufficient conditions of when $P(\xi)$ is a small cover.** We first prove the following general fact:

Theorem 4.3. *Let M be an n-dimensional, compact, connected manifold. Assume that M has a locally standard $\mathbb{Z}_2^\ell$-action for some $1 \leq \ell \leq n$, and this $\mathbb{Z}_2^\ell$-action is a maximal real torus action, i.e., there is no effective $\mathbb{Z}_2^{\ell+1}$-action which is an extension of this $\mathbb{Z}_2^\ell$-action. Let ξ be a k-dimensional, $\mathbb{Z}_2^\ell$-equivariant vector bundle over M. The projective bundle $P(\xi)$ of ξ admits a locally standard $\mathbb{Z}_2^\ell \times \mathbb{Z}_2^{k-1}$-action if and only if the equivariant vector bundle ξ decomposes into the Whitney sum of line bundles, i.e., $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$.*

Proof. Assume that $P(\xi)$ admits a locally standard $\mathbb{Z}_2^\ell \times \mathbb{Z}_2^{k-1}$-action. Because ξ is a $\mathbb{Z}_2^\ell$-equivariant vector bundle over M with k-dimensional fibers, the projection $\rho : P(\xi) \to M$ is $\mathbb{Z}_2^\ell$-equivariant. Note that the $\mathbb{Z}_2^{k-1}$-factor acts trivially on M because the $\mathbb{Z}_2^\ell$-action on M is maximal. It follows from this fact that $\rho : P(\xi) \to M$ is also $(\mathbb{Z}_2^\ell \times \mathbb{Z}_2^{k-1})$-equivariant, where $\mathbb{Z}_2^{k-1}$ acts on M trivially. Therefore, each fibre $P_x(\xi)$ over $x \in M$ has an effective $\mathbb{Z}_2^{k-1}$-action. This implies that there is the $\mathbb{Z}_2^k$-action on $F_x(\xi)$ such that $(F_x(\xi) \setminus \{0\})/\mathbb{R}^*$ is $\mathbb{Z}_2^{k-1}$-equivariantly homeomorphic to $P_x(\xi)$, where $F_x(\xi) \cong \mathbb{R}^k$ is the fibre of $E(\xi)$ over $x \in M$. Hence, the total space $E(\xi)$ of ξ has a $(\mathbb{Z}_2^\ell \times \mathbb{Z}_2^k)$-action and the restricted $\mathbb{Z}_2^\ell$-action is induced from the lift of the $\mathbb{Z}_2^\ell$-action on M. Let $\{U_i\}_{i\in\mathcal{I}}$ be a $\mathbb{Z}_2^\ell$-equivariant open covering of M. Then, by using the local triviality condition of the vector bundle, we may denote ξ as the gluing of $U_i \times \mathbb{R}^k$ for $i \in \mathcal{I}$, say $\amalg_{i\in\mathcal{I}}(U_i \times \mathbb{R}^k)/\sim$. Here, the symbol $\sim$ represents the identification $(u, x) \sim (u, y)$ for $u \in U_i \cap U_j$ by $x = A(u)y \in \mathbb{R}^k$ for some transition function $A(u) \in GL(k; \mathbb{R})$. Here, because M is a smooth closed manifold, we may reduce the structure group into the orthogonal group $O(k)$ and we can take $A(u) \in O(k)$. Therefore, if the $\mathbb{Z}_2^k$-action on the $\mathbb{R}^k$-factor in $U_i \times \mathbb{R}^k$ extends to the global action on $\amalg(U_i \times \mathbb{R}^k)/\sim$, then the transition function $A(u) \in O(k)$ must commute with $\mathbb{Z}_2^k$ for all $u \in U_i \cap U_j$. Note that we may regard $\mathbb{Z}_2^k$ as the diagonal subgroup in $O(k)$ up to conjugation. Because the centralizer of $\mathbb{Z}_2^k$ (the diagonal subgroup) in $O(k)$ is $\mathbb{Z}_2^k$ (the diagonal subgroup) itself, we have $A(u) \in \mathbb{Z}_2^k \subset O(k)$ for all $u \in U_i \cap U_j$. This implies that the structure group of ξ is $\mathbb{Z}_2^k$. This is nothing but $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$.

Conversely, if $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$, then we can easily check that this vector bundle has the $\mathbb{Z}_2^k$-action along fibre and $P(\xi)$ has the induced locally standard $(\mathbb{Z}_2^\ell \times \mathbb{Z}_2^{k-1})$-action. □

As a corollary of this theorem, we have the following criterion for the projective bundle $P(\xi)$ to be a small cover:

Corollary 4.4 (Theorem 1.1)**.** *Let $P(\xi)$ be a real projective bundle over a small cover. $P(\xi)$ is a small cover if and only if the equivariant vector bundle ξ decomposes into the Whitney sum of equivariant line bundles, i.e., $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$.*

As is well known, $P(\xi \otimes \gamma) \cong P(\xi)$ (homeomorphic) for all line bundle γ (e.g. see [20]). Hence, by using the above Corollary 4.4, the following corollary holds.

Corollary 4.5. *Let M be a small cover, and ξ be a Whitney sum of k line bundles over M. Then the small cover $P(\xi)$ is homeomorphic to*

$$P(\gamma_1 \oplus \cdots \gamma_{k-1} \oplus \epsilon),$$

where ϵ is the trivial line bundle over M.

Proof. Assume $\xi \equiv \gamma_1' \oplus \cdots \oplus \gamma_{k-1}' \oplus \gamma_k'$. Then we have that

$$P(\gamma_1' \oplus \cdots \oplus \gamma_{k-1}' \oplus \gamma_k') \cong P((\gamma_1' \otimes \gamma_k') \oplus \cdots \oplus (\gamma_{k-1}' \otimes \gamma_k') \oplus \epsilon),$$

because $\gamma_k' \otimes \gamma_k' \equiv \epsilon$. This establishes the statement. □

In this paper, the projective bundle in Corollary 4.5 (also see Section 1) is said to be the *projective bundle over small cover.*

In addition, by Corollary 4.4, we also have the following two corollaries:

Corollary 4.6. *Each generalized real Bott manifold is a small cover. In particular, each Stong manifold is a small cover.*

The generalized Bott manifold is defined in [8] as a special class of toric manifolds. The generalized real Bott manifold is its real analogue. It was shown in [28] that Stong manifolds can be chosen as generators of $\mathfrak{N}_*$.

Corollary 4.7. *Each class of $\mathfrak{N}_*$ contains a small cover as its representative.*

The fact of Corollary 4.7 has been proved in [4] with a different argument, and in addition, the fact that each class of complex cobordism contains a quasitoric manifold as its representative was also proved in [4]. For the equivariant case, see [16], [17].

From the next subsection to Section 6, we assume that M is an n-dimensional small cover, and ξ is a k-dimensional, $\mathbb{Z}_2^n$-equivariant vector bundle over M.

