EQUIVARIANT COHOMOLOGY OF TORUS ORBIFOLDS WITH TWO FIXED POINTS

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Abstract. The main purpose of this paper is to introduce the main result of [6] and apply it to show a ring structure of the equivariant cohomology of a certain class of torus orbifolds which have exactly two fixed points. We also show that a torus orbifold over the suspension of an $(n-1)$-dimensional simplex is equivariantly homeomorphic to the $2n$-dimensional sphere quotiented by a product of cyclic groups.

1. Introduction

The relationship between the equivariant cohomology of certain smooth manifolds M^{2n}, with half-dimensional torus actions $\mathbb{T}^n = S^1 \times \cdots \times S^1$, and the combinatorics of their quotient spaces is well-known. A toric manifold, defined as a compact non-singular toric variety, admits the locally standard action of a half-dimensional torus. This implies that its quotient space is a manifold with faces (see Definition 2.3). The equivariant cohomology is then simply given as the *face-ring* of this quotient space. This can also be realized as the face-ring of the complete regular fan associated to the toric variety. A quasitoric manifold, first defined in [9], is a topological generalization of a toric manifold whose quotient space is a simple polytope (an example of a manifold with faces) but which is more general than a projective toric variety. Its equivariant cohomology is then given by the face-ring associated to the simple polytope.

Torus manifolds, appearing in [14], are a wider class of manifolds M^{2n}, with half-dimensional torus action, that contain both toric and quasitoric manifolds. The main example of a torus manifold that is neither a toric nor a quasitoric manifold is the even-dimensional sphere $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$, where the torus acts coordinatewise on the first n coordinates. To any torus manifold we can associate a combinatorial object, called a *torus graph* (see [17]), which is an n-valent graph whose vertices correspond to the fixed points of M and whose edges are labelled by irreducible torus representations. The underlying graph for S^{2n} is the one with exactly two vertices and n edges between them. We can then calculate the *graph equivariant cohomology* of the torus graph, which is a ring of *piecewise polynomials* on the graph. When the ordinary cohomology of M is trivial in all odd degrees, then the torus action is locally standard and the equivariant cohomology of M is isomorphic to the graph cohomology of the associated torus graph. It is also possible to give explicit generators and relations for this ring via *Thom classes* of the torus graph and we can see that this is isomorphic to the face-ring of the quotient space which is, in this case, a manifold with faces.

When we move from manifolds to orbifolds the picture slightly changes. In the case of singular toric varieties, including toric varieties having orbifold singularities, it was proved in [10] that its equivariant cohomology with integer coefficients is given by the piecewise polynomials on its fan if its ordinary odd degree cohomology vanishes. For toric manifolds the ring of piecewise polynomials on the fan is isomorphic to the face-ring of its quotient but this is not true for orbifolds in general. In [5] it is shown that the equivariant cohomology of projective toric orbifolds and quasitoric orbifolds, under the condition of vanishing odd degree cohomology, can be realized as a subring of the usual face-ring of the orbit space/fan that satisfies an *intergrality condition.*

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In this paper we give the main result from [6] where we focus on *torus orbifolds*. Let T^n be an n-dimensional torus, i.e., an n-dimensional commutative, compact, connected Lie group. A torus orbifold is a $2n$-dimensional, closed, oriented orbifold with an effective T^n-action whose fixed point set is non-empty. This notion is a generalization of torus manifolds as introduced by Hattori-Masuda in [14].

In Section 2 we introduce some basic definitions that we need including that of an orbifold with group action which our main objects, torus orbifolds, are an example of. We then give a constructive definition of an orbifold over a combinatorial object known as a manifold with faces. These will be examples of torus orbifolds and their quotient spaces are manifolds with faces. We restrict out attention to these cases.

The suspension of an $(n-1)$-simplex gives the simplest example of an n-dimensional manifold with faces – one with exactly two vertices. In Section 3 we show that every manifold with exactly two vertices is equivalent, in a combinatorial sense, to the suspension of a simplex and then show that a torus orbifold over it is equivariantly homeomorphic to the quotient of an even dimensional sphere by a product of cyclic groups.

In Section 4 we give the main result from [6] which computes the equivariant cohomology of a torus orbifold X whose odd degree ordinary cohomology is trivial. To do this we associate to each torus orbifold over a manifold with faces a labelled graph, called a *torus orbifold graph*, which generalizes the torus graphs of [17]. We can then describe the equivariant cohomology of X as the ring of piecewise polynomials on the associated torus orbifold graph if $H^{\mathrm{odd}}(X) = 0$. We show that this is isomorphic to a *weighted face ring* that gives us a description of $H^*_T(X)$ in terms of generators and relations. Restricting to torus orbifolds X over the suspension of a simplex we compute these ring structures explicitly.

2. Torus orbifolds over manifolds with faces

In this section, we introduce $2n$-dimensional torus orbifolds over a manifold with faces Q. We first prepare some notation.

We set

$$[n] := \{1, \dots, n\}$$

and

$$\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \text{ for all } i = 1, \dots, n\}.$$

A compact, connected, commutative, Lie group is called a *torus*, often denoted by T. We use the following symbols for a torus T:

- $\mathfrak{t} := \mathrm{Lie}(T)$, the Lie algebra of T;
- $\mathfrak{t}_{\mathbb{Z}} := \exp^{-1}(e) \subset \mathfrak{t}$ is the lattice, where $e \in T$ is the identity element and $\exp : \mathfrak{t} \to T$ is the exponential map;
- $\mathfrak{t}_{\mathbb{Q}} := \mathfrak{t}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

We also use the following symbol as the *standard n-dimensional torus*:

$$\mathbb{T}^n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1\}.$$

This is nothing but the product of n unit circles in $\mathbb{C}$. It is well-known that every n-dimensional torus T is isomorphic to $\mathbb{T}^n$.

In this paper, we often identify $\mathfrak{t}^*_{\mathbb{Z}} := \mathrm{Hom}(\mathfrak{t}_{\mathbb{Z}}, \mathbb{Z})$ with $H^2(BT; \mathbb{Z}) \cong \mathbb{Z}^n$. This identification is canonically defined as follows. It is well-known that elements in $H^2(BT; \mathbb{Z})$ are given by the first Chern classes of line bundles $ET \times_T \mathbb{C}_\rho \to BT$, where $\mathbb{C}_\rho$ is the 1-dimensional representation space defined by $\rho : T \to S^1$. Moreover, the representation ρ lifts to the Lie algebra homomorphism $\hat{\rho} : \mathfrak{t} \to \mathbb{R}$; therefore, $\hat{\rho} \in \mathfrak{t}^*$. Since T and S^1 are abelian groups, we can take $\hat{\rho} \in \mathfrak{t}^*_{\mathbb{Z}}$. This gives a homomorphism from $H^2(BT; \mathbb{Z})$ to $\mathfrak{t}^*_{\mathbb{Z}}$. By comparing their dimensions, we can check this homomorphism is isomorphism.

Moreover, in this paper, we use the symbol $\cong$ for the equivalence between two algebraic objects (i.e. group isomorphism or algebra isomorphism) and the symbol $\simeq$ for the equivalence between two geometric objects (i.e. homeomorphism or equivariant homeomorphism).

2.1. **Torus orbifolds.** We first briefly recall the notion of a torus orbifold defined in [14, Section 12]. Let $\mathcal{X} = (X, \mathcal{U})$ be a $2n$-dimensional orbifold whose underlying topological space is X and whose (maximal) orbifold atlas is $\mathcal{U} = \{(U_\alpha, V_\alpha, H_\alpha, p_\alpha)\}$, i.e., $\mathcal{U}$ satisfies the following conditions:

- $\{U_\alpha\}$ is an open covering of X;
- V_α is an open subset in $\mathbb{R}^{2n}$;
- H_α is a finite subgroup of $O(2n)$ acting on V_α;
- a continuous map $p_\alpha : V_\alpha \to U_\alpha$ which is an H_α-equivariant map that induces a homeomorphism $V_\alpha/H_\alpha \simeq U_\alpha$, where H_α acts on U_α trivially.

