Algebraic PL-Invariants and Cluster Algebras

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A *PL-map* $\varphi: K_1 \to K_2$ is a simplicial map from a subdivision of $K_1$ to a subdivision of $K_2$. So $K_1$ and $K_2$ are *PL-homeomorphic* iff there exists a simplicial complex isomorphic to a subdivision of the both of them.
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A \textit{PL-sphere} is a triangulated sphere which is PL-homeomorphic to the boundary of a simplex.

- In dim $\leq 3$, any triangulated sphere is PL.
- In dim $= 4$, the question is open.
- In dim $\geq 5$, there exist non–PL-sphere triangulations.
All manifolds are assumed to be connected, closed and oriented.

Let $\alpha \in K$. Then $\text{lk}_K \alpha = \{ \alpha' \in K \mid \alpha \cup \alpha' \in K, \alpha \cap \alpha' = \emptyset \}$. 
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**Definition**

A *PL-manifold* of dim \( n \) is a simplicial complex \( K \) of dim \( n \) such that \( \text{lk}_K \alpha \) is a PL-sphere of dim \( n - |\alpha| \), for all non-empty \( \alpha \in K \).
PL-Manifolds

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- In dim $\leq 3$, Smooth = PL = Top
- In dim = 4, Smooth = PL $\subset$ Top
- In dim $\geq 4$, Smooth $\subset$ PL $\subset$ Top
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\textbf{Problem}

Is a PL-structure on $S^4$ unique?
Bistellar Moves

Let $K$ be a triangulated manifold of dim $n$. 
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$$\text{bm}_\alpha K = (K \setminus (\alpha \ast \partial \beta)) \cup (\partial \alpha \ast \beta)$$

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Let $f = (f_0, f_1, \ldots, f_n)$ be the $f$-vector of $K$. Then

**Lemma**

$$f(\text{bm}_\alpha K) = f(K) \iff n = 2j.$$
Bistellar Moves

Two simplicial complexes are \textit{bistellarly equivalent} if one can be transformed into another by a finite sequence of bistellar moves.
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**Theorem (Pachner '87)**

Two PL-manifolds are bistellarly equivalent if and only if they are PL-homeomorphic.
Cluster Algebras

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- A distinguished family of generators $X$ called *cluster variables*;
- Such that $X$ is grouped into overlapping subsets called *clusters* that all have cardinality $n$. 

$x_1 = \{x_{11}, \ldots, x_{1n}\}, x_2 = \{x_{21}, \ldots, x_{2n}\}, \ldots, X = \bigcup x_i$ non-disjoint union.
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- a distinguished family of generators \( X \) called *cluster variables*;
- such that \( X \) is grouped into overlapping subsets called *clusters* that all have cardinality \( n \).

\[
x_1 = \{x_1^1, \ldots, x_n^1\}, \quad x_2 = \{x_1^2, \ldots, x_n^2\}, \ldots
\]

\[
X = \bigcup x_i \quad \text{non-disjoint union}
\]
The clusters have the following exchange property:

For every cluster $\mathbf{x}$ and $x \in \mathbf{x}$, there exists another cluster $\mathbf{x}'$ and $x' \in \mathbf{x}'$ such that

$$x' = (\mathbf{x} \setminus \{x\}) \cup \{x'\},$$
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where $x$ and $x'$ are related by the exchange relation

$$xx' = M + M',$$

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Exchange Property

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Furthermore, any two clusters can be obtained from each other by a sequence of exchanges.
Let $A = \mathbb{C}[SL_2] = \mathbb{C}[a, b, c, d]/(ad - bc - 1)$ be the coordinate ring, where we write an element of $SL_2$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ with } ad - bc = 1.$$
Rank 1

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with $ad - bc = 1$.

Consider $a, d$ as cluster variables and $b, c$ as scalers. Then we just have two clusters $\{a\}, \{d\}$ and $\mathcal{A}$ is the algebra over $\mathbb{C}[b, c]$ generated by cluster variables $a, d$ subject to the exchange relation

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$$ad = 1 + bc,$$

which we can write as the ring $\mathbb{C}[b, c][a, a^{-1}]$ of Laurent polynomials.
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The cluster variables are elements $x_m, m \in \mathbb{Z}$, defined recursively by the exchange relations:

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1, & \text{if } m \text{ is odd;} \\ x_m^c + 1, & \text{if } m \text{ is even.} \end{cases}$$
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Each $x_m$ is then a rational function of $x_1$ and $x_2$. 
Rank 2