4.2. **Structures of projective bundles over small covers.** In this subsection, we show the quotient construction of the projective bundles of small covers. First, we recall the general facts on the (real) moment-angle manifolds of simple convex polytopes (see [3], [10]). Let P be a simple, convex polytope and $\mathcal{F}$ the set of its facets $\{F_1, \cdots, F_m\}$. We denote by $\mathcal{Z}_P$ the manifolds

$$\mathcal{Z}_P = \mathbb{Z}_2^m \times P/\sim,$$

where $(t,\ p) \sim (t',\ p)$ is defined by $t^{-1}t' \in \prod_{p\in F_i} \mathbb{Z}_2(i)$ ($\mathbb{Z}_2(i) \subset \mathbb{Z}_2^m$ is the rank 1 subgroup generated by the i-th factor), and we call it a (real) *moment-angle manifold* over P. We note that if $P = M^n/\mathbb{Z}_2^n$ (i.e., there is a small cover M^n over P), then there is the subgroup $K \subset \mathbb{Z}_2^m$ such that $K \simeq \mathbb{Z}_2^{m-n}$ and K acts freely on $\mathcal{Z}_P$. Therefore, in this case we can denote the small cover $M = \mathcal{Z}_P/\mathbb{Z}_2^l$ by the free $\mathbb{Z}_2^l$-action on $\mathcal{Z}_P$ for $l = m - n$.

Now assume that there is a small cover M over P. Since $[M,\ B\mathbb{Z}_2] = H^1(M;\ \mathbb{Z}_2) \simeq \mathbb{Z}_2^l$ (see [10], [27]), we see that all line bundles γ can be written as follows:

$$\gamma \equiv \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}_\alpha,$$

where $\mathbb{Z}_2^l$ acts on $\mathbb{R}_\alpha = \mathbb{R}$ by some representation $\alpha : \mathbb{Z}_2^l \to \mathbb{Z}_2$. Moreover, its total Stiefel-Whitney class is $w(\mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}) = 1 + \delta_1 x_1 + \cdots + \delta_l x_l$, where $(\delta_1, \cdots, \delta_l) \in \{0,\ 1\}^l$ is induced by a representation $\mathbb{Z}_2^l \to \mathbb{Z}_2$, i.e.,

$$(\epsilon_1,\ \cdots,\ \epsilon_l) \mapsto \epsilon_1^{\delta_1} \cdots \epsilon_l^{\delta_l},$$

for $\epsilon_i \in \mathbb{Z}_2$, and $x_1,\ \ldots,\ x_l$ are the degree 1 generators of $H^*(M)$ introduced in Section 2.3. Therefore, by using Corollary 4.5, all projective bundles over the small cover M are as follows:

$$P(\xi) = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} (\mathbb{R}^k \setminus \{0\})/\mathbb{R}^* = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}P^{k-1}, \tag{4.1}$$

where

$$\xi = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}^k$$

with the $\mathbb{Z}_2^l$-representation space $\mathbb{R}^k = \mathbb{R}_{\alpha_1} \oplus \cdots \oplus \mathbb{R}_{\alpha_k}$ such that

$$\alpha_i : \mathbb{Z}_2^l \to \mathbb{Z}_2$$

where $i = 1,\ \cdots,\ k$ and α_k is the trivial representation. Then, we may denote each projective bundle over the small cover M by

$$\mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}P^{k-1} = P(\gamma_1 \oplus \cdots \gamma_{k-1} \oplus \epsilon),$$

where $\gamma_i = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}_{\alpha_i}$ ($i = 1,\ \cdots,\ k-1$) satisfies $w(\gamma_i) = 1+\delta_{1i}x_1+\cdots+\delta_{li}x_l$ for $(\delta_{1i},\ \cdots,\ \delta_{li}) \in (\mathbb{Z}/2\mathbb{Z})^l$, which is induced by the representation $\alpha_i : \mathbb{Z}_2^l \to \mathbb{Z}_2$. This is also denoted by the following form:

$$\mathcal{Z}_P \times_{\mathbb{Z}_2^l} P(\mathbb{R}_{\alpha_1} \oplus \cdots \oplus \mathbb{R}_{\alpha_k}).$$

Let $\Lambda_P \in M(m,n;\mathbb{Z}/2\mathbb{Z})$ be the characteristic matrix of M (see Section 2.2, 2.3), where $M(m,n;\mathbb{Z}/2\mathbb{Z})$ is the set of $(n \times m)$-matrices with $\mathbb{Z}/2\mathbb{Z}$ entries. Due to the arguments in Section 2.3, up to D-J equivalence, Λ_P is equivalent to $(I_n|\Lambda) \in M(m,n;\mathbb{Z}/2\mathbb{Z})$ for some $\Lambda \in M(l,n;\mathbb{Z}/2\mathbb{Z})$, where $l = m - n$. Using the above constriction of projective bundles and computing their characteristic functions, we have the following proposition.

Proposition 4.8. *Let $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon)$ be the projective bundle over M. Then its orbit polytope is $P^n \times \Delta^{k-1}$, and its characteristic matrix can be written by*

$$\begin{pmatrix} \Lambda_P & O \\ \Lambda'_P & Y \end{pmatrix} \in M(m+k, n+k-1; \mathbb{Z}/2\mathbb{Z}) \tag{4.2}$$

for some $\Lambda'_P \in M(m, k-1; \mathbb{Z}/2\mathbb{Z})$ and a characteristic matrix $Y \in M(k, k-1; \mathbb{Z}/2\mathbb{Z})$ of $\mathbb{R}P^{k-1}$, where $P^n = M/\mathbb{Z}_2^n$, $\Delta^{k-1} = \mathbb{R}P^{k-1}/\mathbb{Z}_2^{k-1}$ (i.e., $(k-1)$-dimensional simplex). Moreover, up to D-J equivalence (and taking an appropriate order of facets), this is equivalent to the following characteristic matrix:

$$\begin{pmatrix} I_n & \Lambda & O & \boldsymbol{0} \\ O & \Lambda_\xi & I_{k-1} & \boldsymbol{1} \end{pmatrix}, \tag{4.3}$$

with

$$\Lambda_\xi = \begin{pmatrix} \delta_{11} & \cdots & \delta_{l1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k-1} & \cdots & \delta_{l,k-1} \end{pmatrix},$$

where $\boldsymbol{0} \in M(1, n; \mathbb{Z}/2\mathbb{Z})$ is the 0 vector, $\boldsymbol{1} \in M(1, k-1; \mathbb{Z}/2\mathbb{Z})$ is the vector whose all entries are 1 and δ_{ij}'s are coefficients appeared in $w(\gamma_i) = 1 + \delta_{1i}x_1 + \cdots + \delta_{li}x_l$ for $i = 1, \ldots, k-1$.

Therefore, we have the following corollary by using Proposition 3.1.

Corollary 4.9. *Let M be a small cover, and ξ a Whitney sum of k line bundles over M. Then the following two statements are equivalent:*

(1) $H^*(P(\xi)) \simeq H^*(M \times \mathbb{R}P^{k-1})$;
(2) $w(\xi) = w(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon) = \prod_{j=1}^{k-1}(1 + \sum_{i=1}^{l} \delta_{ij}x_i) = (1+X)^k$ for some $X \in H^1(M)$.