We often denote $\mathcal{X}$ by X and call $(U_\alpha, V_\alpha, H_\alpha, p_\alpha)$ an *orbifold chart* of X. A continuous map $f : X \to X'$ from an orbifold X to another orbifold X' is called an *orbifold map* if for every $x \in X$ there are two orbifold charts $(U_\alpha, V_\alpha, H_\alpha, p_\alpha)$ for X around x and $(U'_\alpha, V'_\alpha, H'_\alpha, p'_\alpha)$ for X' around $f(x)$ and a continuous map $f_\alpha : V_\alpha \to V'_\alpha$ such that the following diagram commutes:

$$\begin{tikzcd} V_\alpha \arrow[r, "f_\alpha"] \arrow[d, "p_\alpha"'] & V'_\alpha \arrow[d, "p'_\alpha"] \\ U_\alpha \arrow[r, "f|_{U_\alpha}"] & U'_\alpha \end{tikzcd}$$

If f_α can be taken as a smooth map for every orbifold chart, then we call an orbifold map f a *smooth map*.

Let G be a Lie group. A *G-action on an orbifold* X is a smooth map $\varphi : G \times X \to X$ which satisfies the usual rules of being a group action. We often denote an orbifold (or any topological space) X with G-action by (X, G) or (X, G, φ) if we emphasize the action $\varphi : G \times X \to X$. A $2n$-dimensional closed oriented orbifold (X, T) with an effective n-dimensional torus T-action is called a *torus orbifold* if it has a nonempty fixed point set X^T.

Remark 2.1. In this paper, an *equivariant map* $f : (X, G, \varphi) \to (X', G', \varphi')$ means a *weak equivariant map*, i.e., there is a group homomorphism $h : G \to G'$ such that $f(\varphi(g, x)) = \varphi'(h(g), f(x))$ for all $(g, x) \in G \times X$.

Let (X, T) be a torus orbifold. A *special orbifold chart* (U_x, V_x, H_x, p_x), (also called a *good local chart*, see e.g. [1]), around $x \in X$ is an orbifold chart satisfying the condition that $p_x^{-1}(x)$ is a single point $\widetilde{x} \in V_x$. The following fact is well-known (see, for instance, [11, Proposition 2.12]).

Lemma 2.2. *Let $x \in X$ be a fixed point in a torus orbifold (X, T). Then, there exists a T-invariant open neighborhood U_x of x and a special orbifold chart (U_x, V_x, H_x, p_x) around x which satisfies the following conditions:*

1. *there is a finite covering $\widetilde{T}_x$ of T such that $\widetilde{T}_x/H_x \simeq T$ and $\widetilde{T}_x$ acts on V_x;*
2. *the continuous map p_x is an equivariant map between $(V_x, \widetilde{T}_x)$ and (U_x, T) which induces an equivariant homeomorphism between $(V_x/H_x, \widetilde{T}_x/H_x)$ and (U_x, T).*

A connected component of the fixed point set of a circle subgroup S_i of T is a suborbifold, say X_i. This suborbifold X_i is called a *characteristic suborbifold* if X_i is a $(2n-2)$-dimensional orbifold and contains at least one fixed point of the T-action. In the definition of torus orbifolds in [14], we also need to choose an "invariant normal orientation" for every characteristic suborbifold X_i. We will explain this in Section 2.3 for the case of torus orbifolds over manifolds with faces.

2.2. **Manifolds with faces.** In order to define a torus orbifold over Q, we next recall the definition of an *n-dimensional manifold with faces*, also called a *face acyclic nice manifold with corners* (see [7, Chapter 10], [18, Section 5] or [16, Section 2.2] for more details). Let $Q^n (= Q)$ be an n-dimensional topological manifold with boundary (we will always assume that Q is connected). A *chart with corners*, or simply a *chart*, for Q^n is a pair (V, ψ_V), where V is an open subset of Q^n and

$$\psi_V : V \to \mathbb{R}^n_+$$

is a homeomorphism from V to an open subset $\Omega_V \subset \mathbb{R}^n_+$. Two charts with corners (V, ψ_V), (W, ψ_W) are said to be *compatible* if the transition function $\psi_V \circ \psi_W^{-1} : \psi_W(V \cap W) \to \psi_V(V \cap W)$ is a strata-preserving homeomorphism. We call a collection of compatible charts with corners $\{(V, \psi_V)\}$ which covers Q^n an *atlas*. A maximal atlas is called the *structure with corners* of Q^n. A topological manifold with boundary together with a structure with corners is called a *(topological) manifold with corners*.

Let $p \in Q^n$ be a point of an n-dimensional manifold with corners Q^n. If an element $p \in V$ is in a chart (V, ψ_V), then we can assign a number, say $d(p) \in [n] \cup \{0\}$, to p by the number of zero-coordinates of $\psi_V(p) \in \mathbb{R}^n_+$. This number $d(p)$ is called the *depth* of p. We call the closure of a connected component of $d^{-1}(k)$ $(0 \leq k \leq n)$ a *codimension-k face*. The codimension 1, $(n-1)$ and n faces are called *facets*, *edges* and *vertices*, respectively. The set of all edges and vertices is called the *one-skeleton* (or the *graph*) of Q^n. By restricting the structure with corners on Q^n to faces, we may regard each codimension-k face as an $(n-k)$-dimensional (sub)manifold with corners. If each face F of a manifold with corners Q^n is acyclic, i.e. $\widetilde{H}_*(F) = \{0\}$, then we call Q^n a *face acyclic manifold with corners*.

Definition 2.3 (Manifold with faces)**.** An n-dimensional face acyclic manifold with corners Q is said to be an n-dimensional *manifold with faces* (or a *face acyclic nice manifold with corners*) if Q satisfies the following conditions:

1. for every $k \in [n]$, there exists a codimension-k face;
2. a connected component of $\bigcap_{i=1}^k F_i$ for distinct k facets is a codimension-k face whenever $\bigcap_{i=1}^k F_i \neq \emptyset$; conversely, for each codimension-k face H, there are exactly k distinct facets $F_1, \ldots, F_k$ such that H is a connected component of $\bigcap_{i=1}^k F_i$.

Let Q_1, Q_2 be manifolds with faces. If there is a homeomorphism $f : Q_1 \to Q_2$ which preserves faces for two manifolds with faces Q_1 and Q_2, then we call Q_1 and Q_2 *isomorphic* (in the sense of manifolds with faces). We can also define a weaker equivalence relation called a *combinatorially equivalence* among manifolds with faces as follows. We may regard the set of faces of a manifold with faces Q as a partially ordered set by the inclusions of faces, say $\mathcal{S}(Q)$. If there is a bijective map between $\mathcal{S}(Q_1)$ to $\mathcal{S}(Q_2)$ which preserves the order, then we call two manifolds with faces Q_1 and Q_2 *combinatorially equivalent*. It is easy to check that if Q_1 and Q_2 are isomorphic, then Q_1 and Q_2 are combinatorially equivalent. However, the converse is not true, see Remark 3.3.

2.3. **Torus orbifolds over manifolds with faces.** In this paper, we introduce a constructive definition of a torus orbifold X over a manifold with faces Q (also see [9, 19] for details).

Let Q be an n-dimensional manifold with faces. We write the set of facets of Q as

$$\mathcal{F}(Q) := \{F_1, \ldots, F_m\}.$$

Let T be an n-dimensional torus. We identify the lattice $\mathfrak{t}_\mathbb{Z}$ of the Lie algebra of T with $\mathbb{Z}^n$. A function

$$\lambda \colon \mathcal{F}(Q) \to \mathfrak{t}_\mathbb{Z}$$

is called a *characteristic function* if it satisfies the following condition:

- $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\}$ is linearly independent whenever $F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset$, for $1 \leq k \leq n$.

