On projective bundles over small covers

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1. Basic facts of small covers

**Small cover** (Davis-Januszkiewicz, 1990)

\[\text{def} \quad \iff \quad \text{a compact } n\text{-dimensional manifold } M^n \text{ with the following two conditions:} \]

1. \(M^n\) has an effective, **locally standard** \((\mathbb{Z}_2)^n\)-action, i.e., locally looks like the standard \((\mathbb{Z}_2)^n \acts \mathbb{R}^n\);

2. the orbit space is an **\(n\)-dimensional simple polytope** \(M^n/(\mathbb{Z}_2)^n = P^n\), i.e., each vertex is constructed by the intersection of just \(n\) facets.

![Simple polytope](image1)

![Non-simple polytope](image2)
Example 1. Let $\mathbb{R}P(n)$ be the $n$-dimensional real projective space with the following $\mathbb{Z}_{2}^{n}$-action:

$$(t_1, \ldots, t_n) \cdot [r_0 : r_1 : \cdots : r_n] = [r_0 : t_1 r_1 : \cdots : t_n r_n].$$

Then this action is locally standard and $\mathbb{R}P(n)/\mathbb{Z}_{2}^{n} = \Delta^{n}$.
From the small cover, we have the following two datas.

1. $P^n$: an $n$-dimensional simple polytope,

2. $\lambda : \mathcal{F} \to \{0, 1\}^n$: a characteristic function, that is,
   \[ \det(\lambda(F_1), \ldots, \lambda(F_n)) = 1 \text{ (mod 2)} \text{ for } \bigcap_{i=1}^n F_i = \{v\}, \]
where $\mathcal{F} = \{F_1, \ldots, F_m\}$ denotes the set of all facets (codimension-one faces) of $P$. 
Let $\mathbb{R}P(n)$ be the $n$-dimensional real projective space with the following $\mathbb{Z}_2^n$-action:

$$(t_1, t_2) \cdot [r_0 : r_1 : r_2] = [r_0 : t_1 r_1 : t_2 r_2].$$

The isotropy group of $[r_0 : 0 : r_2]$ is $\mathbb{Z}_2 \times \{1\} \implies e_1 \in \{0, 1\}^2$.

The isotropy group of $[r_0 : r_1 : 0]$ is $\{1\} \times \mathbb{Z}_2 \implies e_2 \in \{0, 1\}^2$.

The isotropy group of $[0 : r_1 : r_2]$ is $\Delta \implies e_1 + e_2 \in \{0, 1\}^2$. 
Characterization of small covers

Small covers can be reconstructed from the two datas $(P^n, \lambda)$.

1. $P^n$: an $n$-dimensional simple polytope,

2. $\lambda: \mathcal{F} \to \{0, 1\}^n$: a characteristic function.

Then

$$M(P, \lambda) = (\mathbb{Z}_2)^n \times P^n / \sim_\lambda$$

is small cover, where

$$(t, p) \sim_\lambda (t', q) \Leftrightarrow p = q, \text{ and } t't^{-1} \in T(p) = \langle (-1)^{\lambda(F)} \mid p \in F \rangle \subset (\mathbb{Z}_2)^n.$$ 

Here, $-1 = (-1, \ldots, -1) \in \mathbb{Z}_2^n$. 

Example 2. In the following figures, the left and right pair are called by \((\Delta^2, \lambda_2)\) and \((I^2, \lambda_1^2)\) respectively (where \(e_1\) and \(e_2\) are standard basis in \(\{0, 1\}^2\)).

\[
M(\Delta^2, \lambda_2) = RP(2)
\]

\[
M(I^2, \lambda_1^2) = T^2
\]
In summary we have the following correspondence.

The function \( \lambda \) is also denoted by the following matrix

\[
(\lambda(F_1), \ldots, \lambda(F_m)) = (I_n \ \land) \in M(n, m; \mathbb{Z}_2),
\]

where \( \land \in M(n, m-n; \mathbb{Z}_2) \). We call \((I_n \ \land)\) a characteristic matrix.
2. Motivation

Cohomological rigidity problem for small cover

Assume $H^*(M; \mathbb{Z}_2) \simeq H^*(M'; \mathbb{Z}_2)$ for two small covers $M$ and $M'$.

**Problem**: Are $M$ and $M'$ homeomorphic?

\[\downarrow\]

**Answer**: No!

There are counter examples in the above class.
Masuda’s counter examples

\[ M(q) = P(q\gamma \oplus (b - q)\epsilon) \]: the projective bundle over \( \mathbb{R}P(a) \), where \( \gamma \) is the canonical line bundle, \( \epsilon \) is the trivial bundle and \( 0 \leq q \leq b \).

**Theorem 1** (Masuda). The following two statements hold:

1. \( H^*(M(q); \mathbb{Z}_2) \simeq H^*(M(q'); \mathbb{Z}_2) \) \( \iff \) \( q' \equiv q \) or \( b - q \mod 2^h(a) \),
   where \( h(a) = \min\{n \in \mathbb{N} \cup \{0\} \mid 2^n \geq a\} \);

2. \( M(q) \cong M(q') \) \( \iff \) \( q' \equiv q \) or \( b - q \mod 2^k(a) \),
   where \( k(a) = \#\{n \in \mathbb{N} \mid 0 < n < a \text{ and } n \equiv 0, 1, 2, 4 \mod 8\} \).
Put $a = 10$, then we have $h(10) = 4$, $k(10) = 5$.

Put $b = 17$ and $q = 1$ and $q' = 0$.