4.3. **New characteristic function of projective bundles over small covers.** In order to show the construction theorem of projective bundles over 2-dimensional small covers, we introduce a new characteristic function (matrix). By using Proposition 4.8, the characteristic matrix of $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon)$ is

$$\begin{pmatrix} \Lambda_P & O \\ \Lambda'_P & Y \end{pmatrix} \tag{4.4}$$

which satisfies that there exists $A \in GL(n+k-1; \mathbb{Z}/2\mathbb{Z})$ such that

$$A \begin{pmatrix} \Lambda_P \\ \Lambda'_P \end{pmatrix} = \begin{pmatrix} I_n & \Lambda \\ O & \Lambda_\xi \end{pmatrix}. \tag{4.5}$$

Here, we may put

$$\left(\begin{array}{c} \Lambda_P \\ \hline \Lambda'_P \end{array} \right) = \left(\begin{array}{ccc} \mathbf{a}_1 & \cdots & \mathbf{a}_m \\ \hline \mathbf{b}_1 & \cdots & \mathbf{b}_m \end{array} \right),$$

for some $\mathbf{a}_i \in \{0,\ 1\}^n$ and $\mathbf{b}_i \in \{0,\ 1\}^{k-1}$ $(i = 1,\ \cdots,\ m)$. Therefore, in order to characterize the projective bundles over M^n, it is sufficient to attach the following $((n+k-1) \times m)$-matrix which satisfies (4.5)

$$\begin{pmatrix} \Lambda_P \\ \Lambda'_P \end{pmatrix} \tag{4.6}$$

on the facets of P. Namely, it is sufficient to consider the following function on facets: for P and its facets $\mathcal{F}$ with the fixed appropriate order, the function

$$\lambda_P : \mathcal{F} = \{F_1,\ \cdots,\ F_m\} \to \{0,\ 1\}^n \times \{0,\ 1\}^{k-1}$$

which satisfies that the projection to the 1st factor $p_a : \{0,\ 1\}^n \times \{0,\ 1\}^{k-1} \to \{0,\ 1\}^n$ satisfies that

$$\det(p_a \circ \lambda_P(F_{i_1}) \cdots p_a \circ \lambda_P(F_{i_n})) = 1 \tag{4.7}$$

if $F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset$, i.e., $p_a \circ \lambda_P$ is the usual characteristic function on P^n. We call this function λ_P a *projective characteristic function* over P^n; or a $(k-1)$-*dimensional projective characteristic function* when we emphasize the dimension of the fibre. One can easily show that the $((n+k-1)\times m)$-matrix $(\lambda_P(F_1)\cdots\lambda_P(F_m))$ is identified with the matrix (4.6). We call this matrix a (*(k-1)-dimensional*) *projective characteristic matrix* over P. Figure 1 is an illustration of projective characteristic functions over 2-dimensional small covers.

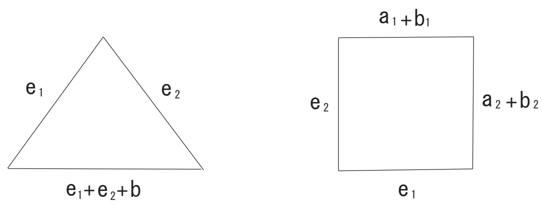

FIGURE 1. The examples of $(k-1)$-dimensional projective characteristic functions. Here, $e_1 = (1,0)\times \mathbf{0}$ and $e_2 = (0,1)\times\mathbf{0}$ are the generators of $(\mathbb{Z}/2\mathbb{Z})^2\times\mathbf{0}$. For the left triangle, we may take an arbitrary element $b \in (0,0)\times(\mathbb{Z}/2\mathbb{Z})^{k-1}$ and one can easily show that each of those corresponds with a projective bundle over $\mathbb{R}P^2$ whose fibre is $\mathbb{R}P^{k-1}$. For the right square, $a_1,\ a_2$ are elements in $(\mathbb{Z}/2\mathbb{Z})^2\times\mathbf{0}$ which satisfy (4.7) on each vertex, and $b_1,\ b_2 \in (0,0)\times(\mathbb{Z}/2\mathbb{Z})^{k-1}$ determine the bundle structure.

Note that in Figure 1, if we put $b=0$ and $b_1=b_2=0\in\mathbb{Z}_2^{k-1}$, then this gives ordinary characteristic functions on the triangle and the square. It is easy to check that this is the general fact for any projective characteristic functions. Therefore, we may regard that such a forgetful map of the $(\mathbb{Z}/2\mathbb{Z})^{k-1}$ part, say $f:(P^n,\lambda_P)\to(P^n,p_a\circ\lambda_P)$, as the equivariant projection $P(\xi)\to M(P,p_a\circ\lambda_P)$.

Using Proposition 4.8 and the construction method of small covers (see Section 2.2), one can easily show that the pair $(P^n,\ \lambda_P)$ corresponds with the projective bundle over the n-dimensional small cover whose orbit polytope is P^n. More precisely, for the projective bundle $P(\xi)$ over the small cover $M(P,\lambda)$, there exists the projective characteristic function (P,λ_P) such that $p_a\circ\lambda_P=\lambda$. On the other hand, for the projective characteristic function (P,λ_P) there exists the projective bundle $\mathcal{Z}_P\times_{\mathbb{Z}_2^l}\mathbb{R}P^{k-1}=P(\gamma_1\oplus\cdots\oplus\gamma_{k-1}\oplus\epsilon)$ over $\mathcal{Z}_p/\mathbb{Z}_2^l=M(P,p_a\circ\lambda_P)$, where the line bundle γ_i is determined by the ith column vector of Λ_ξ appeared in (4.5) (also see Section 4.1).

Summing up, we have the following bijective correspondence:

$$\boxed{\begin{matrix}\text{Projective bundles}\\ \text{over small cover } M^n\end{matrix}} \ \begin{matrix}\longrightarrow\\ \longleftarrow\end{matrix}\ \boxed{P^n \text{ with projective characteristic functions } \lambda_P}$$

5. New operations and main theorem

Before stating our main theorem, we introduce a new operation in this section.

5.1. **Combinatorial interpretation of the fibre sum.** For two polytopes with projective characteristic functions, we can do the connected sum operation which is compatible with projective characteristic functions as indicated in Figure 2. Then we get a new polytope with the projective characteristic function. We call this operation a *projective fibre sum* and denote it by $\sharp_{\Delta^{k-1}}$.

More precisely, the operation is defined as follows. Let p and q be vertices in n-dimensional polytopes with $(k-1)$-dimensional projective characteristic functions (P,λ_P) and $(P',\lambda_{P'})$, respectively. Here, we assume that the target spaces of the maps λ_P and $\lambda_{P'}$ are the same $(\mathbb{Z}/2\mathbb{Z})^n\times(\mathbb{Z}/2\mathbb{Z})^{k-1}$, i.e., the corresponding projective bundles have the same fibre $\mathbb{R}P^{k-1}$. Moreover, we assume that $\lambda_P(F_i)=\lambda_{P'}(F_i')$ for all facets $\{F_1,\ldots,F_n\}$ around p and $\{F_1',\ldots,F_n'\}$ around q, i.e., $\cap_{i=1}^n F_i=\{p\}$ and $\cap_{i=1}^n F_i'=\{q\}$. Then we can do the connected sum of two polytopes P and P' at these vertices by gluing each pair of facets F_i and F_i'. Thus, we get a combinatorial object (might not be a convex polytope) with a function $(P\sharp_{\Delta^{k-1}}P',\ \lambda_{P\sharp_{\Delta^{k-1}}P'})$ from

(P, λ_P) and $(P', \lambda_{P'})$ (also see Figure 2). Note that $P_1\sharp_{\Delta^{k-1}}P_2$ is a combinatorial simple convex polytope if $n \leq 3$ by using *the Steinitz' theorem*: the graph Γ is a graph of the 3-dimensional polytope P if and only if Γ is 3-connected and planer (see [29, Chapter 4]). Moreover, it is easy to check that the function $\lambda_{P\sharp_{\Delta^{k-1}}P'}$ is a projective characteristic function, i.e., it satisfies (4.7). By following the converse of the above definition, we may define the inverse operation $\sharp^{-1}_{\Delta^{k-1}}$.