So $A(b, c)$ is the subring generated by $x_m$, $m \in \mathbb{Z}$, inside the field of rational functions

$$\mathbb{Q}(x_1, x_2) := \left\{ \frac{f(x_1, x_2)}{g(x_1, x_2)} \mid f, g \text{ are polynomials such that } g \neq 0 \right\}.$$
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Starting with $\{x_1, x_2\}$ we can reach any other cluster by a series of exchanges

$$
\cdots \leftrightarrow \{x_0, x_1\} \leftrightarrow \{x_1, x_2\} \leftrightarrow \{x_2, x_3\} \leftrightarrow \cdots
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Each cluster variable $x_k$ can be exchanged from $\mathbf{x}$ to form a new cluster $\mathbf{x}' = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\}$,
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Each cluster variable $x_k$ can be exchanged from $\mathbf{x}$ to form a new cluster $\mathbf{x}' = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\}$, where $x_k$ and $x'_k$ are related by an exchange relation

$$x_k x'_k = M + M',$$

where $M, M'$ are disjoint monomials in the variables $\mathbf{x} \cap \mathbf{x'}$. 
The exponents in $M$ and $M'$ are encoded in an $(n \times n)$-integer matrix $B = (b_{ij})$ (usually skew-symmetric) called the exchange matrix.
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$$x_k x'_k = \prod_i x_i^{[b_{ik}]+} + \prod_i x_i^{[-b_{ik}]+},$$

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where $[b]^+ = \max\{b, 0\}$.

A pair $(x, B)$ is called a seed, which can be realised as a quiver.
For each index $k$ we can extend $x \mapsto x'$ to

$$(x, B) \mapsto (x', B')$$
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called a seed mutation in direction $k$, where we use the matrix mutation $\mu_k : B \rightarrow B' = (b'_{ij})$ given by

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{ik}]_+ [-b_{kj}]_+, & \text{otherwise}. \end{cases}$$
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Note that $\mu_k(\mu_k(B)) = B$. 
The cluster algebra is then defined as the subring of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables,
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Theorem (Fomin, Zelevinsky 2003)

Cluster algebras of finite type can be classified in terms of the Dynkin diagrams of finite-dimensional simple Lie algebras.
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Surprising since every cluster variable appears as the denominator of the expression used for producing a new one. Conjecture: All coefficients in these Laurent polynomials are positive.
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Conjecture: All coefficients in these Laurent polynomials are positive.
Suppose we have a pair \((S, M)\) where \(S\) is a closed oriented connected surface and \(M\) is a set of marked points on \(S\) with \(|M| = m\). We want to consider all triangulations \(K\) of \(S\) with \(m\) vertices as the marked points.
Suppose we have a pair \((S, M)\) where \(S\) is a closed oriented connected surface and \(M\) is a set of marked points on \(S\) with \(|M| = m\). We want to consider all triangulations \(K\) of \(S\) with \(m\) vertices as the marked points.

Consider the category of PL-surfaces \(K\) with a fixed number of vertices.

Then

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Consider the category of PL-surfaces \(K\) with a fixed number of vertices.

Then

\[ K \cong K' \iff \text{related by a finite sequence of bistellar 1-moves}. \]

Given \((S, M)\) we can form a cluster algebra \(A(S, M)\).
The set of cluster variables $X$ is the set of potential edges of $K$. Every triangulation $K$ will have

$$n = 6g + 3m - 6$$

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For every triangulation $K$ we form the $(n \times n)$-exchange matrix $B$ as follows:

$$B(K) = B = \sum_{\Delta \in S} B^\Delta,$$

where $\Delta$ is a 2-simplex of $S$. 
Using the following matrix mutation \( B' = \mu_k(B) \) in direction \( k \):

\[
\begin{align*}
    b'_{ij} &= \begin{cases} 
        -b_{ij}, & \text{if } i = k \text{ of } j = k; \\
        b_{ij} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{ik}]_+ [-b_{kj}]_+, & \text{otherwise},
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\end{align*}
\]

we obtain the following theorem:

\textbf{Theorem (Fomin, Shapiro, Thurston '08)}

Suppose \( K' \) is obtained from \( K \) by a 1-move on the edge labelled \( k \). Then

\[ \mu_k(B(K)) = B(K') \]

Therefore, \( A(K) \sim A(K') \) as cluster algebras of rank \( 6g + 3m - 6 \).
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Suppose $K'$ is obtained from $K$ by a 1-move on the edge labelled $k$. Then $\mu_k(B(K)) = B(K')$.