Let us define the following quotient space:

$$X(Q, \lambda) := (Q \times T)/\sim, \tag{2.1}$$

where the equivalence relation $\sim$ is given by

$$(x, t) \sim (y, s) \text{ if and only if } x = y \text{ and } t^{-1}s \in T_{F(x)}, \tag{2.2}$$

where $F(x)$ is the unique face of Q containing x in its relative interior and $T_{F(x)}$ is the k-dimensional torus generated by $\lambda(F_{i_1}), \ldots, \lambda(F_{i_k}) \in \mathfrak{t}_\mathbb{Z}$, where $F(x) = (F_{i_1} \cap \cdots \cap F_{i_k})^o$ is a codimension-k face (the symbol F^o denotes a connected component of the set F). To be more precise, $T_{F(x)}$ is the k-dimensional torus subgroup of T defined by

$$T_{F(x)} := (\mathbb{R}\lambda(F_{i_1}) \times \cdots \times \mathbb{R}\lambda(F_{i_k})) \,/\, (\mathbb{Z}\lambda(F_{i_1}) \times \cdots \times \mathbb{Z}\lambda(F_{i_k}))$$

Note that if $x \in Q$ is a vertex then $T_{F(x)} \cong T$.

It is easy to check that $X := X(Q, \lambda)$ has the natural T-action induced from the T-action on the second factor of $Q \times T$, and its projection onto the orbit space $X \to X/T$ is induced from the projection onto the first factor $Q \times T \to Q$, i.e., X/T is equipped with the structure of a manifold with faces by the induced homeomorphism $X/T \simeq Q$. In this sense, we may identify the orbit space of X with Q, and we call the projection $\pi : X \to Q$ the *orbit projection* of X. Note that if $F(x)$ (for $x \in Q$) is a codimension-k face then the inverse $\pi^{-1}(x)$ is an $(n-k)$-dimensional orbit which is homeomorphic to $T/T_{F(x)}$. More precisely, one can define an orbifold structure on $X(Q, \lambda)$ similarly as in [19, Section 2.1] (also see [6] for a more precise account). Moreover, one can see that $\{\pi^{-1}(v) \mid v \in V(Q)\}$ is the set of fixed points. Hence, we conclude that $X(Q, \lambda)$ is a torus orbifold. Conversely, let (X, T) be a $2n$-dimensional torus orbifold. We assume that the orbit space X/T is isomorphic to some n-dimensional manifold with faces Q and that the projection $\pi : X \to X/T \simeq Q$ can be regarded as an orbit projection, i.e., $\pi^{-1}(F_i)$ is a characteristic suborbifold X_i for every facet $F_i \in \mathcal{F}(Q)$. Let S_i be the circle subgroup of T fixing X_i. Then, we can define the characteristic function $\lambda : \mathcal{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}$ by a choice of a nonzero vector $v_i \in \mathfrak{t}_{\mathbb{Z}}$ such that $S_i = \exp\mathbb{R}v_i$. Note that there are infinitely many choices of such nonzero vectors, i.e. if we take a primitive vector s_i such that $S_i = \exp\mathbb{R}s_i$, then for any $r \in \mathbb{Z} \setminus \{0\}$ the equality $S_i = \exp\mathbb{R}(rs_i)$ holds. In order to determine the nonzero vector, we need to choose a "normal orientation" of X_i in the following way. Due to [14, Lemma 12.1], for every $x \in X_i$ there is a special chart (U_x, V_x, H_x, p_x) around x which satisfies the following properties:

(1) the tangent space of $\widetilde{x} = p_x^{-1}(x)$ in V_x, say $V_x := T_{\widetilde{x}}V_x \simeq \mathbb{R}^{2n}$ again, splits into $W_{ix} \oplus W_{ix}^{\perp}$, where W_{ix} is tangent to $p_x^{-1}(U_x \cap X_i)$ and $W_{ix}^{\perp}$ may be regarded as the normal vector space of $x \in X_i$;
(2) H_x acts on $W_{ix}^{\perp}$;
(3) there is a (connected) finite cover $\widetilde{S}_i$ of S_i and a lifting of the action of S_i to the action of $\widetilde{S}_i$ on V_x for any point $x \in X_i$;
(4) the lifted action of $\widetilde{S}_i$ preserves (acts on) the normal vector space $W_{ix}^{\perp}(\subset V_x)$ non-trivially;
(5) $\widetilde{S}_i$ acts on $V_x = W_{ix} \oplus W_{ix}^{\perp}$ effectively.

We choose an orientation of $W_{ix}^{\perp}$ for every $x \in X_i$. We call this orientation a *normal orientation* of X_i. On the other hand, there are exactly two primitive vectors s_i and $-s_i$ such that $\exp\mathbb{R}s_i = \exp\mathbb{R}(-s_i) = S_i$. Note that the choice of signs determines the orientation of S_i. We take the orientation of S_i as the orientaion whose lifted $\widetilde{S}_i$-action preserves the orientation of the given normal orientation of X_i. Therefore, we can determine the primitive vector $s_i \in \mathfrak{t}_{\mathbb{Z}}$ for X_i without the sign ambiguity. Moreover, the continuous map $p_x : V_x \to U_x \simeq V_x/H_x$ is an equivariant map with respect to the finite covering $\widetilde{S}_i \to S_i \simeq \widetilde{S}_i/H_x$, i.e., the $r_i(>0)$-times rotation map between circles. Thus we may regard $\widetilde{S}_i = \exp\mathbb{R}(r_is_i)$ and $S_i = \exp\mathbb{R}s_i$ (also see (3.2)). Consequently, once we choose a normal orientation for each of the characteristic suborbifolds X_i in the torus orbifold X (we call a torus orbifold with normal orientations of each X_i an *omnioriented* torus orbifold), the characteristic function $\lambda : \mathcal{F}(Q) \to \mathfrak{t}_{\mathbb{Z}}$ is uniquely determined without ambiguity of scalar multiplications, i.e., $\lambda(F_i) = v_i := r_is_i$. We also have the following lemma:

Lemma 2.4. *Given a torus orbifold (X, T) with $X/T \simeq Q$, let λ be the characteristic function determined by the appropriate choice of omniorientation as above. Then, X is equivariant homeomorphic to $X(Q, \lambda)$.*

We call a(n) (omnioriented) torus orbifold X a *torus orbifold over Q* if X is equivariantly homeomorphic to $X(Q, \lambda)$ for some characteristic function λ.

Remark 2.5. Note that a choice of scalar multiplications of the $\lambda(F_i)$'s changes the orbifold structure on X; however, it does not change the equivariant homeomorphism type of X. Namely, if $\lambda(F_i) = r_i\lambda'(F_i)$ $(F_i \in \mathcal{F}(Q))$ for some $r_i \in \mathbb{Z} \setminus \{0\}$, then $X(Q, \lambda) \simeq X(Q, \lambda')$ (equivariantly homeomorphic).

Example 2.6 (Spindle)**.** We denote the cyclic k-group by C_k and consider the natural surjection $p_k : \mathbb{C} \to \mathbb{C}/C_k$. We define a *spindle* $S^2(m, n)$, for $m, n \neq 0$, as follows. The underlying topological space of $S^2(m,n)$ is homeomorphic to the 2-dimensional sphere S^2. Denote the northpole in S^2 as N and the southpole as S. The orbifold structure of $S^2(m,n)$, for $m,n>0$, is the maximal orbifold atlas $\mathcal{U}=\{(U_\alpha,V_\alpha,H_\alpha,p_\alpha)\}$ which contains the following two orbifold charts: (1) $(U_S,\mathbb{C},C_m,p_m)$ for $m>0$ around S whose open neighborhood U_S is defined by $S^2\setminus\{N\}$; (2) $(U_N,\mathbb{C},C_n,p_n)$ for $n>0$ around N whose open neighborhood U_N is defined by $S^2\setminus\{S\}$. The orbifold $S^2(m,m)$ is also called a *rugby ball* and $S^2(m,1)$ or $S^2(1,n)$ is also called a *tear drop*. If m is negative, then we consider the orbifold chart on U_S as $(U_S,\overline{\mathbb{C}},C_{-m},p_{-m})$, where $\overline{\mathbb{C}}$ is $\mathbb{C}$ with the reversed orientation, C_{-m} acts on it by the multiplication and $p_{-m}:\overline{\mathbb{C}}\to\overline{\mathbb{C}}/C_{-m}$ is the natural surjection. Similarly, we define the orbifold chart on U_N when $n<0$.