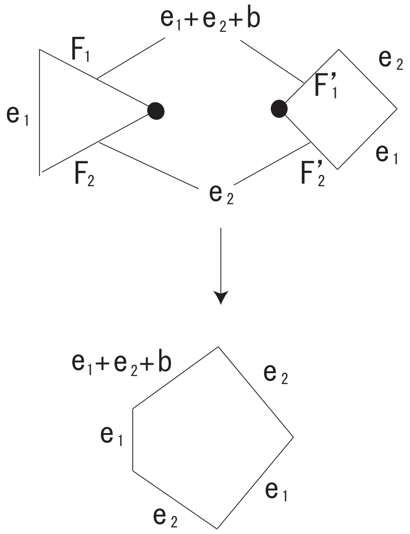

FIGURE 2. The projective fibre sum $\sharp_{\Delta^{k-1}}$ along the same labeled vertices.

From the geometric point of view, the inverse image of vertices of polytopes with projective characteristic functions corresponds to the projective space $\mathbb{R}P^{k-1}$. Therefore, this operation is nothing but an equivariant gluing along the fibre $\mathbb{R}P^{k-1}$, i.e., fibre sum of two fibre bundles.

Remark 5.1. If $k = 1$, then the projective characteristic function is the ordinary characteristic function, i.e., the dimension of fibres is 0. Therefore, we can regard the 0-dimensional projective fibre sum $\sharp_{\Delta^0}$ as the ordinary (equivariant) connected sum $\sharp$ appeared in [13], [14], [18], [25].

If $P\sharp_{\Delta^{k-1}}P'$ is a convex simple polytope, then $(P\sharp_{\Delta^{k-1}}P',\ \lambda_{P\sharp_{\Delta^{k-1}}P'})$ defines the $(k-1)$-dimensional projective bundle over $M\sharp M'$ (connected sum), where $M = M(P, p_a \circ \lambda_P)$ and $M' = M(P', p_a \circ \lambda_{P'})$. We note that if λ_P and $\lambda_{P'}$ are

$$\begin{pmatrix} X & \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ Y & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{pmatrix}, \quad \begin{pmatrix} X' & \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ Y' & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{pmatrix},$$

respectively, then $\lambda_{P\sharp_{\Delta^{k-1}}P'}$ is

$$\begin{pmatrix} X & \mathbf{a}_1 & \cdots & \mathbf{a}_n & X' \\ Y & \mathbf{b}_1 & \cdots & \mathbf{b}_n & Y' \end{pmatrix},$$

where the same n column vectors above correspond to the projective characteristic functions on $\{F_1, \ldots, F_n\}$ and $\{F'_1, \ldots, F'_n\}$.

5.2. **Construction theorem of projective bundles over 2-dimensional small covers.** In this subsection, we prove one of the main results of this paper. The standard $\mathbb{Z}_2^n$-action on $\mathbb{R}P^n$ is defined by

$$(t_1, \ldots, t_n) \cdot [r_0 : r_1 \cdots : r_n] \mapsto [r_0 : t_1r_1 \cdots : t_nr_n]$$

where $(t_1, \ldots, t_n) \in \mathbb{Z}_2^n$ and $[r_0 : r_1 \cdots : r_n] \in \mathbb{R}P^n$, and we regard T^2 as the product of two $\mathbb{R}P^1$ with the standard $\mathbb{Z}_2$-actions (and we call it the standard $\mathbb{Z}_2^2$-action on T^2). Put $P(\kappa_i)$ and $P(\zeta_j)$ equivariant projective bundles over $\mathbb{R}P^2$ and T^2 with the standard $\mathbb{Z}_2^2$-actions, respectively (also see Proposition 6.1 and 6.2). Here, κ_i and ζ_j are products of k line bundles, i.e., $P(\kappa_i)$ and $P(\zeta_j)$ have the same fibre $\mathbb{R}P^{k-1}$ (for $i,\ j = 1, 2, \ldots$). Then, we have the following theorem.

Theorem 5.2. *Let $P(\xi)$ be a projective bundle over 2-dimensional small cover M. Then, $P(\xi)$ is D-J equivalent to*

$$P(\kappa_1)\sharp_{\Delta^{k-1}}\cdots\sharp_{\Delta^{k-1}}P(\kappa_{l_1})\sharp_{\Delta^{k-1}}P(\zeta_1)\sharp_{\Delta^{k-1}}\cdots\sharp_{\Delta^{k-1}}P(\zeta_{l_2}),$$

for some vector bundles $\kappa_1, \ldots, \kappa_{l_1}$ and $\zeta_1, \ldots, \zeta_{l_2}$.

Proof. Let P be the orbit polytope of M. Because $\dim M = 2$, we may assume that P is an m-gon for some $m \geq 3$, where m is the number of facets in P and we may put them $\{F_1, \ldots, F_m\}$ such that $F_i \cap F_{i+1} \neq \emptyset$ ($i = 1, \ldots, m-1$) and $F_1 \cap F_m \neq \emptyset$.

We first claim that, in m-gon for $m \geq 5$, there are two separated facets F and F' whose projective characteristic functions satisfy (4.7). Assume $m \geq 5$. Put the projective characteristic function λ_P on F_i and F_j (where $F_i \cap F_j = \emptyset$) as follows:

$$\begin{pmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{a}_j \\ \mathbf{b}_j \end{pmatrix},$$

respectively, where $\mathbf{a}_i,\ \mathbf{a}_j \in \{0,\ 1\}^2 = (\mathbb{Z}/2\mathbb{Z})^2$ and $\mathbf{b}_i,\ \mathbf{b}_j \in \{0,\ 1\}^{k-1}$. Because the projection $p_a \circ \lambda_P$ is nothing but the "ordinary" characteristic function λ on P induced from the 2-dimensional small cover M, we have that $\mathbf{a}_i \neq 0$ for all $i = 1, \ldots, m$. Therefore, we see that $\det(\mathbf{a}_i, \mathbf{a}_j) = 1$ if and only if $\mathbf{a}_i \neq \mathbf{a}_j$. If $\det(\mathbf{a}_i, \mathbf{a}_j) = 1$, we can take F_i and F_j as F and F' we want. Assume $\det(\mathbf{a}_i, \mathbf{a}_j) = 0$, i.e, $\mathbf{a}_i = \mathbf{a}_j$. Since $m \geq 5$, we may assume that the facet F_{j+1} which is next to F_j, i.e, $F_{j+1} \cap F_j \neq \emptyset$, satisfies that $F_{j+1} \cap F_i = \emptyset$. Therefore, by $\mathbf{a}_i = \mathbf{a}_j$, we have $\det(\mathbf{a}_{j+1}, \mathbf{a}_j) = \det(\mathbf{a}_{j+1}, \mathbf{a}_i) = 1$. Thus, we can take F_i and F_{j+1} as F and F' we want. This establishes the claim.

