Toric Topology 2019
in Okayama

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Sheaves on GKM-graphs
and the group of axial functions

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Motivation

We study the upper estimate on the GKM-torus T dimension acting on a fixed smooth mfd M^{2n} by the rank of the axial function group [Ku]. In particular, we develop:

- Computational methods for the axial function group in terms of sheaf theory;
- Necessary conditions on invariant functions of GKM-graphs ("monodromy relations" on weighted cycle graphs).

GKM-action

Smooth (usually compact) mfd M^{2n}, smooth effective action $T^k = (S^1)^k$ on M^{2n}, T^k-invariant almost complex structure on M, $|M^T| < \infty$.

$$T_x X^n \simeq \bigoplus_{j=1}^{n} V(w_j),\ w_j = w_j(x) \in \mathrm{Hom}(T^k, T^1) \simeq \mathbb{Z}^k.$$

Weights at $x \in M^T$ are called q-linearly independent if $w_{j_1}(x), \ldots, w_{j_q}(x)$ are linearly independent for any pairwise different $j_1, \ldots, j_q \in [n]$.

Definition

$T^k : M$ is called a GKM-action, if the weights at $x \in M^T$ are 2-linearly independent.

GKM-manifolds and GKM-graphs

GKM-action $T^k : M^{2n} \rightsquigarrow$ GKM-data (Γ, α, ∇)

Theorem ([GuZa])

Let (Γ, α, ∇) *be a GKM-graph. Then there exists a GKM-action* $T : M$ *with the GKM-graph* (Γ, α, ∇)*.*

Remark

In this theorem, the manifold M is complex, non-compact and is not equivariantly formal.

GKM-graph-I

Simple (i.e. no loops, no multiple edges), connected, finite n-valent graph $\Gamma = (V(\Gamma), E(\Gamma))$, $e = \big(i(e), t(e)\big)$, $\overline{e} = \big(t(e), i(e)\big)$, $E_v(\Gamma) = \{e \in E(\Gamma) | \ i(e) = v\}$.

Definition

A collection of bijections $\nabla_e : \ E_{i(e)}(\Gamma) \to E_{t(e)}(\Gamma)$ is called a connection on Γ, if

$$\nabla_{\overline{e}} = (\nabla_e)^{-1}, \ \nabla_e(e) = \overline{e}.$$

A connected subgraph $\Gamma' \subseteq \Gamma$ is called a face of Γ, if $\nabla_e e' \in E(\Gamma')$ for any $e, e' \in E_v(\Gamma')$, $v \in V(\Gamma')$.

GKM-graph-II

Definition

Axial function $\alpha: \ E(\Gamma) \to \mathrm{Hom}(\mathbb{T}^k, \mathbb{T}) \simeq \mathbb{Z}^k$ on Γ satisfies

$$\alpha(\bar{e}) = -\alpha(e), \ \alpha(\nabla_e e') - \alpha(e') = c_e(e') \cdot \alpha(e), \tag{1}$$

for any $e \in E(\Gamma)$, $e' \in E_{i(e)}(\Gamma)$ and some $c_e(e') \in \mathbb{Z}$.

The rank $\mathrm{rk}\,\alpha := \mathrm{rk}\,\mathbb{Z}\langle \alpha(e) |\ e \in E_v(\Gamma)\rangle = k$ of axial function is well-defined. We require $\mathrm{rk}\,\alpha = k$.
(Γ, α, ∇) is called the GKM-graph [GuZa].

Proposition

If α is $(q+1)$-linearly independent, then for any $v \in V(\Gamma)$, any pairwise-different edges $e_1, \ldots, e_q \in E_v(\Gamma)$ span a q-face in Γ. (We call such a connection ∇ the q-complete connection.)

Invariant and axial functions

Definition

An invariant function on (Γ, ∇) is $(c(e))_{e\in E(\Gamma)}$, $c(e) \in \mathbb{Z}E_{i(e)}(\Gamma) \simeq \mathbb{Z}^n$ s.t.

$$\langle c(e), e\rangle = -2,\ \nabla_e c(e) = c(\bar{e})\ (\langle e, e'\rangle = \delta_e^{e'}).$$

(Γ, α, ∇) defines the invariant function c as $c(e) := \sum_{e'\in E_{i(e)}(\Gamma)} c_e(e') \cdot e'$.

Definition (Different denotation from [Ku])

The group $\mathcal{A}(\Gamma, \alpha, \nabla)$ of axial functions is the set of $(f(v))_{v\in V(\Gamma)}$ s.t.

$$f(t(e)) = \nabla_e\big(f_{i(e)} + \langle f(i(e)), e\rangle \cdot c(e)\big),\ e \in E(\Gamma),$$

with vertex-wise summation.

