

A TOPOLOGICAL DEFINITION OF HYPERTORIC MANIFOLDS AND ITS EQUIVARIANT COHOMOLOGY

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1. INTRODUCTION

In [2], Bielawski and Dancer defined the hyperKähler analogue of symplectic toric manifolds, called a *toric hyperKähler manifold* or *hypertoric manifold* (also see [1, 6, 7, 8, 9, 10]). On the other hand, there is the topological definition of toric manifolds introduced by Davis-Januszkiewicz [4], called a *quasitoric manifold* (also see [3, 5]).

In this article, we give a constructive definition of a *topological hypertoric manifold* (see Section 2), and compute its equivariant cohomology (see Theorem 3.2).

The details of this article will be appeared in the forthcoming paper [13].

2. DEFINITION

Let T^n be the n -dimensional compact abelian group, i.e., n -dimensional torus, and $\mathbb{R}^n \times \mathbb{C}^n$ be the product of n -dimensional real and complex Euclidean spaces. We let $\mathcal{S} = \{S_1, \dots, S_m\}$ denote a hypersurface arrangement in \mathbb{R}^n , i.e., each $S_i \in \mathcal{S}$, $i = 1, \dots, m$, is a smooth surface in \mathbb{R}^n which is diffeomorphic to \mathbb{R}^{n-1} . Note that $S_i \in \mathcal{S}$ might not be the affine space in \mathbb{R}^n . Assume that if $\cap_{i \in I} S_i \neq \emptyset$ then $\cap_{i \in I} S_i \cong \mathbb{R}^{n-|I|}$, where $|I|$ is the cardinality of $I \subset [m]$.

We attach a label to each of S_i 's by using the *characteristic function*

$$\lambda : \mathcal{S} \rightarrow \mathbb{Z}^n$$

such that

$$\det(\lambda(S_{i_1}) \cdots \lambda(S_{i_n})) = \pm 1 \quad \text{when } \cap_{j=1}^n S_{i_j} \neq \emptyset.$$

Now we may define the space M as follows:

$$M = T^n \times \mathbb{R}^n \times \mathbb{C}^n / \sim_\lambda.$$

Here, the equivalence relation \sim_λ is defined by the following relations:

$$\begin{aligned} (t_1, r, z) \sim_\lambda (t_2, r, z) &\Leftrightarrow t_1^{-1}t_2 \in T_{\lambda(S)} \quad \text{if } (r, z) \in S \times \lambda(S)^\perp; \\ (t_1, r, z) \sim_\lambda (t_2, r, z) &\Leftrightarrow t_1 = t_2 \quad \text{otherwise,} \end{aligned}$$

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where $T_{\lambda(S)}$ is the connected 1-dimensional subgroup in T^n whose Lie algebra is generated by $\lambda(S) \in \mathbb{Z}^n = \mathfrak{t}_{\mathbb{Z}}^n$ for $S \in \mathcal{S}$, and $\lambda(S)^\perp$ is the normal space of the vector $\lambda(S)$ in \mathbb{C}^n with respect to the standard Hermitian inner product on \mathbb{C}^n , i.e., the inner product $\langle z, w \rangle = \sum_{i=1}^n z_i \overline{w_i}$ for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. Note that $\lambda(S)^\perp$ is a complex $(n-1)$ -dimensional linear subspace in \mathbb{C}^n .

Then we have the following theorem.

Theorem 2.1. *For the space M defined above, the following statements hold.*

- (1) *M is a $4n$ -dimensional non-compact manifold.*
- (2) *M has the natural T^n -action on the T^n -factor in $T^n \times \mathbb{R}^n \times \mathbb{C}^n$.*
- (3) *M has the extra S^1 -action on the \mathbb{C}^n -factor in $T^n \times \mathbb{R}^n \times \mathbb{C}^n$ by the scalar product.*

We call M defined as above a *topological hypertoric manifold*.

Remark 2.2. Let X be a hypertoric manifold. Then, X satisfies the above conditions, i.e., X is a topological hypertoric manifold. Moreover, in this case, all S_i 's are codimension-one affine subspaces in \mathbb{R}^n .

Remark 2.3. For all $2n$ -dimensional quasitoric manifolds V , there is a $4n$ -dimensional topological hypertoric manifold M such that M is T^n -homotopy equivalent to V .

It follows from Remark 2.2 and 2.3 that the set of all topological hypertoric manifolds is strictly wider than that of all hypertoric manifolds.

3. EQUIVARIANT COHOMOLOGY

In order to compute $T^n \times S^1$ -equivariant cohomologies of topological hypertoric manifolds M , we first prepare the following lemma.

Lemma 3.1. *A topological hypertoric manifold with the extra S^1 -action $(M^{4n}, T^n \times S^1)$ is a GKM manifold whose GKM graph is a $2n$ -valent hypertorus graph.*

Furthermore, the hypertorus graph defined by $(M^{4n}, T^n \times S^1)$ satisfies the following conditions:

- (1) *For each $L \in \mathcal{L}$, there is the unique pair of a hyperfacet H and its opposite side \overline{H} such that $H \cap \overline{H} = L$;*
- (2) *For each subset $\mathcal{L}' \subset \mathcal{L}$, the intersection of all elements in \mathcal{L}' is empty or connected.*

where $\mathcal{L} = \{L_1, \dots, L_m\}$ is the set of all $(2n-2)$ -valent hypertorus subgraphs.

Here, a *hypertorus graph* is the special type of GKM graphs defined in [12] (also see [11]), and all notations appeared in Lemma 3.1 are defined in [12].

Using Lemma 3.1 and the main theorem in [12], we have the following theorem.

Theorem 3.2. *Let $(M^{4n}, T^n \times S^1)$ be the topological hypertoric manifold defined by the labeled, hypersurface arrangement $\mathcal{S} = \{S_1, \dots, S_m\}$ in \mathbb{R}^n . Then its equivariant cohomology $H_{T^n \times S^1}(M; \mathbb{Z})$ is isomorphic to the following ring:*

$$\mathbb{Z}[\tau_1, \dots, \tau_m, \tau_{m+1}, \dots, \tau_{2m}, \chi] / \mathcal{I},$$

where $\deg \tau_i = \deg \chi = 2$, $i = 1, \dots, 2m$, and the ideal \mathcal{I} is generated by the following homogeneous polynomials:

$$\begin{aligned} & \tau_i + \tau_{m+i} - \chi \quad \text{for all } i = 1, \dots, m; \\ & \prod_{i \in I} \tau_i \quad \text{for } \cap_{i \in I} H_i = \emptyset \quad (I \subset [2m]). \end{aligned}$$

Here, H_j in Theorem 3.2 is defined as follows. Now \mathbb{R}^n can be divided into two half (topological) spaces by S_j , $j = 1, \dots, m$; we denote one of them and another by H_j and H_{j+m} , respectively. That is, H_j and H_{j+m} are diffeomorphic to the half spaces \mathbb{R}_+^n and \mathbb{R}_-^n (respectively) in \mathbb{R}^n ; moreover, they satisfy that $H_j \cup H_{j+m} = \mathbb{R}^n$ and $H_j \cap H_{j+m} = S_j$.

See [13] for further studies.

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