For such facets F and F', we can do $\sharp^{-1}_{\Delta^{k-1}}$, because there are two m_1-gon P_1 and m_2-gon P_2 (where $m = m_1 + m_2 - 2$) with vertices generated by two facets which have the same projective characteristic functions of F and F' (see Figure3). This implies that (P, λ_P) can be constructed

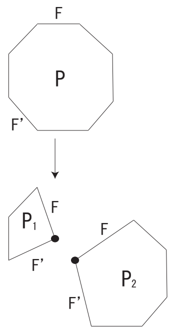

FIGURE 3. We can do $\sharp^{-1}_{\Delta^{k-1}}$ always for m-gon P ($m \geq 5$). This figure illustrates the 8-gon P decomposes into the 4-gon P_1 and the 6-gon P_2. Here, each F (resp. F') has the same projective characteristic function, and every corresponding facets also have the same projective characteristic functions.

from (P_1, λ_{P_1}) and (P_2, λ_{P_2}) by using $\sharp_{\Delta^{k-1}}$, where P is an m-gon ($m \geq 5$), P_1 is an m_1-gon and P_2 is an m_2-gon. Note that m_1 and m_2 are strictly less than m. Iterating this argument, finally we have the finite number of 3-gons and 4-gons (see Figure 4).

It is easy to see that we can not do $\sharp^{-1}_{\Delta^{k-1}}$ for 3-gons any more. However, there are two 4-gons; one can not do $\sharp^{-1}_{\Delta^{k-1}}$ (such as the left in Figure 5, because $\det(e_1, e_1) = 0$ and $\det(e_2, e_2) = 0$), and another can do $\sharp^{-1}_{\Delta^{k-1}}$ (such as the right in Figure 5). If we can do $\sharp^{-1}_{\Delta^{k-1}}$ on a 4-gon, then we get two 3-gons (see the right in Figure 5). Consequently, we get 3-gons and 4-gons which we can not do $\sharp^{-1}_{\Delta^{k-1}}$ any more from an m-gon ($m \geq 3$). We can easily classify such 3-gons and 4-gons are equivalent (i.e. up to D-J equivalence) to the characteristic functions illustrated in Figure 6. Here, the "ordinary" characteristic functions obtained from the projections $p_a \circ \lambda_P$ of the projective characteristic functions λ_P in Figure 6 are nothing but those of $\mathbb{R}P^2$ and T^2 with

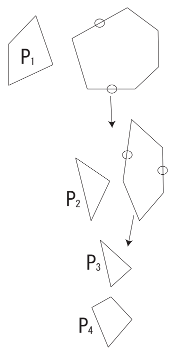

FIGURE 4. Iterating $\sharp_{\Delta^{k-1}}^{-1}$, finally, we have finite 3-gons and 4-gons; $P_1, \ldots, P_4$ in this case.

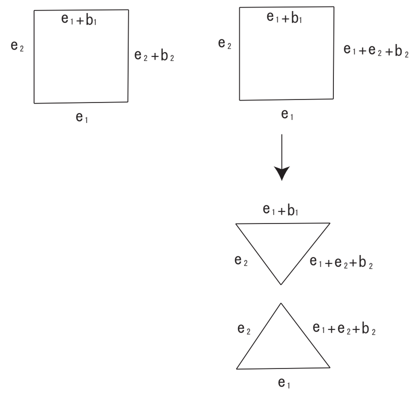

FIGURE 5. We can not do $\sharp_{\Delta^{k-1}}^{-1}$ for the left 4-gon; however, we can do $\sharp_{\Delta^{k-1}}^{-1}$ for the right 4-gon.

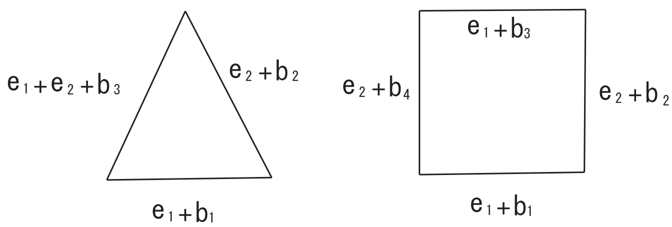

FIGURE 6. Characteristic functions on 3-gons Δ^2 and 4-gons I^2 in the final step.

the standard $\mathbb{Z}_2^2$-actions. Therefore, by using the finite times $\sharp_{\Delta^{k-1}}^{-1}$, the projective bundle over M can be decomposed into projective bundles over $\mathbb{R}P^2$ and T^2. It follows from the converse of this argument that we establish the statement of this theorem. $\square$

By using Remark 5.1 and Theorem 5.2, we have the following well-known fact.

Corollary 5.3. *Let M^2 be a 2-dimensional small cover. Then M^2 is equivariantly homeomorphic to an equivariant connected sum of finite $\mathbb{R}P^2$'s and T^2's with the standard $\mathbb{Z}_2^2$-actions.*

Remark 5.4. Recall that the one-dimensional small cover is just $\mathbb{R}P^1 \cong S^1$, and its real line bundle over can be written as the quotient space $S^1 \times_{\mathbb{Z}_2} \mathbb{R}_\alpha$ by the free $\mathbb{Z}_2$ action on S^1 and the representation $\alpha : \mathbb{Z}_2 \to \mathbb{Z}_2 \in (\mathbb{Z}/2\mathbb{Z})$ (i.e., trivial or non-trivial) and that all vector bundles over S^1 can be split into line bundles. Therefore, all projectivization of vector bundles over S^1 is homeomorphic to

$$S^1 \times_{\mathbb{Z}_2} P(\mathbb{R}_{\alpha_1} \oplus \cdots \oplus \mathbb{R}_{\alpha_{k-1}} \oplus \underline{\mathbb{R}})$$

for some vector $(\alpha_1, \ldots, \alpha_{k-1}) \in (\mathbb{Z}/2\mathbb{Z})^{k-1}$, where $S^1 \times_{\mathbb{Z}_2} \underline{\mathbb{R}} = S^1 \times \mathbb{R}$ (i.e., the trivial bundle). This implies that in the case of 1-dimensional small cover there is not the construction theorem such as Theomre 5.2 but the direct classification explained as above.

6. Topological classification of projective bundles over $\mathbb{R}P^2$ and T^2

In this section, we give the topological classification of $P(\kappa)$ and $P(\zeta)$ appeared in Theorem 5.2, i.e., the classification of the topological types of projective bundles over $\mathbb{R}P^2$ and T^2. As we assumed before, all vector bundles in this section are split into the Whitney sum of line bundles.