Note that the standard $\mathbb{T}^1$-action on $\mathbb{C}$ induces $(U_S,\mathbb{T}^1/C_m)\simeq(U_S,T^1)$ and $(U_N,\mathbb{T}^1/C_n)\simeq(U_N,T^1)$ by p_m and p_n respectively. Moreover, because the underlying space S^2 has an effective T^1-action, $S^2(m,n)$ also has an effective T^1-action. Then, there are two fixed point $\{N,S\}$. Therefore, $(S^2(m,n),T^1)$ is a torus orbifold and its orbit space $S^2(m,n)/T^1$ is the interval $[-1,1]$ such that $\{-1\}$ (resp. $\{1\}$) corresponds to S (resp. N). Then, $(U_S,\mathbb{C},C_m,p_m)$ (resp. $(U_S,\overline{\mathbb{C}},C_{-m},p_{-m})$) has the natural extension of the T^1-action whose normal orientation is determined by the standard orientation of $\mathbb{C}$ (resp. $\overline{\mathbb{C}}$) and $p_m:\mathbb{C}\to U_S\simeq\mathbb{C}/C_m$ (resp. $p_{-m}:\overline{\mathbb{C}}\to U_S\simeq\mathbb{C}/C_{-m}$) induces the homomorphism $\widetilde{T}^1\to T^1\simeq\widetilde{T}^1/C_{\pm m}$ by the m-times rotation. Similarly we have the T^1-extension on the orbifold chart on U_N. Hence, the characteristic function $\lambda_{m,n}:\{-1,1\}\to\mathbb{Z}\setminus\{0\}$ is defined by $\lambda_{m,n}(-1)=m$ and $\lambda_{m,n}(1)=n$. This means that the pair $([-1,1],\lambda_{m,n})$ defines the spindle $S^2(m,n)$ (see Figure 1), i.e., $S^2(m,n)$ is a torus orbifold over the manifold with corners $[-1,1]$.

Note that for any $m,n\in\mathbb{Z}\setminus\{0\}$, $(S^2(m,n),T^1)$ is equivariantly homeomorphic to $(S^2,T^1)\simeq(S^2(1,1),T^1)$. We also note that $(S^2(1,1),T^1)$, $(S^2(1,-1),T^1)$, $(S^2(-1,1),T^1)$, $(S^2(-1,-1),T^1)$ are torus manifolds with four omniorientations (they give four invariant stably complex structures on S^2), also see Remark 2.5.

FIGURE 1. The characteristic pair of the spindle $S^2(m,n)$.

3. The topology of a torus orbifold over the suspension of a simplex

Let Q be a manifold with faces with two vertices, we call such Q a *manifold with two vertices* for short. It is easy to check that there exists a characteristic function λ on Q. By definition, the T-action on $X(Q,\lambda)$ has exactly two fixed point. Conversely, if there are exactly two fixed points on $X(Q,\lambda)$ then its orbit space has exactly two vertices.

Definition 3.1. A torus orbifold X is said to be a *torus orbifold with two fixed points* if there is a manifold with two vertices Q such that X is a torus orbifold over Q.

In this section, we introduce some topological properties of the special class of torus orbifolds with two fixed points. By the definition of a manifold with faces, an n-dimensional manifold Q with two vertices, for $n\geq 2$, has the following properties:

(1) there exist exactly n facets, say $\mathcal{F}(Q)=\{F_1,\dots,F_n\}$;
(2) if $0<k<n$, the intersection $\bigcap_{j=1}^k F_{i_j}$ is connected, i.e., a codimensioin-k face of Q;
(3) conversely, for every codimensional k face H ($k\neq 0,n$), there exist exactly k distinct facets $F_{i_1},\dots,F_{i_k}$ such that $H=\bigcap_{j=1}^k F_{i_j}$;
(4) $\bigcap_{j=1}^n F_j=\{p,q\}$ (the set of all vertices of Q).

If $n=1$, Q is nothing but the interval $[-1,1]$, i.e., there are only two facets $\{-1\}$ and $\{1\}$.

The typical example of an n-dimensional manifold with two vertices is the suspension $\Sigma\Delta^{n-1}$ of the $(n-1)$-dimensional simplex Δ^{n-1}. Here, $\Sigma\Delta^{n-1}$ is defined by

$$\Delta^{n-1}\times[-1,1]/\sim,$$

where $[-1,1]$ is the interval and the equivalence relation $\sim$ is defined by collapsing $\Delta^{n-1}\times\{-1\}$ (resp. $\Delta^{n-1}\times\{1\}$) to the vertex p (resp. q). Note that a codimension-k face H of $\Sigma\Delta^{n-1}$, for $k=0,\dots,n-1$, is determined by the suspension ΣF of some codimension-k face F in Δ^{n-1} and codimension-n faces are the two vertices $p,\ q$ in $\Sigma\Delta^{n-1}$. It is easy to check the following property:

Proposition 3.2. *Let Q be an n-dimensional manifold with two vertices. Then Q is combinatorially equivalent to $\Sigma\Delta^{n-1}$.*

Remark 3.3. We define the topology on Δ^{n-1} by the induced topology from $\mathbb{R}^n_+$, i.e., $\Delta^{n-1} := \{(x_1,\dots,x_n)\in\mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$. By the connected sum with a homology sphere and $\Sigma\Delta^{n-1}$, we can construct an n-dimensional manifold with two vertices which is not *isomorphic* to $\Sigma\Delta^{n-1}$ (also see [8]).

Remark 3.4. We also note that $\Sigma\Delta^{n-1}$ can also be thought of as taking Δ^n and collapsing a facet.

If $n\geq 2$, the characteristic function $\lambda:\mathcal{F}(\Sigma\Delta^{n-1})\to\mathfrak{t}_{\mathbb{Z}}\cong\mathbb{Z}^n$ is often illustrated by the following $(n\times n)$-square matrix in $GL(n;\mathbb{Q})\cap M_n(\mathbb{Z})$:

$$\Lambda = (\lambda(F_1)\mid\cdots\mid\lambda(F_n)) = \begin{pmatrix}\lambda_{11} & \cdots & \lambda_{1n}\\ \vdots & \ddots & \vdots\\ \lambda_{n1} & \cdots & \lambda_{nn}\end{pmatrix} \tag{3.1}$$

where $\lambda(F_i)$, $i=1,\dots,n$, is a nonzero vector in $\mathbb{Z}^n$. Therefore, $X(\Sigma\Delta^{n-1},\lambda)$ (i.e. if we fix the topology of the manifold with two vertices) is completely determined by the above matrix Λ. We denote the torus orbifold over $\Sigma\Delta^{n-1}$ (for $n\geq 2$) determined by the matrix Λ as $X(\Lambda)$. The goal of this section is to describe the equivariant topological type of a torus orbifold $X(\Lambda)$.

3.1. The equivariant topological type of $X(\Lambda)$.

Let $\pi : X(\Lambda)\to\Sigma\Delta^{n-1}$ be the orbit projection of the T^n-action on $X(\Lambda)$ and X_i be the $(2n-2)$-dimensional torus suborbifold defined by $\pi^{-1}(F_i)$, for $i=1,\dots,n$, i.e. the characteristic suborbifold associated to F_i. Since the characteristic function $\lambda_i := \lambda(F_i)$ is a vector in $\mathfrak{t}_{\mathbb{Z}}$, one can define the circle subgroup $\exp\mathbb{R}\lambda_i\subset T^n$, say S_i. Then $X_i = X(\Lambda)^{S_i}$ is the fixed point set of the restricted S_i-action.

