A necessary and sufficient condition for the extension of an axial function of GKM graph

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Torus Actions in Geometry, Topology, and Applications
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GKM mfd [Guillemin-Zara, 2000~]

\((M^{2m}, T^n)\) (almost cpx) is a GKM mfd if \(M^T \neq \emptyset\) and \(\dim M_1 = 2\) \((M_1/T\) is a graph), where \(M_1 = \{x \in M \mid \dim T(x) \leq 1\}\).

E.g., Toric mfds, \(G_2/SU(3) \simeq S^6\), \(SU(n + 1)/T^n \simeq FL(C^{n+1})\).

**Figure:** GKM graphs induced from \((S^6, T^2)\) and \((FL(C^3), T^2)\).
GKM mfd [Guillemin-Zara, 2000∼]

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where $M_1 = \{ x \in M \mid \dim T(x) \leq 1 \}$.

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\[ \begin{array}{c}
\alpha \\
-\alpha \\
-\alpha - \beta \\
\beta \\
\alpha + \beta \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\alpha \\
-\alpha \\
-\alpha - \beta \\
-\beta \\
-\alpha - \beta \\
\alpha + \beta \\
\beta \\
\alpha \\
\alpha + \beta \\
\end{array} \]

\textbf{Figure:} GKM graphs induced from $(S^6, T^2)$ and $(FL(\mathbb{C}^3), T^2)$.

\textbf{Problem}

\textit{When does $(M^{2m}, T^n)$ extend to $(M^{2m}, T^\ell)$? ($n \leq \ell \leq m$)}
(Abstract) GKM graph

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an $m$-valent graph, i.e., $\#E_p(\Gamma) = m$ for all $p \in V(\Gamma)$.

![Figure: Two 3-valent graphs and one 4-valent graph.](image)
(Abstract) GKM graph

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**Figure:** Two 3-valent graphs and one 4-valent graph.

**Definition**

A GKM graph is a labelled graph $(\Gamma, A)$, where a label $A : E(\Gamma) \rightarrow H^2(BT^n) \cong \mathbb{Z}^n$ (for $1 \leq n \leq m$) satisfies the following conditions:
Axial function $\mathcal{A}$ (I)

$\mathcal{A} : E(\Gamma) \to H^2(BT^n) \cong \mathbb{Z}^n$ (called axial function) satisfies the following 3 conditions:

1. $\mathcal{A}(pq) = -\mathcal{A}(qp)$

2. $\{\mathcal{A}(e) \mid e \in E_p(\Gamma)\}$ spans $\mathbb{Z}^n$ and pairwise linearly independent

where $H^2(BT^3) = \langle \alpha, \beta, \gamma \rangle$. 
Axial function $A$ (II) and Examples

(3) $\forall pq \in E(\Gamma), \exists$ a bijection $\nabla_{pq} : E_p(\Gamma) \to E_q(\Gamma)$ which satisfies $\forall e \in E_p(\Gamma), \exists c_{pq}(e) \in \mathbb{Z}$ s.t. $A(\nabla_{pq}(e)) - A(e) = c_{pq}(e)A(pq)$. ($\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ is called a connection)
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Example
Problem from comb. point of view

Let \((\Gamma, A)\) be an \(m\)-valent GKM graph.

Problem (Combinatorial interpretation of \((M^{2m}, T^n) \Rightarrow (M^{2m}, T^\ell)\))

When does \(A : E(\Gamma) \to H^2(BT^n)\) \(((m, n)\text{-type})\) extend to \(\tilde{A} : E(\Gamma) \to H^2(BT^\ell)\) \(((m, \ell)\text{-type})\)?

Here, \(n \leq \ell \leq m\).

Example
Key fact

Let \((\Gamma, \tilde{\mathcal{A}})\) be an \((m, \ell)\)-type extension of \((m, n)\)-type \((\Gamma, \mathcal{A})\).

**KEY FACT [Shunji Takuma]**

The integer \(c_{pq}(e)\) of the condition (3) does **NOT** change! Namely, 
\[\forall e \in E_p(\Gamma), \exists c_{pq}(e) \in \mathbb{Z} \text{ s.t.}\]

\[\mathcal{A}(\nabla_{pq}(e)) - \mathcal{A}(e) = c_{pq}(e)\mathcal{A}(pq) \quad \text{for } (\Gamma, \mathcal{A}),\]

\[\tilde{\mathcal{A}}(\nabla_{pq}(e)) - \tilde{\mathcal{A}}(e) = c_{pq}(e)\tilde{\mathcal{A}}(pq) \quad \text{for } (\Gamma, \tilde{\mathcal{A}}).\]
**Key fact**