6.1. **Topological classification of projective bundles over $\mathbb{R}P^2$.** The classification of projective bundles over $\mathbb{R}P^2$ is known by Masuda's paper [19]. Due to [19], we have $q \equiv q'$ or $k - q'$ (mod 4) if and only if $S^2 \times_{\mathbb{Z}_2} P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(q'\gamma \oplus (k-q')\epsilon)$, where $\mathbb{Z}_2$ acts on S^2 diagonally and γ represents the tautological line bundle over $\mathbb{R}P^2$, i..e, $E(\gamma) \equiv S^2 \times_{\mathbb{Z}_2} \mathbb{R}$ such that $\mathbb{Z}_2$ acts on $\mathbb{R}$ standardly. Note that a line bundle over $\mathbb{R}P^2$ is γ or the trivial line bundle ϵ. By using this fact (and comparing the cohomology rings), we can easily check the following proposition:

Proposition 6.1. *Let $P(\kappa) \cong P(q\gamma \oplus (k-q)\epsilon)$ be a projective bundle over $\mathbb{R}P^2$. Then, it is homeomorphic to one of the following distinct manifolds.*

(1) *The case $k \equiv 0$ (mod 4):*
  (a) *if $q \equiv 0$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*
  (b) *if $q \equiv 1,\ 3$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$;*
  (c) *if $q \equiv 2$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\epsilon)$.*

(2) *The case $k \equiv 1$ (mod 4):*
  (a) *if $q \equiv 0,\ 1$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*
  (b) *if $q \equiv 2,\ 3$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\epsilon)$.*

(3) *The case $k \equiv 2$ (mod 4):*
  (a) *if $q \equiv 0,\ 2$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*
  (b) *if $q \equiv 1$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$;*
  (c) *if $q \equiv 3$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(3\gamma \oplus (k-3)\epsilon)$.*

(4) *The case $k \equiv 3$ (mod 4):*
  (a) *if $q \equiv 0,\ 3$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P^2 \times \mathbb{R}P^{k-1}$;*
  (b) *if $q \equiv 1,\ 2$ (mod 4), then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$.*

Note that the moment-angle manifold over Δ^2 is S^2.

6.2. **Topological classification of projective bundles over T^2.** Next we classify projective bundles over T^2. Let γ_i be the pull back of the canonical line bundle over S^1 by the ith factor projection $\pi_i : T^2 \to S^1$ ($i = 1,\ 2$). We can easily show that line bundles over T^2 is completely determined by its 1st Stiefel-Whitney classes via $[T^2,\ B\mathbb{Z}_2] \simeq H^1(T^2;\ \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, all of the line bundles over T^2 are ϵ, γ_1, γ_2 and $\gamma_1 \otimes \gamma_2$. By the definition of γ_i, we can easily show that

$$\gamma_i \oplus \gamma_i = \pi_i^*(\gamma \oplus \gamma) = \pi_i^*(2\epsilon) = 2\epsilon. \tag{6.1}$$

Therefore, we also have

$$(\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \gamma_2) = \gamma_1 \otimes (\gamma_2 \oplus \gamma_2) = \gamma_1 \otimes 2\epsilon = \gamma_1 \oplus \gamma_1 = 2\epsilon. \tag{6.2}$$

Let ζ be a k-dimensional vector bundle ($k \geq 2$). Because $\dim T^2 = 2$, if $k \geq 2$ then ζ is in the stable range. Therefore, we have that

$$\zeta \equiv \zeta^2 \oplus (k-2)\epsilon,$$

where ζ^2 is a 2-dimensional vector bundle over T^2. Hence, if ζ is a Whitney sum of k line bundles then ζ is isomorphic to one of the followings by computing the Stiefel-Whitney class:

$$\begin{aligned} k\epsilon;\\ \gamma_1 \oplus (k-1)\epsilon;\\ \gamma_2 \oplus (k-1)\epsilon;\\ (\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon;\\ \gamma_1 \oplus \gamma_2 \oplus (k-2)\epsilon;\\ \gamma_1 \oplus (\gamma_1 \otimes \gamma_2) \oplus (k-2)\epsilon;\\ \gamma_2 \oplus (\gamma_1 \otimes \gamma_2) \oplus (k-2)\epsilon. \end{aligned}$$

By using this classification, we can prove the following proposition.

Proposition 6.2. *Let $P(\zeta)$ be a projective bundle over T^2. Then it is homeomorphic to one of the following manifolds:*

(1) The trivial bundle $T^2 \times \mathbb{R}P(k-1)$;
(2) The non-trivial bundle of type $T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$;
(3) The non-trivial bundle of type $T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1}) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-1})$,

where $\rho_i : \mathbb{Z}_2^2 \to \mathbb{Z}_2$ is the ith projection and $\underline{\mathbb{R}}$ is the trivial representation space.

When $k > 2$, each manifold above has different topological types; however, when $k = 2$, both of two non-trivial bundles above are isomorphic to the non-trivial bundle $T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}})$.

Proof. Recall that $P(\zeta \otimes \gamma) = P(\zeta)$ for all line bundles γ. Therefore, by using the classification of vector bundles over T^2 just before this proposition and the relations (6.1), (6.2), it is easy to check that the topological types of $P(\zeta)$ are one of the followings.

(1) The case $k \equiv 0 \pmod 2$:
  (a) $P(k\epsilon) \cong T^2 \times \mathbb{R}P(k-1)$;
  (b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon) \cong P(\gamma_1 \oplus \gamma_2 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$;
  (c) $P(\gamma_1 \oplus (k-1)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;
  (d) $P(\gamma_2 \oplus (k-1)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-1})$;
(2) The case $k \equiv 1 \pmod 2$:
  (a) $P(k\epsilon) \cong T^2 \times \mathbb{R}P(k-1)$;
  (b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon) \cong P(\gamma_1 \oplus (k-1)\epsilon) \cong P(\gamma_2 \oplus (k-1)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;
  (c) $P((\gamma_1 \oplus \gamma_2) \oplus (k-2)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$.

By using the Borel-Hirzebruch formula, we have the cohomology ring of $P(\zeta)$ as the following list:

$P(\zeta)$	$H^*(\cdot)$
$T^2 \times \mathbb{R}P(k-1)$	$\mathbb{Z}/2\mathbb{Z}[x,\ y,\ z]/\langle x^2,\ y^2,\ z^k\rangle$
$T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$	$\mathbb{Z}/2\mathbb{Z}[x,\ y,\ z]/\langle x^2,\ y^2,\ z^k + z^{k-1}(x+y) + z^{k-2}xy\rangle$
$T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$	$\mathbb{Z}/2\mathbb{Z}[x,\ y,\ z]/\langle x^2,\ y^2,\ z^k + z^{k-1}x\rangle$
$T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-1})$	$\mathbb{Z}/2\mathbb{Z}[x,\ y,\ z]/\langle x^2,\ y^2,\ z^k + z^{k-1}y\rangle$

for $\deg x = \deg y = \deg z = 1$. This implies that the bundles as above are not homeomorphic to each other except (1)-(c) and (1)-(d) when $k > 2$. It is easy to check that

$$T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1}) \cong S^1 \times (S^1 \times_{\mathbb{Z}_2} P(\mathbb{R}_\rho \oplus \underline{\mathbb{R}}^{k-1})) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-1}),$$

where $S^1 \times_{\mathbb{Z}_2} \mathbb{R}_\rho$ is the canonical line bundle over $\mathbb{R}P(1)$. This establishes the statement except the case when $k = 2$.