Note that the integer square matrix Λ defined as in (3.1) induces the isomorphism $\Lambda:\mathbb{R}^n\to\mathbb{R}^n$; therefore, this induces the isomorphism $\mathbb{R}^n/\mathbb{Z}^n\to\mathbb{R}^n/\Lambda(\mathbb{Z}^n)$, which we also call Λ. Hence, we can define the following surjective homomorphism:

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \xrightarrow[\cong]{\Lambda} \mathbb{R}^n/\Lambda(\mathbb{Z}^n)\cong\mathfrak{t}/\Lambda(\mathfrak{t}_{\mathbb{Z}}) = \textstyle\prod_{i=1}^n S_i \xrightarrow{\iota} T^n = \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}} \tag{3.2}$$

where $\mathbb{T}^n$ is the standard n-dimensional torus in $\mathbb{C}^n$ and the surjective homomorphism ι is induced from the product of the injective homomorphisms $\iota_i : S_i\to T^n$. Put

$$\widetilde{T}^n := \prod_{i=1}^n S_i,$$

then via the isomorphism Λ we may regard $\widetilde{T}^n$ as the standard torus $\mathbb{T}^n$. By this identification, the standard $\widetilde{T}^n$-action on the unit $2n$-dimensional sphere $S^{2n}(\oplus_{i=1}^n\mathbb{C}\lambda_i\oplus\mathbb{R})$ may be regarded as the standard $\mathbb{T}^n$-action on $\mathbb{S}^{2n} := S^{2n}(\mathbb{C}^n\oplus\mathbb{R})$, where the symbol $S^{2n}(V(\rho)\oplus\mathbb{R})$ for a complex n-dimensional T^n-representation space $V(\rho)$ represents the unit sphere (with respect to the standard metric) in $\mathbb{C}^n\times\mathbb{R}\simeq V(\rho)\oplus\mathbb{R}$ with the torus action induced from the representation $\rho : T^n\to T^n$. This is known as one of the torus manifolds with two fixed points, and its characteristic submanifold $M_i\subset\mathbb{S}^{2n}$, $i=1,\dots,n$, is defined as

$$M_i = \{(z_1,\dots,z_n,r)\in\mathbb{S}^{2n}\subset\mathbb{C}^n\oplus\mathbb{R}\mid z_i=0\}. \tag{3.3}$$

Therefore, its normal orientaion can be canonically determined by the orientation of the ith complex space. Denote

$$G(\Lambda) := \ker\iota\circ\Lambda\subset\mathbb{T}^n.$$

We also denote the complex one-dimensional representation $\iota_i \circ \Lambda$ by

$$\mu_i := \iota_i \circ p_i \circ \Lambda : \mathbb{T}^n \to \widetilde{T}^n \to S_i \hookrightarrow T^n.$$

It is well-known that the following group is a product of cyclic groups, i.e., a finite abelian group (see Remark 3.11):

$$G(\Lambda) = \bigcap_{i=1}^{n} \ker \mu_i \cong \mathfrak{t}_{\mathbb{Z}}/\Lambda(\mathfrak{t}_{\mathbb{Z}}).$$

Then, we can consider the following action induced from the standard action $(\mathbb{S}^{2n}, \mathbb{T}^n)$:

$$(\mathbb{S}^{2n}/G(\Lambda), \mathbb{T}^n/G(\Lambda)) \simeq (\mathbb{S}^{2n}/G(\Lambda), T^n).$$

It is easy to check that there are exactly two fixed points in this action and that the characteristic suborbifolds $X_i \subset \mathbb{S}^{2n}/G(\Lambda)$, $i = 1, \ldots, n$, are given by

$$X_i = M_i/G(\Lambda),$$

where the M_i's are defined in (3.3). Since only the ith factor of $\mathbb{T}^n$ fixes M_i, X_i is fixed by $\mu_i(\mathbb{T}^n) = S_i = \exp\mathbb{R}\lambda_i \subset T^n$. Here, we define the normal orientation of X_i as the normal orientation of M_i. This shows that the characteristic function of $(\mathbb{S}^{2n}/G(\Lambda), T^n)$ coincides with that of $(X(\Lambda), T^n)$. Hence by using Lemma 2.4, we have the following theorem:

Theorem 3.5. *Assume $n \geq 2$. Let $X(\Lambda)$ be a torus orbifold over $Q = \Sigma\Delta^{n-1}$ with a characteristic function Λ. Then, $(X(\Lambda), T^n)$ is equivariantly homeomorphic to $(\mathbb{S}^{2n}/G(\Lambda), \mathbb{T}^n/G(\Lambda))$.*

Remark 3.6. If $G(\Lambda)$ is the identity group, then $(\mathbb{S}^{2n}/G(\Lambda), \mathbb{T}^n/G(\Lambda)))$ is the torus manifold $(\mathbb{S}^{2n}, \mathbb{T}^n)$.

If $n = 1$, then $G(\Lambda)$ is the cyclic group C_m. Therefore, $(\mathbb{S}^2/G(\Lambda), \mathbb{T}^1/G(\Lambda)))$ is the rugby ball with T^1-action $(\mathbb{S}^2(m, m), T^1)$. It is well-known that the spindle $S^2(m, n)$ for $|m| \neq |n|$ is a bad orbifold, i.e., it can not be obtained by the global quotient of S^2. This shows that Theorem 3.5 does not hold for $n = 1$.

Theorem 3.5 leads us to the following corollary which can be obtained by applying Lemma 2.2 to the case of a torus orbifold over Q:

Corollary 3.7. *Let $U_x = X(\mathbb{R}^n_+, \lambda|_{\mathbb{R}^n_+})$ be the open invariant neighborhood around a fixed point x of a torus orbifold $X(Q, \lambda)$, i.e., the λ restricts to a facet around x, say $\lambda|_{\mathbb{R}^n_+}$. Let Λ be the $(n \times n)$-matrix as in* (3.1) *which defines the characteristic function $\lambda|_{\mathbb{R}^n_+}$. Then, the following holds:*

$$(U_x, T^n) \simeq (\mathbb{C}^n/G(\Lambda), \mathbb{T}^n/G(\Lambda)),$$

where $\simeq$ represents an equivariant homemorphism. Furthermore, there is the following special orbifold chart around x:

$$(U_x, V_x, H_x, p_x) = (U_x, \mathbb{C}^n, G(\Lambda), p_x : \mathbb{C}^n \to \mathbb{C}^n/G(\Lambda) \simeq U_x).$$

Note that $G(\Lambda)$ acts on $S^{2n-1} = \{(z_1, \ldots, z_n, 0) \in S^{2n}(\mathbb{C}^n \oplus \mathbb{R})\}$. Denote its orbit space by

$$L(\Lambda) := S^{2n-1}/G(\Lambda),$$

which is called an *orbifold lens space* in [5]. We note that this orbifold $L(\Lambda)$ has a natural T^n-action.

Remark 3.8. When $G(\Lambda)$ is isomorphic to a cyclic group C_p and acts freely on S^{2n-1}, then $L(\Lambda)$ is a lens space. Kawasaki [15] considers the case when a cyclic group C_p acts almost freely on S^{2n-1} and calls the quotient space S^{2n-1}/C_p the *twisted lens space* which is an orbifold in general (in [2, 4], a twisted lens space is also called a *weighted lens space*).

We also have the following:

Corollary 3.9. *The torus orbifold $(X(\Lambda), T^n)$ is equivariantly homeomorphic to $(\Sigma L(\Lambda), T^n)$, where the T^n-action on the suspension $\Sigma L(\Lambda)$ is the natural extension of the T^n-action on $L(\Lambda)$.*

Moreover, we have the following lemma:

Lemma 3.10. *Assume $n \geq 2$. Let N be the smallest subgroup of $G(\Lambda)$ which contains all those elements of $G(\Lambda)$ that fix points in S^{2n-1}. Then $H^3(X(\Lambda)) \cong G(\Lambda)/N$.*

Proof. Because $n \geq 2$, S^{2n-1} is simply connected. Moreover, $G(\Lambda)$ acts on S^{2n-1} effectively. Therefore, by [3], we have that $\pi_1(S^{2n-1}/G(\Lambda)) \cong G(\Lambda)/N \cong H_1(L(\Lambda))$. This shows that $H_1(L(\Lambda))$ is torsion if $G(\Lambda)/N$ is not the identity group; namely, $G(\Lambda) \neq N$. By using the universal coefficient theorem, we have $H_1(L(\Lambda)) \cong H^2(L(\Lambda))$. Therefore, it follows from the Mayer-Vietoris exact sequence that $H^2(L(\Lambda)) \cong H^3(\Sigma L(\Lambda)) = H^3(X(\Lambda))$. □

In particular, if $(S^{2n-1})^{G(\Lambda)} \neq \emptyset$, then $H^3(X(\Lambda)) = 0$. Moreover, $H^{odd}(X(\Lambda)) = 0$ if $G(\Lambda) = \{e\}$. Namely, if $\det \Lambda = \pm 1$, then $H^{odd}(X(\Lambda)) = 0$.

