Algebraic PL-Invariants and Cluster Algebras

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The 43rd Symposium on Transformation Groups

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Joint work with Zhi Lü

November 18, 2016

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PL-Manifolds

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A *PL-sphere* is a triangulated sphere which is PL-homeomorphic to the boundary of a simplex.

- In dim ≤ 3, any triangulated sphere is PL.
- In dim $= 4$, the question is open.
- In dim ≥ 5, there exist non–PL-sphere triangulations.

PL-Manifolds

All manifolds are assumed to be connected, closed and oriented.

Let $\alpha \in K$. Then $\mathrm{lk}_K \alpha = \{\alpha' \in K \mid \alpha \cup \alpha' \in K,\ \alpha \cap \alpha' = \emptyset\}$.

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- In dim ≤ 3, Smooth $=$ PL $=$ Top
- In dim $= 4$, Smooth $=$ PL $\subset$ Top
- In dim ≥ 4, Smooth $\subset$ PL $\subset$ Top

PL-Spheres

A PL-sphere is not the same as a *PL-manifold homeomorphic to a sphere* but to the 'standard' sphere, i.e. the PL-structure is given by the boundary of a simplex. These notions coincide in dimensions other than 4.

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Problem

Is a PL-structure on S^4 unique?

Bistellar Moves

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A 0-move adds a vertex and an n-move deletes one. All other moves don't change the number of vertices.
Let $f = (f_0, f_1, \ldots, f_n)$ be the f-vector of K. Then

Lemma

$$f(bm_\alpha K) = f(K) \iff n = 2j.$$

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Theorem (Pachner '87)

Two PL-manifolds are bistellarly equivalent if and only if they are PL-homeomorphic.

Cluster Algebras

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- a distinguished family of generators X called *cluster variables*;
- such that X is grouped into overlapping subsets called *clusters* that all have cardinality n.

$$\mathbf{x}_1 = \{x_1^1, \dots, x_n^1\}, \quad \mathbf{x}_2 = \{x_1^2, \dots, x_n^2\}, \dots$$

$$X = \bigcup \mathbf{x}_i \qquad \text{non-disjoint union}$$

Exchange Property

The clusters have the following *exchange property*:

For every cluster $\mathbf{x}$ and $x \in \mathbf{x}$, there exists another cluster $\mathbf{x}'$ and $x' \in \mathbf{x}'$ such that

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where x and x' are related by the *exchange relation*

$$xx' = M + M',$$

where M, M' are monomials without common divisors in the variables $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} \setminus \{x\}$.

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Furthermore, any two clusters can be obtained from each other by a sequence of exchanges.

Rank 1

Let $\mathcal{A} = \mathbb{C}[SL_2] = \mathbb{C}[a, b, c, d]/(ad - bc - 1)$ be the coordiante ring, where we write an element of SL_2 as

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Consider a, d as cluster variables and b, c as scalers. Then we just have two clusters $\{a\}, \{d\}$ and $\mathcal{A}$ is the algebra over $\mathbb{C}[b, c]$ generated by cluster variables a, d subject to the exchange relation

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which we can write as the ring $\mathbb{C}[b, c][a, a^{-1}]$ of Laurent polynomials.

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The cluster variables are elements x_m, $m \in \mathbb{Z}$, defined recursively by the exchange relations:

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1, & \text{if } m \text{ is odd;} \\ x_m^c + 1, & \text{if } m \text{ is even.} \end{cases}$$

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Each x_m is then a rational function of x_1 and x_2.

Rank 2

So $\mathcal{A}(b, c)$ is the subring generated by x_m, $m \in \mathbb{Z}$, inside the *field of rational functions*

$$\mathbb{Q}(x_1, x_2) := \{\frac{f(x_1, x_2)}{g(x_1 . x_2)} \mid f, g \text{ are polynomials such that } g \neq 0\}.$$

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The clusters are pairs $\{x_m, x_{m+1}\}$, $m \in \mathbb{Z}$.

Starting with $\{x_1, x_2\}$ we can reach any other cluster by a series of exchanges

$$\cdots \longleftrightarrow \{x_0, x_1\} \longleftrightarrow \{x_1, x_2\} \longleftrightarrow \{x_2, x_3\} \longleftrightarrow \cdots$$

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Each cluster variable x_k can be exchanged from $\mathbf{x}$ to form a new cluster $\mathbf{x}' = (\mathbf{x} \setminus \{x_k\}) \cup \{x_k'\}$, where x_k and x_k' are related by an *exchange relation*

$$x_k x_k' = M + M',$$

where M, M' are disjoint monomials in the variables $\mathbf{x} \cap \mathbf{x}'$.