Therefore, $\mathcal{A}(K) \cong \mathcal{A}(K')$ as cluster algebras of rank $6g + 3m - 6$. 
For each surface $S$ we can now form a family of cluster algebras $A_m$, where $m$ corresponds to the number of vertices in the triangulations of $S$. 
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We have a canonical inclusion $\mathcal{A}_m \hookrightarrow \mathcal{A}_{m+1}$ induced by performing a zero move on a 2-simplex.
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This forms a direct system and we can define

$$\mathcal{A} = \lim_{m} \mathcal{A}_m.$$
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$$A = \varinjlim_{m} A_m.$$

So to compare two triangulations with a different number of vertices we can embed their cluster algebras into ones of higher rank.
We now generalise the exchange matrix $B$ to higher-dimensional triangulated manifolds. Consider a simplicial complex $K$ on the vertex set $[m]$. Using the ordering $1 < \cdots < m$ on the vertices we impose the lexicographical ordering $<$ on $\binom{m}{2}$, e.g. when $m = 3$ we have $\emptyset < \{1\} < \{2\} < \{3\} < \{1, 2\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}$. Let $\alpha = (a_0, a_1, \ldots, a_n)$ be an oriented $n$-simplex. Then the standard boundary operator is defined as $\partial(\alpha) = \sum_{j=0}^{n} (-1)^{j+1} (\hat{a}_j, a_0, \ldots, \hat{a}_j, \ldots, a_n)$. 
We now generalise the exchange matrix $B$ to higher-dimensional triangulated manifolds. Consider a simplicial complex $K$ on the vertex set $[m]$. Using the ordering $1 < \cdots < m$ on the vertices we impose the *lexographical ordering* $\prec$ on $2^m$, e.g. when $m = 3$ we have

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Let $\alpha = (a_0, a_1, \ldots, a_n)$ be an oriented $n$-simplex. Then the standard boundary operator is defined as

$$\partial(a_0, a_1, \ldots, a_n) = \sum_{j=0}^{n} (-1)^{j+1} (a_0, \ldots, \hat{a_j}, \ldots, a_n).$$
We generalise this as follows: for any $0 \leq k \leq n$ define

$$
\partial^{(k)}(a_0, a_1, \ldots, a_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} (-1)^{j_1+\cdots+j_k+1}(\ldots, \hat{a}_{j_1}, \ldots, \hat{a}_{j_k}, \ldots).
$$

Obviously, we have $\partial^{(1)} = \partial$. 

We now define a pair ordering $\prec$ on all $(n-1)$-faces of a simplex $\alpha = (a_0, a_1, \ldots, a_n)$. Take two facets $f$ and $g$ of $\alpha$ such that $f \prec g$. Define $f \setminus g = (f \cup g) \setminus (f \cap g) = \alpha \setminus (f \cap g)$. Note that $\dim (f \setminus g) = 1$. Consider the coefficient $c_{fg}$ of $f \setminus g$ in $\partial^{(n-1)}(\alpha)$. 
Pair Ordering

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Note that $\dim f \setminus g = 1$. Consider the coefficient $c_{fg}$ of $f \setminus g$ in $\partial^{(n-1)} \alpha$. 
Pair Ordering

Set

\[ f \prec g \quad \text{if} \quad c_{fg} = +1 \]

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Example

Take \( a = (1, 2, 3) \). Then the one faces have the lexicographical ordering \((12) < (13) < (23)\). By considering

\[ \partial(1, 2, 3) = (23) - (13) + (12) \]

we obtain the pair ordering for all one faces as follows:

\((12) \prec (13);\) \quad (23) \prec (13); \quad (13) \prec (23).\)
Consider a triangulated manifold $K$ of dimension $n$. Let $\mathcal{F}$ denote the set of $(n - 1)$-faces of $K$. 