When $k = 2$, we have that

$$T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2}) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho'} \oplus \underline{\mathbb{R}}),$$

where $\rho' : T^2 \to S^1$ is the representation $(t_1, t_2) \mapsto t_1 t_2$. By using the kernel of this representation $\Delta = \{(t, t^{-1}) \mid t \in S^1\}$, we also have the following homeomorphism:

$$T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho'} \oplus \underline{\mathbb{R}}) \cong \Delta \times (S^1 \times_{\mathbb{Z}_2} P(\mathbb{R}_\rho \oplus \underline{\mathbb{R}})) \cong S^1 \times (S^1 \times_{\mathbb{Z}_2} P(\mathbb{R}_\rho \oplus \underline{\mathbb{R}})),$$

where $S^1 \times_{\mathbb{Z}_2} \mathbb{R}_\rho$ is the canonical line bundle over $\mathbb{R}P(1)$. Similarly, we have the following homeomorphisms:

$$T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}) \cong S^1 \times (S^1 \times_{\mathbb{Z}_2} P(\mathbb{R}_\rho \oplus \underline{\mathbb{R}})) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}).$$

This also establishes the case when $k = 2$. □

Note that the moment-angle manifold over $\Delta^1 \times \Delta^1 (= I^2)$ is T^2 itself.

It also follows from the proof of Proposition 6.2 that the following corollary holds.

Corollary 6.3. *Let $\mathcal{P}(T^2)$ be the set of all projective bundles over T^2 and $P(\zeta_1),\ P(\zeta_2) \in \mathcal{P}(T^2)$. Then, $H^*(P(\zeta_1)) \simeq H^*(P(\zeta_2))$ if and only if $P(\zeta_1) \cong P(\zeta_2)$ (homeomorphic), i.e., $\mathcal{P}(T^2)$ satisfies cohomological rigidity.*

Remark 6.4. Let $\mathcal{P}(\mathbb{R}P^2)$ be the set of all projective bundles over $\mathbb{R}P^2$. Due to [19, Theorem 3.3], $\mathcal{P}(\mathbb{R}P^2)$ also satisfies cohomological rigidity.

7. Bundle triviality of some projective bundles over real projective spaces

In this final section, we shall prove the following theorem:

Theorem 7.1. *The projectivization $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is diffeomorphic to $\mathbb{R}P^n \times \mathbb{R}P^n$ if and only if $n = 0,\ 2$ or 6.*

In order to prove Theorem 7.1, we first show when cohomology ring of $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is isomorphic to that of $\mathbb{R}P^n \times \mathbb{R}P^n$:

Lemma 7.2. *The $\mathbb{Z}/2\mathbb{Z}$-cohomology ring of $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is isomorphic to that of $\mathbb{R}P^n \times \mathbb{R}P^n$ if and only if $n + 2 = 2^r$ for some $r \in \mathbb{N}$.*

Proof. Because of (1.1), $\gamma \oplus \tau_{\mathbb{R}P^n} \oplus \epsilon = (n+2)\gamma$. Therefore, we have that

$$\omega(\gamma \oplus \tau_{\mathbb{R}P^n}) = (1+x)^{n+2} \equiv \sum_{i=0}^{n} \binom{n+2}{i} x^i$$

for $x \in H^1(\mathbb{R}P^n)$. Together with the Borel-Hirzebruch formula, we see that the cohomology ring of $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is as follows:

$$H^*(P(\gamma \oplus \tau_{\mathbb{R}P^n})) \simeq \mathbb{Z}/2\mathbb{Z}[x, y]/\langle x^{n+1}, Y\rangle.$$

Here,

$$Y = \sum_{i=0}^{n} \binom{n+2}{i} y^{n+1-i} x^i$$

Note that $n + 2 = 2^r$ if and only if $Y = y^{n+1}$ (e.g. see [20, Corollary 4.6]). Therefore, if $n+2 = 2^r$ then $Y = y^{n+1}$ and the cohomology ring is isomorphic to $H^*(\mathbb{R}P^n \times \mathbb{R}P^n)$. On the other hand, if the cohomology ring is isomorphic to $H^*(\mathbb{R}P^n \times \mathbb{R}P^n)$, then it is easy to check that Y must be y^{n+1} or $(x+y)^{n+1}$. However, if $Y = (x+y)^{n+1}$ then

$$Y = \sum_{i=0}^{n} \binom{n+2}{i} y^{n+1-i} x^i = \sum_{i=0}^{n} \binom{n+1}{i} y^{n+1-i} x^i.$$

This gives a contradiction. Therefore, $Y = y^{n+1}$ and $n + 2 = 2^r$. This establishes the statement of this lemma. □

Lemma 7.2 tells us that if $n + 2 \neq 2^r$ for all $r \in \mathbb{N}$ then $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is not homeomorphic to $\mathbb{R}P^n \times \mathbb{R}P^n$.

Assume $n + 2 = 2^r$ for some $r \in \mathbb{N}$. If $r = 1$ then $n = 0$, so this case is the trivial case. We may assume $r \geq 2$.

7.1. **“if” part of Theorem 7.1.** We next discuss when $\gamma \oplus \tau_{\mathbb{R}P^n}$ is the trivial bundle. To show this, we need the fact about the stable KO group in [1] (also see [19]). Before we state Lemma 7.4, we need to prepare some notation. Let $k(2^r-1) = \#\{s \in \mathbb{N} \mid 0 < s \leq 2^r-2, s \equiv 0,1,2,4 \ (mod\ 8)\}$. For example, $k(3) = 2$ when $r = 2$, $k(7) = 3$ when $r = 3$, $k(15) = 7$ when $r = 4$, $k(31) = 15$ when $r = 5$, e.t.c. We have the following lemma:

Lemma 7.3. *If $r = 2$ or 3, then $k(2^r - 1) = r$. If $r \geq 4$, then $k(2^r - 1) = 2^{r-1} - 1$.*

Proof. The first statement is easy. The 2nd statement is proved by induction. When $r = 4$, then $k(15) = 7$. Assume the statement is true until $r - 1$, i.e., $k(2^{r-1} - 1) = 2^{r-2} - 1$. Because of the definition of $k(2^r - 1)$, the number of s such that $0 < s \leq 2^r - 2$ and $s \equiv 0, 1, 2, 4 \pmod 8$ is

$$k(2^r - 1) = (2^{r-2} - 1) + 4 \cdot 2^{r-4} = 2^{r-1} - 1.$$

This establishes the statement. □

Together with the stable KO group of real projective space proved in [1], we have the following lemma:

Lemma 7.4. *When $r = 2, 3$, $\widetilde{KO}(\mathbb{R}P^{2^r-2})$ is a cyclic group generated by $\gamma - \epsilon$ with order $4, 8$, respectively. When $r \geq 4$, $\widetilde{KO}(\mathbb{R}P^{2^r-2})$ is a cyclic group generated by $\gamma - \epsilon$ with order $2^{(2^{r-1}-1)}$.*

Note that $\gamma \oplus \tau_{\mathbb{R}P^n}$ is in the stable range, i.e., the dimension of fibre is strictly greater than n. Because of the stable range theorem (i.e., for vector bundles κ and η in the stable range, $\kappa \oplus \epsilon^a \equiv \eta \oplus \epsilon^a$ if and only if $\kappa \equiv \eta$, see [11, Chapter 9]), $\gamma \oplus \tau_{\mathbb{R}P^n}$ is the trivial bundle if and only if it is the trivial bundle in $\widetilde{KO}(\mathbb{R}P^n)$. By this fact, we have the following proposition:

Lemma 7.5. *Assume $n = 2^r - 2$. Then $\gamma \oplus \tau_{\mathbb{R}P^n} \equiv (n + 1)\epsilon$ if and only if $n = 2, 6$.*

Proof. By using (1.1), we have that

$$\gamma \oplus \tau_{\mathbb{R}P^n} \oplus \epsilon \equiv 2^r\gamma. \tag{7.1}$$

It follows from Lemma 7.4 that when $r \geq 4$

$$2^{(2^{r-1}-1)}\gamma \equiv 2^{(2^{r-1}-1)}\epsilon. \tag{7.2}$$

Because $r < 2^{r-1} - 1$, together with (7.1), this case is not the trivial bundle. On the other hand, when $r = 2, 3$, we have that

$$2^{(2^{r-1}-1)}\gamma = 2^r\gamma \equiv 2^{(2^{r-1}-1)}\epsilon = 2^r\epsilon.$$

Therefore, by (7.1) and the stable range theorem, $\gamma \oplus \tau_{\mathbb{R}P^n}$ is the trivial bundle. This establishes the statement. □

Hence, by Lemma 7.5, the projectivization $P(\gamma \oplus \tau_{\mathbb{R}P^n})$ is the trivial bundle when $n = 2, 6$. This establishes the “if” part of Theorem 7.1.