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The exponents in M and M' are encoded in an $(n \times n)-$integer matrix $B = (b_{ij})$ (usually skew-symmetric) called the *exchange matrix*.

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A pair $(\mathbf{x}, B)$ is called a *seed*, which can be realised as a *quiver*.

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For each index k we can extend $\mathbf{x} \mapsto \mathbf{x}'$ to

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called a *seed mutation in direction* k, where we use the *matrix mutation* $\mu_k \colon B \to B' = (b'_{ij})$ given by

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & \text{otherwise.} \end{cases}$$

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Note that $\mu_k(\mu_k(B)) = B$.

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Theorem (Fomin, Zelevinsky 2003)

Cluster algebras of finite type can be classified in terms of the Dynkin diagrams of finite-dimensional simple Lie algebras.

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- One of the main consequences of the definition of matrix mutations.
- Every cluster variable (a rational function in the elements of a given cluster) is a Laurent polynomial with integer coefficients.
- Surprising since every cluster variable appears as the denominator of the expression used for producing a new one.
- Conjecture: All coefficients in these Laurent polynomials are positive.

Triangulated Surfaces

Suppose we have a pair (S, M) where S is a closed oriented connected surface and M is a set of marked points on S with $|M| = m$. We want to consider all triangulations K of S with m vertices as the marked points.

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Then

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Then

$$K \cong K' \iff \text{related by a finite sequence of bistellar 1-moves.}$$

Given (S, M) we can form a cluster algebra $\mathcal{A}(S, M)$.

Triangulated Surfaces

The set of cluster variables X is the set of potential edges of K. Every triangulation K will have

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edges, where g is the genus of S. These will form our clusters.

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edges, where g is the genus of S. These will form our clusters.

For every triangulation K we form the $(n \times n)$-exchange matrix B as follows:

$$B(K) = B = \sum_{\Delta \in S} B^{\Delta},$$

where Δ is a 2-simplex of S.

Triangulated Surfaces

Using the following matrix mutation $B' = \mu_k(B)$ in direction k:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ of } j = k; \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & \text{otherwise,} \end{cases}$$

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we obtain the following theorem:

Theorem (Fomin, Shapiro, Thurston '08)

Suppose K' is obtained from K by a 1-move on the edge labelled k. Then $\mu_k(B(K)) = B(K')$.
Therefore, $\mathcal{A}(K) \cong \mathcal{A}(K')$ as cluster algebras of rank $6g + 3m - 6$.

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For each surface S we can now form a family of cluster algebras $\mathcal{A}_m$, where m corresponds to the number of vertices in the triangulations of S.

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This forms a direct system and we can define

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So to compare two triangulations with a different number of vertices we can embed their cluster algebras into ones of higher rank.

Pair Ordering

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We now generalise the exchange matrix B to higher-dimensional triangulated manifolds. Consider a simplicial complex K on the vertex set $[m]$. Using the ordering $1 < \cdots < m$ on the vertices we impose the *lexographical ordering* $<$ on $2^{[m]}$, e.g. when $m = 3$ we have

$$\emptyset < \{1\} < \{2\} < \{3\} < \{1,2\} < \{1,3\} < \{2,3\} < \{1,2,3\}.$$

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$$\emptyset < \{1\} < \{2\} < \{3\} < \{1,2\} < \{1,3\} < \{2,3\} < \{1,2,3\}.$$

Let $\alpha = (a_0, a_1, \ldots, a_n)$ be an oriented n-simplex. Then the standard boundary operator is defined as

$$\partial(a_0, a_1, \ldots, a_n) = \sum_{j=0}^{n} (-1)^{j+1} (a_0, \ldots, \hat{a}_j, \ldots, a_n).$$

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We generalise this as follows: for any $0 \leq k \leq n$ define

$$\partial^{(k)}(a_0, a_1, \ldots, a_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} (-1)^{j_1 + \cdots + j_k + 1} (\ldots, \hat{a_{j_1}}, \ldots, \hat{a_{j_k}}, \ldots).$$

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We now define a *pair ordering* $\prec$ on all $(n-1)$-faces of a simplex $\alpha = (a_0, a_1, \ldots, a_n)$.

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We now define a *pair ordering* $\prec$ on all $(n-1)$-faces of a simplex $\alpha = (a_0, a_1, \dots, a_n)$.
Take two facets f and g of α such that $f < g$. Define

$$f \setminus g = (f \cup g) \setminus (f \cap g) = \alpha \setminus (f \cap g).$$

Note that $\dim f \setminus g = 1$. Consider the coefficient c_{fg} of $f \setminus g$ in $\partial^{(n-1)}\alpha$.