Remark 3.11. By using the Smith normal form, there are $P, Q \in GL(n;\mathbb{Z})$ such that the sequence (3.2) can be written as follows:

$$\begin{tikzcd} \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \arrow[r, "\Lambda", "\cong"'] \arrow[d, "P"'] & \mathbb{R}^n/\Lambda(\mathbb{Z}^n) \cong \widetilde{T}^n \arrow[r, two heads, "\iota"] \arrow[d, "Q"'] & T^n = \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}} \arrow[d, "\cong"'] \\ \mathbb{R}^n/P(\mathbb{Z}^n) \arrow[r, "\Lambda'", "\cong"'] & \mathbb{R}^n/(r_1\mathbb{Z} \oplus \cdots \oplus r_n\mathbb{Z}) \arrow[r, two heads, "\iota"] & \mathbb{T}^n/(C_{r_1} \times \cdots \times C_{r_n}) \end{tikzcd} \tag{3.4}$$

where

$$\Lambda' = Q\Lambda P^{-1} = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}$$

for some positive integers $r_1, \ldots, r_n$ such $r_1|r_2|\cdots|r_n$, and $C_r \cong \mathbb{Z}/r\mathbb{Z}$ is the cyclic subgroup in $\mathbb{T}^1$. Here, we can compute r_i, $i = 1, \ldots, n$, as

$$r_i = \frac{m_i(\Lambda)}{m_{i-1}(\Lambda)},$$

where $m_0(\Lambda) := 1$ and $m_i(\Lambda)$ is the ith *determinant divisor*, i.e., the greatest common divisor of all $i \times i$ minors of Λ.

Remark 3.12. As $X(\Lambda)$ is simply connected, $H^1(X(\Lambda)) = 0$.

Suppose that the product of cyclic groups $G = C_{r_1} \times \cdots \times C_{r_n}$ acts on $S^{2n-1} \subset \mathbb{C}^n$ coordinatewise. Let N be the smallest subgroup in G which contains all those elements of G which have fixed points. Then, $N = G$. Therefore, by Lemma 3.10, $H^3(S^{2n}/G) = 0$.

4. The equivariant cohomology of $X(\Lambda)$ with $H^{odd}(X(\Lambda)) = 0$

In this section, we compute the equivariant cohomology of $X(\Lambda)$ with $H^{odd}(X(\Lambda)) = 0$ by using the formula in [6].

4.1. **Orbifold torus graph of (Q, λ) and its equivariant graph cohomology.** Let (Q, λ) be a pair of a manifold with faces and its characteristic function. We shall define an *orbifold torus graph* ((n, n)-type *GKM-graph* with rational axial function) $\Gamma(Q, \lambda) := (\Gamma, \alpha)$ of (Q, λ) as follows. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be the one-skeleton of Q. By the definition of a manifold with faces, this becomes an n-valent graph. Moreover, we see that each edge e is a connected component of the intersection of exactly $(n-1)$ facets, say $F_{k_1}, \ldots, F_{k_{n-1}}$; and we also denote one of the normal facets of e by F_{k_n}, i.e., $F_{k_n} \cap e$ contains the initial vertex $i(e)$ with the appropriate orientation. Now, we define a function

$$\alpha : E(\Gamma) \to \mathfrak{t}^*_{\mathbb{Q}}$$

by the following system of equations:

$$\begin{cases} \langle \alpha(e), \lambda(F_{k_1}) \rangle = \cdots = \langle \alpha(e), \lambda(F_{k_{n-1}}) \rangle = 0; \\ \langle \alpha(e), \lambda(F_{k_n}) \rangle = 1, \end{cases} \tag{4.1}$$

where $\langle\ ,\ \rangle$ denotes the natural paring between a vector space $\mathfrak{t}_\mathbb{Q}$ and its dual space $\mathfrak{t}_\mathbb{Q}^*$. Note that $\lambda(F_i) \in \mathfrak{t}_\mathbb{Z} \subset \mathfrak{t}_\mathbb{Q}$. We call such a labeled graph (Γ, α) an *orbifold torus graph* of (Q, λ) and often denote it by $\Gamma(Q, \lambda)$.

Remark 4.1. In [6], we define an *abstract orbifold torus graph* like [17] without (Q, λ).

Example 4.2. The spindle $S^2(m, n)$ is defined by $([-1, 1], \lambda_{m,n})$ in Example 2.6. Set the generator of $\mathfrak{t}_\mathbb{Z}^*$ by x, i.e., $\mathfrak{t}_\mathbb{Z}^* = \mathbb{Z}x$ and $\mathfrak{t}_\mathbb{Q}^* = \mathbb{Q}x$. In this case, $[-1, 1]$ itself, say e, is the edge in this manifold with corners and the two vertices $i(e) = \{-1\}$ and $t(e) = \{1\}$ are the only facets. It follows from the definition of $\Gamma([-1, 1], \lambda_{m,n})$ that we have the following axial function (see Figure 2):

$$\alpha(e) = \frac{1}{m}x, \quad \alpha(\overline{e}) = \frac{1}{n}x.$$

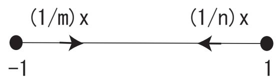

FIGURE 2. The orbifold torus graph of $([-1, 1], \lambda_{m,n})$.

Example 4.3. Assume $n \geq 2$. Suppose that $Q = \Sigma\Delta^{n-1}$ and consider the torus orbifold $X(\Lambda)$ over Q. Let $\widetilde{\Lambda}$ be the transpose of the cofactor matrix of Λ, i.e., $\widetilde{\Lambda}\Lambda = (\det \Lambda)I_n$. We may put

$$\widetilde{\Lambda} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \tag{4.2}$$

where $\mu_i \in \mathfrak{t}_\mathbb{Z}^*$ is the one-dimensional representation of $\mathbb{T}^n$ defined by $\iota_i \circ \Lambda$ as in Section 3.1. Then, by equation (4.1), the vector $\mu_i \in \mathfrak{t}_\mathbb{Z}^*$ satisfies the following equation:

$$\alpha(e_i) = \frac{1}{\det \Lambda}\mu_i, \tag{4.3}$$

where e_i is the edge of $\Sigma\Delta^{n-1}$ which is not contained in the facet F_i (see Figure 3).

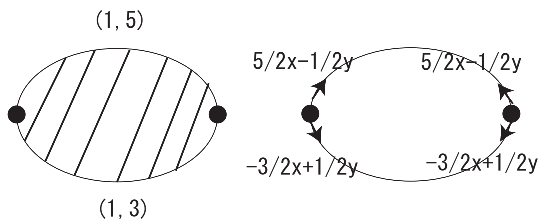

FIGURE 3. The case when $n = 2$. The left figure is $(\Sigma\Delta^1, \lambda)$ and the right one is its orbifold torus graph, where $\mu_1 = 5x - y$ and $\mu_2 = -3x + y$ for some generators x, y in $\mathfrak{t}_\mathbb{Z}^*$.

For an orbifold torus graph (Γ, α), we define the following rings.