Then $H^*(M(1)) \sim H^*(M(0))$ (by $q' \equiv 17 - q \mod 2^{h(10)} = 16$), but $M(1) \not\approx M(0)$ (by $q' \not\equiv 17 - q \mod 2^{k(10)} = 32$).

**Problem:** Characterize (or classify) the topological types of projective bundles over small covers.
3. Projective bundles over small covers

Let \( \xi = (E(\xi), \pi, M, \mathbb{R}^k) \) be an equivariant \( k \)-dimensional vector bundle over a small cover \( M^n \).

Put \( \sigma_0(M) \) is the image of the zero section and
\[
P(\xi) = E(\xi) - \sigma_0(M)/\mathbb{R}^*,
\]
then \( P(\xi) \) is the \( \mathbb{R}P^{k-1} \)-bundle over \( M \).

Lemma 1. \( P(\xi) \) is a small cover \( \iff \xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_k \) where \( \gamma_i \) is a line bundle.

We call such \( P(\xi) \) a projective bundle over small cover (or projective bundle).
Lemma 2. \( P(\xi) \) has the following two properties:

1. **the orbit space** is \( P^n \times \Delta^{k-1} \) (where \( M/\mathbb{Z}_2^n = P^n \));

2. **the characteristic matrix** of \( P(\xi) \) can be denoted by

\[
\begin{pmatrix}
I_n & O & \wedge & 0 \\
O & O & I_{k-1} & \wedge' & 1
\end{pmatrix}
\]

Therefore, in order to consider the projective bundle over small cover, we may only consider the following matrix:

\[
\begin{pmatrix}
I_n & \wedge \\
O & \wedge'
\end{pmatrix} \in M(n + k - 1, m; \mathbb{Z}_2)
\]
Idea: Attach this matrix to the facets of $P^n$ directly.

For example, for $\mathbf{r} = (r_1, \ldots, r_{k-1}) \in \{0, 1\}^{k-1}$,

The following matrix

$$
\begin{pmatrix}
I_2 & \mathbf{1} \\
\mathbf{O} & \mathbf{r}
\end{pmatrix} \in M(k + 1, 3; \mathbb{Z}_2),
$$

corresponds with

$$P(\gamma^{r_1} \oplus \cdots \oplus \gamma^{r_{k-1}} \oplus \epsilon),$$

where $\gamma^0 = \epsilon$ and $\gamma^1 = \gamma$ over $\mathbb{R}P(2)$.

$$\mathbf{r} = r_1 e_1' + \cdots + r_{k-1} e_{k-1}'$$
Projective characteristic functions

$\lambda_P : \mathcal{F}_P \to \{0, 1\}^n \times \{0, 1\}^{k-1}$: projective characteristic functions such that

$$\det(\lambda_P(F_{i_1}), \ldots, \lambda_P(F_{i_n}), X_1, \ldots, X_{k-1}) = 1$$

for $F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset$ and $\{X_1, \ldots, X_{k-1}\} \subset \{e'_1, \ldots, e'_{k-1}, 1\}$, where $e'_i$ is the standard basis of $\{0, 1\}^{k-1}$.

Then $(P, \lambda_P)$ characterizes the projective bundle over small cover.
New operation on projective characteristic functions

In order to prove the construction theorem of projective bundles over 2-dim small covers, we introduce an operation $\# \Delta^{k-1}$ on the projective characteristic functions as follows.

Remark: This operation corresponds with the fibre some of projective bundles (gluing along the fibres).
Theorem 2. Let $P(\xi)$ be a projective bundle over 2-dimensional small cover $M^2$. Then $P(\xi)$ can be constructed from projective bundles $P(\zeta)$ over the real projective space $\mathbb{R}P^2$ and $P(\kappa)$ over the torus $T^2$ by using $\#\Delta^{k-1}$.

\[
P(\zeta) = P(\gamma^{r_1} \oplus \cdots \oplus \gamma^{r_{k-1}} \oplus \epsilon)
P(\kappa) = P(\gamma^{r_1} \otimes \gamma^{r_1'} \oplus \cdots \oplus \gamma^{r_{k-1}} \otimes \gamma^{r_{k-1}'} \oplus \epsilon)
\]
Outline of proof

**Step 1:** Prove there are two edges $F_i$, $F_j$ such that

$$\det(\lambda_P(F_i), \lambda_P(F_j), X_1, \ldots, X_{k-1}) = 1.$$  

**Step 2:** Then we can do the converse of the operation $\#\Delta^{k-1}$ along $F_i$ and $F_j$.

**Step 3:** Iterating the above argument, finally $P$ decomposes into the sum of $\Delta^2$'s and $I^2$'s.
Finally we list up all topological types of projective bundles over $\mathbb{R}P(2)$ and $T^2$.

**Proposition 1.** The topological type of $P(\zeta)$ is one of the following 4 topological types:

$$S^2 \times \mathbb{Z}_2 P(q\mathbb{R} \oplus (k - q)\mathbb{R}),$$

for $q = 0, 1, 2, 3$.

**Proposition 2.** The topological type of $P(\kappa)$ is one of the following 4 topological types:

$$T^2 \times \mathbb{Z}_2 P(R_1 \oplus R_2 \oplus (k - 2)\mathbb{R});$$
$$T^2 \times \mathbb{Z}_2 P(R_1 \oplus (k - 1)\mathbb{R});$$
$$T^2 \times \mathbb{Z}_2 P(R_2 \oplus (k - 1)\mathbb{R});$$
$$T^2 \times \mathbb{R}P(k - 1),$$

where $T^2 \times \mathbb{Z}_2 R_i$ is the canonical bundle of the $i$-th $S^1 \subset T^2$ ($i = 1, 2$).