Pair Ordering

Set

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Example

Take $a = (1, 2, 3)$. Then the one faces have the lexographical ordering $(12) < (13) < (23)$. By considering

$$\partial(1, 2, 3) = (23) - (13) + (12)$$

we obtain the pair ordering for all one faces as follows:

$$(12) \prec (13); \quad (23) \prec (13); \quad (13) \prec (23).$$

Exchange Matrix

Consider a triangulated manifold K of dimension n. Let $\mathcal{F}$ denote the set of $(n-1)$-faces of K.

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$$b^\alpha_{fg} = \begin{cases} +1, & \text{if } f \prec g; \\ -1, & \text{if } g \prec f; \\ 0, & \text{otherwise,} \end{cases}$$

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for $f, g \subset \alpha$.
Then we define

$$B(K) = \sum_{\alpha\in\mathcal{F}} B^\alpha K.$$

Note that $B(K)$ is skew-symmetric with entries belonging to $\{-1, 0, +1\}$.

Matrix Mutations

We now restrict to the case of $2j = n$, i.e. bisteller n-moves on a manifold of dimension $2n$.

Question

How does $B(K)$ change into $B(\mathrm{bm}_\alpha K)$? We only need to consider the local structure around α.

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WLOG assume that $\alpha = (1, \ldots, j+1)$ and $\beta = (j+2, \ldots, 2j+2)$. Set

$$F_i = (-1)^i (1, \ldots, j+1, j+2, \ldots, \widehat{2j+3-i}, \ldots, 2j+2),$$

for $i = 1, \ldots, j+1$.

Matrix Mutations

We now restrict to the case of $2j = n$, i.e. bisteller n-moves on a manifold of dimension $2n$.

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for $i = 1, \ldots, j+1$. Note $\alpha = \bigcap F_i$. Let $\Lambda(F) = \bigcup F_i$.

Matrix Mutations

Then

$$\mathrm{bm}_\alpha \Lambda(F) = \Lambda(H),$$

where

$$H_i = (-1)^i(1, \ldots, \hat{i}, \ldots, j+1, \ldots, 2j+2)$$

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Lemma

For any n-face γ not in $\Lambda(F)$ we have that $b_{fg} = b_{fg}^{bm_\alpha}$, for $f, g \in \mathcal{F}(\gamma)$. Furthermore,

$$\sum_{\gamma \in \mathcal{F}(K)\setminus F} B^\gamma(K) = \sum_{\gamma \in \mathcal{F}(bm_\alpha K)\setminus H} B^\gamma(bm_\alpha K).$$

Matrix Mutations

Let σ be a permutation on $[2j+2]$ such that $\sigma(\alpha) = \beta$. Then σ induces a combinatorial equivalence between $\Lambda(F)$ and $\Lambda(H)$.

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We define a matrix mutation $\mu_\sigma(B) = B'$ with respect to σ by setting

$$b'_{fg} = \begin{cases} b_{fg}, & \text{if } f, g \notin \Lambda(F); \\ -b_{\sigma(f)\sigma(g)}, & \text{if } f, g \in \Lambda(F). \end{cases}$$

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Proposition

$$\mu_\sigma(B(K)) = B(\mathrm{bm}_\alpha(K)).$$

Cluster Algebra

Suppose we have a triangulated manifold K. A *bistellar cluster*

$$\chi(K) = \{x_f \mid f \in \mathcal{F}(K)\}$$

is a set of abstract variables associated to K. Denote by $\mathbb{Q}(\chi(K))$ the field of rational functions over $\chi(K)$.

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We then form a cluster algebra $\mathcal{A}(K)$ by performing *bistellar seed mutations* in all possible directions.

Cluster Algebra

The *bistellar exchange relations* are of the form

$$x_f x_{\sigma(f)} = \prod_{g \in \Lambda(F)} x_g^{[b_f g]_+} + \prod_{g \in \Lambda(F)} x_g^{[-b_{fg}]_+}.$$

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We then have that

Theorem

The associated cluster algebra $\mathcal{A}(K)$ to a PL-manifold is a PL-invariant.

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- Our solution to these problems is to define an algebra similar to a cluster algebra with some differences.
- We have defined generalised matrix mutations for performing bistellar moves on arbitrary dimensional faces.
- So now our goal is to fit this into some algebraic framework similar to cluster algebras to give (possibly complete) invariants of PL-manifolds.

Thank you!