Definition 4.4 (Graph equivariant cohomology)**.** The following ring is said to be the (integral) *graph equivariant cohomology ring*:

$$H_T^*(\Gamma, \alpha) = \{f : V(\Gamma) \to \mathrm{Sym}(\mathfrak{t}_\mathbb{Z}^*) \mid f(i(e)) \equiv f(t(e)) \bmod r_e\alpha(e)\}, \tag{4.4}$$

where r_e is the minimal positive integer such that $r_e\alpha(e) \in \mathrm{Sym}(\mathfrak{t}_\mathbb{Z}^*) \cong H^*(BT; \mathbb{Z})$ and $i(e)$ (resp. $t(e)$) is the initial (resp. terminal) vertex of a directed edge e.

Here, $H_T^*(\Gamma,\alpha)$ may be regarded as an $H^*(BT;\mathbb{Z})$-subalgebra of $\bigoplus_{v\in V(\Gamma)} H^*(BT;\mathbb{Z})$ by identifying $\mathrm{Sym}(\mathfrak{t}_{\mathbb{Z}}^*)$ with $H^*(BT;\mathbb{Z})\cong\mathbb{Z}[x_1,\dots,x_n]$. Then, $r_e\alpha(e)\in H^2(BT;\mathbb{Z})$ and there is a natural grading in $H_T^*(\Gamma,\alpha)$ induced by the grading of $H^*(BT;\mathbb{Z})$.

Denote by $\overline{e}$ the reversed oriented edge of e, i.e., $i(e)=t(\overline{e})$ and $t(e)=i(\overline{e})$. Note that all edges satisfy $r_e\alpha(e)=\pm r_{\overline{e}}\alpha(\overline{e})$ (see [6]). We may define the cohomology of (Γ,α) over rational coefficients as follows:

$$H_T^*(\Gamma,\alpha;\mathbb{Q})=\left\{f:V(\Gamma)\to\mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*)\mid f(i(e))\equiv f(t(e)) \bmod \alpha(e)\right\}. \tag{4.5}$$

Similarly, this has the natural $\mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*)\cong H^*(BT;\mathbb{Q})$-algebra structure. This coincides with the definition of the cohomology ring of a GKM graph (of a symplectic orbifold) in [13, Section 1.7]. One can see that $H_T^*(\Gamma,\alpha)$ is a subring of $H_T^*(\Gamma,\alpha;\mathbb{Q})$. We call $H_T^*(\Gamma,\alpha;\mathbb{Q})$ the *rational graph equivariant cohomology.*

The next theorem is a consequence of applying the main result of [6] restricted to the case of torus orbifolds with two fixed points.

Theorem 4.5. *Assume $X:=X(Q,\lambda)$ satisfies $H^{odd}(X)=0$. Then, there is an isomorphism*

$$H_T^*(X)\cong H_T^*(\Gamma,\alpha),$$

where (Γ,α) is the orbifold torus graph of (Q,λ) and $H_T^*(X):=H^*(ET\times_T X;\mathbb{Z})$ is the equivariant cohomology of X.

Remark 4.6. The above theorem also holds for more general GKM orbifolds with vanishing odd degree cohomology (see [6]).

4.2. **Weighted face ring.** Given an n-valent orbifold torus graph (Γ,α) of (Q,λ) and an $(n-k)$-dimensional face F in Q, there is an $(n-k)$-valent subgraph Γ_F which is the one-skeleton of F. We also call this subgraph an $(n-k)$-*dimensional face* of (Γ,α). Each face F defines a *rational Thom class* $\tau_F\in H_T^{2k}(\Gamma,\alpha;\mathbb{Q})$ as follows:

$$\tau_F(v):=\begin{cases}\prod_{\substack{i(e)=v\\ e\notin E(\Gamma_F)}}\alpha(e) & \text{if } v\in V(\Gamma_F);\\ 0 & \text{otherwise.}\end{cases} \tag{4.6}$$

Note that $\deg\tau_F=2k$ for the codimension-k face F.

We formally define

$$\tau_\emptyset=0,\quad \tau_\Gamma=1.$$

Note that $\deg\tau_\emptyset=\deg\tau_\Gamma=0$.

Example 4.7. The following two figures (Figure 4 and Figure 5) are the examples of rational Thom classes of the orbifold torus graph in Figure 3.

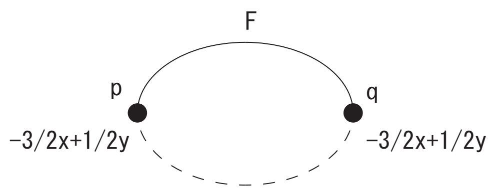

FIGURE 4. The rational Thom classes of the facet F in Figure 3, i.e., $\tau_F(p)=\tau_F(q)=-\frac{3}{2}x+\frac{1}{2}y$.

Let $\mathcal{F}_{all}(Q)$ be the set of all faces in Q and and $\mathscr{Z}_Q:=\mathbb{Z}[\tau_F\mid F\in\mathcal{F}_{all}(Q)]$ be the integer (graded) polynomial ring generated by the rational Thom classes. Set the graded subring of $\mathscr{Z}_Q$ as

$$\mathscr{Z}_{Q,\lambda}:=\left\{f\in\mathscr{Z}_Q\;\middle|\;\forall v\in V(\Gamma),\ f(v)\in H^*(BT^n;\mathbb{Z})\right\}.$$

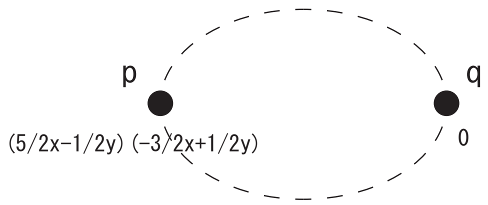

FIGURE 5. The rational Thom classes of the vertex p in Figure 3, i.e., $\tau_p(p) = (\frac{5}{2}x - \frac{1}{2}y)(-\frac{3}{2}x + \frac{1}{2}y)$ and $\tau_p(q) = 0$.

Then, it is easy to see that the elements in $\mathscr{Z}_Q$ of the form

$$\tau_E\tau_F - \tau_{E\vee F}\sum_{G\in E\cap F}\tau_G \tag{4.7}$$

are 0 for all vertices $v \in V(Q)$, where the summation runs through all connected components in $E \cap F$. Here the symbol $E \vee F$ represents the minimal face which contains both E and F; note that if $E \cap F \neq \emptyset$ then the face $E \vee F$ can be uniquely determined (also see [17]). Therefore, we can define the ideal $\mathcal{I}$ of $\mathscr{Z}_{Q,\lambda}$ generated by all elements defined by (4.7). Set

$$\mathbb{Z}[Q,\lambda] := \mathscr{Z}_{Q,\lambda}/\mathcal{I}.$$

We call this ring the *weighted face ring* of (Q, λ).

The following theorem is one of the main theorems in [6]:

Theorem 4.8. *Let (Γ, α) be the orbifold torus graph induced from (Q, λ). Then the following graded rings are isomorphic:*

$$H_T^*(\Gamma, \alpha) \cong \mathbb{Z}[Q, \lambda].$$

Therefore, together with Theorem 4.5, we have the following corollary:

Corollary 4.9. *If the torus orbifold $X(Q,\lambda)$ satisfies $H^{odd}(X(Q,\lambda)) = 0$, then its equivariant cohomology satisfies*

$$H_T^*(X(Q,\lambda)) \cong \mathbb{Z}[Q,\lambda].$$

4.3. **The equivariant cohomology of $X(\Lambda)$ when $H^{odd}(X(\Lambda)) = 0$.** Now we may compute the equivariant cohomology of $X(\Lambda)$ when $H^{odd}(X(\Lambda)) = 0$. Due to Section 4.1, the graph $\Gamma = (V(\Gamma), E(\Gamma))$ of $\Sigma\Delta^{n-1}$ is given by

$$V(\Gamma) = \{p,\ q\}, \quad E(\Gamma) = \{e_1, \dots, e_n, \overline{e_1}, \dots, \overline{e_n}\}$$

where $i(e_i) = p$ and $t(e_i) = q$ for $i = 1, \dots, n$, and the axial function $\alpha : E(\Gamma) \to H^2(BT;\mathbb{Q})$ is given by the equation (4.3), i.e.,

$$\alpha(e_i) = \alpha(\overline{e_i}) = \frac{1}{\det\Lambda}\mu_i.$$

In this subsection, we assume that

$$G(\Lambda) = \{e\}.$$

Then, because $X(\Lambda)$ is a genuine manifold, i.e., $\mathbb{S}^{2n}$, we have $H^{odd}(X(\Lambda)) = 0$. Since $|G(\Lambda)| = |\det\Lambda| = 1$, we have that

$$\alpha(e_i) = \alpha(\overline{e_i}) = \pm\mu_i$$

for each $i = 1, \dots, n$. As $\det\Lambda = \pm1$, we also have $\det\widetilde{\Lambda} = \pm1$, where $\widetilde{\Lambda}$ is defined in (4.2). Hence, μ_i is a (primitive) vector in $\mathfrak{t}^*_{\mathbb{Z}}$. Therefore, the rational Thom class $\tau_i := \tau_{F_i}$ corresponding to a facet F_i, $i = 1, \dots, n$, is an element in $\mathscr{Z}_{Q,\lambda}$. Indeed,

$$\tau_i(p) = \tau_i(q) = \pm\mu_i \in H^2(BT;\mathbb{Z}).$$

Set τ_p and τ_q as the Thom classes of the vertices, i.e.,

$$\tau_p(p) = \prod_{i=1}^{n}(\pm\mu_i) \in H^{2n}(BT;\mathbb{Z}), \quad \tau_p(q) = 0;$$
$$\tau_q(q) = \prod_{i=1}^{n}(\pm\mu_i) \in H^{2n}(BT;\mathbb{Z}), \quad \tau_q(p) = 0.$$

We first prove the following theorem:

Theorem 4.10. *Let $Q(\cong \Sigma\Delta^{n-1})$ be a manifold with two vertices and λ a characteristic function on Q such that $|\det\Lambda| = 1$. Then,*

$$\mathbb{Z}[Q,\lambda] \cong \mathbb{Z}[\tau_1,\ \ldots,\ \tau_n,\ \tau_p,\ \tau_q]/\langle \tau_1\cdots\tau_n - (\tau_p+\tau_q),\ \tau_p\tau_q\rangle.$$

Proof. Due to the combinatorial structure of a manifold with two vertices Q (see Section 3), for every codimension-k face H $(0 < k < n)$, there is a subset $I \subset [n]$ such that $|I| = k$ and $H = \bigcap_{i\in I} F_i$. In this case, we write $H = F_I$ and denote its rational Thom class by τ_I. In particular, we denote the rational Thom class of a facet F_i by τ_i, for convenience.

Note that in $\mathbb{Z}[Q,\lambda]$ the following relation holds for $i \neq j$:

$$\tau_i\tau_j = \tau_\Gamma\tau_{\{i,j\}} = \tau_{\{i,j\}}.$$

Moreover, for $h \in [n] \setminus \{i,j\}$, we have

$$\tau_h\tau_{\{i,j\}} = \tau_{\{i,j,h\}} = \tau_i\tau_j\tau_h.$$

Similarly we have that

$$\tau_I = \prod_{i\in I}\tau_i,$$

for all non-empty proper subsets $I \subsetneq [n]$. This shows that we can reduce the generators in $\mathbb{Z}[Q,\lambda]$ to $\tau_1,\ldots,\tau_n$ and τ_p,τ_q. Namely, the ring homomorphism

$$\varphi : \mathbb{Z}[\tau_1,\ldots,\tau_n,\tau_p,\tau_q] \to \mathbb{Z}[Q,\lambda] = \mathscr{Z}_{Q,\lambda}/\mathcal{I},$$

induced from the injection $\rho : \mathbb{Z}[\tau_1,\ldots,\tau_n,\tau_p,\tau_q] \to \mathscr{Z}_{Q,\lambda}$, is a surjective ring homomorphism. In this case, one can easily see that $\ker\varphi \cong \mathrm{Im}\rho \cap \mathcal{I} = \langle \tau_1,\cdots,\tau_n - (\tau_p+\tau_q)\rangle$ which establishes the statement. ☐

Remark 4.11. Because $\mathbb{Z}[Q,\lambda]$ does not depend on the topology of Q, this theorem also holds for any manifold with two vertices Q (which may not be $\Sigma\Delta^{n-1}$).

Consequently, we have the following corollary:

Corollary 4.12. *If $H^{odd}(X(\Lambda)) = 0$ then*

$$H_T(X(\Lambda)) \cong \mathbb{Z}[\tau_1,\ \cdots,\ \tau_n,\ \tau_p,\ \tau_q]/\langle \tau_1\cdots\tau_n - (\tau_p+\tau_q),\ \tau_p\tau_q\rangle$$

where $\deg\tau_i = 2$, $i = 1,\ldots,n$, *and* $\deg\tau_p = \deg\tau_q = 2n$.

Remark 4.13. If $G(\Lambda) = \{e\}$ then $X(\Lambda) = \mathbb{S}^{2n}$, i.e., the torus manifold. Therefore, we can also obtain the above fact by using the main theorem of [17].

4.4. **The equivariant cohomology of $S^2(m,n)$.** In this final subsection, we apply Corollary 4.9 to the case when $|\det\Lambda| > 1$.

Recall the spindle $S^2(m,n)$. This is homeomorphic to S^2, therefore, $H^{odd}(S^2(m,n)) = 0$. The orbifold torus graph of $S^2(m,n)$ is the one defined in Figure 2.

In this case, the rational Thom classes for the two vertices p and q are defined as follows:

$$\tau_p(p) = \frac{1}{m}x,\ \tau_p(q) = 0 \quad \text{and} \quad \tau_q(p) = 0,\ \tau_q(q) = \frac{1}{n}x, \quad \text{respectively.}$$

Therefore, we have that

$$\mathscr{Z}_{m,n} := \mathscr{Z}_{([-1,1],\lambda_{m,n})} \cong \mathbb{Z}[m\tau_p, n\tau_q, f(\tau_p,\tau_q)\tau_p\tau_q \mid f(\tau_p,\tau_q) \in \mathbb{Z}[\tau_p,\tau_q]].$$

(Note that the generators are duplicated.) Hence, with the method similar to the one demonstrated in Section 4.1, it is easy to show the following theorem:

Theorem 4.14. *The T^1-equivariant cohomology of the spindle $S^2(m,n)$ is isomorphic to the following ring:*

$$H_T^*(S^2(m,n)) \cong \mathscr{Z}_{m,n}/\mathcal{I} \cong \mathbb{Z}[m\tau_p, n\tau_q]/\langle mn\tau_p\tau_q\rangle,$$

where $\deg \tau_p = \deg \tau_q = 2$.

It is well-known that the T^1-action on $S^2(m,n)$ is equivariantly homeomorphic to the standard $\mathbb{T}^1$-action on $\mathbb{S}^2$. Therefore, for any $m, n (\neq 0)$, their equivariant cohomologies are isomorphic. We finally remark the following proposition; the proposition may be regarded as the generalization of this fact for $S^2(m,m) = \mathbb{S}^2/C_m$.

Proposition 4.15. *If Λ is the diagonal matrix (see Λ' in Remark 3.11) then the torus orbifold $X(\Lambda)$ is equivariantly homeomorphic to the torus manifold obtained by $\Lambda = I_n$, i.e., the standard $2n$-dimensional sphere $\mathbb{S}^{2n}$ with $\mathbb{T}^n$-action.*

Proof. If Λ is the diagonal matrix Λ' in Remark 3.11 then $X(\Lambda)$ is the orbifold $S^{2n}/C_{r_1} \times \cdots \times C_{r_n}$, where $C_{r_1} \times \cdots \times C_{r_n}$ acts on the complex coordinates in $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ standardly. Then, because the scalar products on the characteristic functions does not change the topological type of the underlying topological space, this is equivariantly homeomorphic to the torus manifold obtained by $\Lambda = I_n$. This establishes the statement. $\square$

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