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The integer \(c_{pq}(e)\) of the condition (3) does NOT change! Namely, \(\forall e \in E_p(\Gamma), \exists c_{pq}(e) \in \mathbb{Z}\) s.t.

\[
A(\nabla_{pq}(e)) - A(e) = c_{pq}(e)A(pq) \quad \text{for} \quad (\Gamma, A),
\]

\[
\tilde{A}(\nabla_{pq}(e)) - \tilde{A}(e) = c_{pq}(e)\tilde{A}(pq) \quad \text{for} \quad (\Gamma, \tilde{A}).
\]

Thus, the map

\[
c_{(\Gamma, A)} : E(\Gamma) \rightarrow \mathbb{Z}^m \quad \text{s.t.} \quad c_{(\Gamma, A)}(pq) = (c_{pq}(e_1), \ldots, c_{pq}(e_m))
\]

is invariant under the extension! (where \(E_p(\Gamma) = \{e_1, \ldots, e_m\}\))
Example \((c_{\Gamma,A} : E(\Gamma) \to \mathbb{Z}^m)\)

Let \((\Gamma, A)\) be the following \((3, 2)\)-type GKM graph.

Then, the map \(c_{\Gamma,A} : E(\Gamma) \to \mathbb{Z}^3\) is as follows:
Necessary and sufficient condition

IDEA and definition

By defining a sheaf on \((\Gamma, \mathcal{A})\) from \(c_{(\Gamma, \mathcal{A})} : E(\Gamma) \to \mathbb{Z}^m\) and taking its (modified) global sections (in the sense of Braden-MacPherson), we obtain the following \(\mathbb{Z}\)-module from \(c_{(\Gamma, \mathcal{A})} : E(\Gamma) \to \mathbb{Z}^m\):

\[
\mathcal{O}(\Gamma, \mathcal{A}) = \{ f : V(\Gamma) \to \mathbb{Z}^m \mid \nabla_{pq}(f_p) - f_q = f_q(qp)c_{(\Gamma, \mathcal{A})}(qp) \}
\]

where \(f(p) = f_p \in \mathbb{Z}^m = \mathbb{Z}E_p(\Gamma)\) and \(f_q(qp) \in \mathbb{Z}\) is an integer corresponding to the edge \(qp\).
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Theorem (K)

1. \(\mathcal{O}(\Gamma, \mathcal{A})\) is a free \(\mathbb{Z}\)-module with \(n \leq \text{rk}\mathcal{O}(\Gamma, \mathcal{A}) \leq m\);

2. \(\exists\) an \((m, \ell)\)-type extension \(\iff\) \(\ell \leq \text{rk}\mathcal{O}(\Gamma, \mathcal{A})\).
Example \((\mathcal{O}(\Gamma, \mathcal{A}))\)

Let \((\Gamma, \mathcal{A})\) be the following \((3, 2)\)-type GKM graph.

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Example \( \mathcal{O}(\Gamma, A) \)

So, we have

\[
\mathcal{O}(\Gamma, A) = \{ f : \{p, q\} \to \mathbb{Z}^3 | \nabla_{e_i}(f_p) - f_q = f_q(\overline{e_i})c_{(\Gamma, A)}(\overline{e_i}) \} = \{ (f_p, f_q) = ((x, y, z), (-x, -y, -z)) | x + y + z = 0 \} \simeq \mathbb{Z}^2.
\]

Thus, \( \text{rk}\mathcal{O}(\Gamma, A) = 2(< 3) \).

\( \therefore \) \( \nexists \) \( (3, 3) \)-extensions! (Hence, \( (S^6, T^2) \not\cong (S^6, T^3) \))

Remark \( \mathcal{O}(\Gamma, A) \) is generated by

\[\begin{align*}
x & \quad e_1 & -x \\
-x & x & e_2 \\
0 & 0 & e_3
\end{align*}\]
Application (the maximal effective $T$-action on $\mathcal{F}l(\mathbb{C}^{n+1})$)

Let $\mathcal{F}l(\mathbb{C}^{n+1})$ be the flag mfd of type A ($\simeq SU(n + 1)/T^n$). More precisely, $\mathcal{F}l(\mathbb{C}^{n+1})$ is the set of full flags:

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1},$$

where $V_k$ is a $k$-dim subspace in $\mathbb{C}^{n+1}$.

Define an effective $T^n$-action on $\mathcal{F}l(\mathbb{C}^{n+1})$ by acting on the first $n$ coordinates.

Using our theorem, we can prove the following proposition:

**Proposition**

*This $T^n$-action is maximal, i.e., there are no effective $T^{n+1}$-actions on $\mathcal{F}l(\mathbb{C}^{n+1})$ extending this $T^n$-action.*
Thank you for your attention!