7.2. **“only if” part of Theorem 7.1.** We next prove the “only if” part of Theorem 7.1. The idea of this proof is based on the idea of the proof of Theorem 3.2 in [19]. Assume that there exists the following diffeomorphism:

$$f : P = P(\gamma \oplus \tau_{\mathbb{R}P^n}) \to \mathbb{R}P^n \times \mathbb{R}P^n (= P((n + 1)\epsilon)) = T,$$

and we put the projections to the 1st and 2nd factor by $\pi_1 : P \to \mathbb{R}P^n$, $\pi_2 : T \to \mathbb{R}P^n$, respectively. Now $f^*(\tau_T) = \tau_P$ in $\widetilde{KO}(P)$. Recall the following theorem proved in [19, Lemma 3.1]:

Lemma 7.6. *Let $E \to X$ be a real smooth vector bundle over a smooth manifold X. Let $\pi : P(E) \to X$ be its projectivization and η be the tautological real line bundle of $P(E)$. Then the tangent bundle $\tau_{P(E)}$ of $P(E)$ with ϵ^1 added is isomorphic to $\mathrm{Hom}(\eta, \pi^*E) \oplus \pi^*\tau_X$.*

By this lemma and (1.1), we have that

$$\tau_P \oplus \epsilon^1 \oplus \epsilon^1 \equiv \mathrm{Hom}(\eta_P, \pi_1^*(\gamma \oplus \tau_{\mathbb{R}P^n})) \oplus \pi_1^*\tau_{\mathbb{R}P^n} \oplus \epsilon^1 \equiv \mathrm{Hom}(\eta_P, \gamma_P \oplus \pi_1^*\tau_{\mathbb{R}P^n}) \oplus (n+1)\gamma_P$$

and

$$\tau_T \oplus \epsilon^1 \oplus \epsilon^1 \equiv \mathrm{Hom}(\eta_T, \pi_2^*((n+1)\epsilon)) \oplus \pi_2^*\tau_{\mathbb{R}P^n} \oplus \epsilon^1 \equiv \mathrm{Hom}(\eta_T, (n+1)\epsilon) \oplus (n+1)\gamma_T,$$

where $\gamma_P = \pi_1^*\gamma$ and $\gamma_T = \pi_2^*\gamma$ for the tautological line bundle γ over $\mathbb{R}P^n$, and η_P and η_T are the tautological line bundles over $P = P(\gamma \oplus \tau_{\mathbb{R}P^n})$ and $T = P(\epsilon^{n+1})$, respectively. Together with $f^*\tau_T = \tau_P$, we have the following isomorphism:

$$f^*\left(\mathrm{Hom}(\eta_T, (n+1)\epsilon) \oplus (n+1)\gamma_T\right) \equiv \mathrm{Hom}(\eta_P, \gamma_P \oplus \pi_1^*\tau_{\mathbb{R}P^n}) \oplus (n+1)\gamma_P \tag{7.3}$$

By the cohomology ring computed in Lemma 7.2, $f^*w_1(\gamma_T) = w_1(\gamma_P)$, i.e., $f^*\gamma_T = \gamma_P$. Therefore, by (7.3), in $\widetilde{KO}(P)$ we have

$$\mathrm{Hom}(f^*\eta_T, (n+1)\epsilon) \equiv \mathrm{Hom}(\eta_P, \gamma_P \oplus \pi_1^*\tau_{\mathbb{R}P^n}). \tag{7.4}$$

By taking the zero section to $\tau_{\mathbb{R}P^n}$, we have the cross section σ of $\pi_1 : P \to \mathbb{R}P^n$. The induced homomorphism of $\sigma^* : \widetilde{KO}(P) \to \widetilde{KO}(\mathbb{R}P^n)$ sends this identity (7.4) to $\widetilde{KO}(\mathbb{R}P^n)$. Because $\sigma^*\eta_P$ is the trivial bundle over $\mathbb{R}P^n$, we have that

$$\mathrm{Hom}(\sigma^*f^*\eta_T, (n+1)\epsilon) \equiv \mathrm{Hom}(\epsilon, \gamma \oplus \tau_{\mathbb{R}P^n}) \equiv \gamma \oplus \tau_{\mathbb{R}P^n}. \tag{7.5}$$

Now, by the cohomology ring computed in Lemma 7.2 again, we also have the two cases $f^*w_1(\eta_T) = w_1(\eta_P)$ and $w_1(\gamma_P) + w_1(\eta_P)$; these correspond to $f^*\eta_T = \eta_P$ and $\gamma_P \otimes \eta_P$, respectively. If $f^*\eta_T = \eta_P$, then by (7.5), we have that

$$(n+1)\epsilon \equiv \gamma \oplus \tau_{\mathbb{R}P^n}$$

in $\widetilde{KO}(\mathbb{R}P^n)$. By Lemma 7.5, such case is the only $n = 2$ or 6. If $f^*\eta_T = \gamma_P \otimes \eta_P$, then

$$\sigma^*f^*\eta_T = \sigma^*\gamma_P \otimes \sigma^*\eta_P \equiv \gamma \otimes \epsilon \equiv \gamma.$$

Therefore, by (7.5), we have that $(n+1)\gamma \equiv \gamma \oplus \tau_{\mathbb{R}P^n}$. By taking the tensor of γ, we also have that

$$(n+1)\epsilon \equiv \epsilon \oplus (\gamma \otimes \tau_{\mathbb{R}P^n}). \tag{7.6}$$

Because $\gamma \oplus (\gamma \otimes \tau_{\mathbb{R}P^n}) \equiv (n+1)\epsilon$, the vector bundle $\gamma \otimes \tau_{\mathbb{R}P^n}$ is the normal bundle $\gamma^\perp$ of γ in $(n+1)\epsilon$. Therefore, the Stiefel-Whitney class satisfies

$$w(\gamma \otimes \tau_{\mathbb{R}P^n}) = 1 + x + \cdots + x^n.$$

Hence, by (7.6), such case is just $n = 0$. Because this case is the trivial case, we establish the "only if" part.

Finally, we ask the following general question by motivating the above fact.

Problem 3 (the projective bundle triviality problem)**.** Let ξ be a rank k vector bundle over a smooth manifold M. When is its projectivization $P(\xi)$ diffeomorphic to $\mathbb{R}P^{k-1} \times M$?

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