The exchange matrix is defined as follows: for every $n$-simplex $\alpha$ in $K$, we define the skew-symmetric matrix $B_{\alpha}(K) = (b_{\alpha fg})$ of size $(n - 1)(K)$ by setting

$$b_{\alpha fg} = \begin{cases} +1, & \text{if } f \prec g; \\ -1, & \text{if } g \prec f; \\ 0, & \text{otherwise} \end{cases}$$

for $f, g \subset \alpha$. Then we define $B(K) = \sum_{\alpha \in \mathcal{F}} B_{\alpha}(K)$. Note that $B(K)$ is skew-symmetric with entries belonging to $\{-1, 0, +1\}$. 
Consider a triangulated manifold $K$ of dimension $n$. Let $\mathcal{F}$ denote the set of $(n-1)$-faces of $K$. For every $n$-simplex $\alpha$ in $K$ we define the skew-symmetric matrix $B^\alpha(K) = (b_{fg}^\alpha)_{f,g \in \mathcal{F}}$ of size $f_{n-1}(K)$ by setting

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Matrix Mutations

We now restrict to the case of $2j = n$, i.e. bisteller $n$-moves on a manifold of dimension $2n$.

**Question**

How does $B(K)$ change into $B(bm\alpha K)$? We only need to consider the local structure around $\alpha$. 

**Note**

$\alpha = \cap F_i$. Let $\Lambda(F) = \bigsqcup F_i$. 

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Algebraic PL-Invariants and Cluster Algebras
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How does $B(K)$ change into $B(bm_\alpha K)$? We only need to consider the local structure around $\alpha$.

WLOG assume that $\alpha = (1, \ldots, j + 1)$ and $\beta = (j + 2, \ldots, 2j + 2)$. Set

$$F_i = (-1)^i(1, \ldots, j + 1, j + 2, \ldots, 2j + 3 - i, \ldots, 2j + 2),$$

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for $i = 1, \ldots, j + 1$. Note $\alpha = \bigcap F_i$. Let $\Lambda(F) = \bigcup F_i$. 
Then
\[ \text{bm}_\alpha \Lambda(F) = \Lambda(H), \]
where
\[ H_i = (-1)^i (1, \ldots, \hat{i}, \ldots, j + 1, \ldots, 2j + 2) \]
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**Lemma**

For any $$n$$-face $$\gamma$$ not in $$\Lambda(F)$$ we have that $$b_{fg} = b_{fg}^{bm_{\alpha}}$$, for $$f, g \in F(\gamma)$$. Furthermore,

$$\sum_{\gamma \in F(K) \setminus F} B^\gamma(K) = \sum_{\gamma \in F(bm_{\alpha}K) \setminus H} B^\gamma(bm_{\alpha}K).$$
Matrix Mutations

Let $\sigma$ be a permutation on $[2j + 2]$ such that $\sigma(\alpha) = \beta$. Then $\sigma$ induces a combinatorial equivalence between $\Lambda(F)$ and $\Lambda(H)$. 
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$$\sigma = \begin{pmatrix} 1 & \cdots & j + 1 & j + 2 & \cdots & 2j + 2 \\ 2j + 2 & \cdots & j + 2 & j + 1 & \cdots & 1 \end{pmatrix}.$$
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We define a matrix mutation $\mu_{\sigma}(B) = B'$ with respect to $\sigma$ by setting

$$b'_{fg} = \begin{cases} b_{fg}, & \text{if } f, g \notin \Lambda(F); \\ -b_{\sigma(f)\sigma(g)}, & \text{if } f, g \in \Lambda(F). \end{cases}$$
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Proposition

$$
\mu_\sigma(B(K)) = B(bm_\alpha(K)).
$$
Suppose we have a triangulated manifold $K$. A bistellar cluster

$$\chi(K) = \{x_f \mid f \in \mathcal{F}(K)\}$$

is a set of abstract variables associated to $K$. Denote by $\mathbb{Q}(\chi(K))$ the field of rational functions over $\chi(K)$. 
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Associated to $K$ we get a bistellar seed $(\chi(K), B(K))$, where $B(K)$ is the exchange matrix associated to $K$.

We then form a cluster algebra $\mathcal{A}(K)$ by performing bistellar seed mutations in all possible directions.
The *bistellar exchange relations* are of the form

\[
x_f x_{\sigma(f)} = \prod_{g \in \Lambda(F)} x_g^{[bf_g]} + \prod_{g \in \Lambda(F)} x_g^{[-bf_g]}.
\]
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\[ x_f x_{\sigma(f)} = \prod_{g \in \Lambda(F)} x_g^{b_{fg}} + \prod_{g \in \Lambda(F)} x_g^{-b_{fg}}. \]

We then have that

**Theorem**

The associated cluster algebra \( A(K) \) to a PL-manifold is a PL-invariant.
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We have defined generalised matrix mutations for performing bistellar moves on arbitrary dimensional faces.

So now our goal is to fit this into some algebraic framework similar to cluster algebras to give (possibly complete) invariants of PL-manifolds.
Thank you!