$\mathcal{A}(\Gamma, \alpha, \nabla)$ is a free abelian group of rank $\leqslant n$.

Estimate on dim T

Consider GKM-actions $T^k \subseteq T^r : M^{2n}$ with (Γ, α, ∇), $(\Gamma', \alpha', \nabla')$, resp.

One has $\Gamma' = \Gamma$, $\alpha = \pi\alpha'$ for $\pi : \mathbb{Z}^r \to \mathbb{Z}^k$.

α is 3-linearly independent $\Rightarrow (\Gamma, \alpha)$ admits $\leqslant 1$ connection, i.e. $\nabla' = \nabla$.

Theorem (S. Kuroki [Ku])

$r \leqslant \operatorname{rk} \mathcal{A}(\Gamma, \alpha, \nabla)$.

Applications: torus actions of max dimension

- $T^{n+1} : G_2(\mathbb{C}^{n+2})$;
- Milnor hypersurface $T^j : H_{i,j}$, $0 \leqslant i < j$.

Not easy to compute $\mathcal{A}(\Gamma, \alpha, \nabla)$!

Sheaves: basics

Poset category $O(X)$ of open subspaces in top. space X.

Pre-sheaf: contravariant functor $O(X) \to C$ to sets, groups, rings, etc.

Sheaf: pre-sheaf satisfying gluing and locality axioms.

Constant sheaf $\underline{A}$: sheaving of $C(U, A)$, with discrete top. on A.

Locally constant sheaf $\mathcal{F}$: open cover (U_i), $\mathcal{F}|_{U_i}$ isom. to constant sheaf.

Ringed space: $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of rings.

Locally free sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules: stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules (we consider only finitely presented sheaves).

Sheaves on GKM-graphs: Topology

$H_T^*(M)$ isom. to the global sections of a locally free sheaf of graded rings on Γ [BrMc], [Ba].

Graph topology (T. Baird [Ba])

Let $\mathrm{Top}\,\Gamma = V(\Gamma) \sqcup E(\Gamma)$ be the topological space with the topology generated by open sets $U_v = \{v\}$, $U_e = \{e, i(e), t(e)\}$, where $e \in E(\Gamma)$, $v \in V(\Gamma)$. (Open sets=subgraphs, closed sets=unions of edges.)

$\mathrm{Top}\,\Gamma$ is connected, locally path connected, semi-locally simply connected, but not Hausdorff.

Continuous paths $\gamma : [0,1] \to \mathrm{Top}\,\Gamma$ with ends at vertices are step functions with ordered values of the form $\{i(e_1), e_1, t(e_1), e_2, \ldots, t(e_{q-1}), e_q, t(e_q)\}$, where $e_1, \ldots, e_q$ is an edge path in Γ.

$\pi_1(\mathrm{Top}\,\Gamma; v) \simeq \pi_1(\Gamma, v)$ is a free group of rank $E(\Gamma) - E(\Gamma')$ for a maximal tree Γ' in Γ.

Sheaves on GKM-graphs: local systems

Let X be a path connected top. space. Continuous paths $P(X)$ in X.

Definition

A collection $\mathcal{G}$ of groups G_x, $x \in X$, and homomorphisms $\varphi_\gamma : \ G_{x_1} \to G_{x_2}$ depending only on the homotopy type of $\gamma \in P(X)$, is called the local system of groups on X. Composition rule. $\varphi_x = Id_{G_x}$. Fiber G. Morphisms.

Theorem

Cat's of locally constant sheaves of f.g. groups on X and local systems on X are equivalent.

Local system on $\mathrm{Top}\,\Gamma \overset{1:1}{\leftrightarrow}$ Group G and homomorphisms φ_e, $e \in E(\Gamma)$.

Sheaves on GKM-graphs: examples-I

Example

We define the local system $\mathcal{C} = \mathcal{C}(\Gamma, \alpha, \nabla)$ as $\mathcal{C}_u = \mathbb{Z}E_u(\Gamma)$, and

$$\varphi_e : \mathcal{C}_{i(e)} \to \mathcal{C}_{t(e)}, \; x \mapsto \nabla_e(x + \langle x, e\rangle \cdot c(e)). \tag{2}$$

It is well-defined and $\mathcal{A}(\Gamma, \alpha, \nabla) = \mathcal{C}(\Gamma)$.

Definition

Define $\mathcal{A}' = \mathcal{A}'(\Gamma, \nabla, c)$ to be the group of homomorphisms $\alpha : E(\Gamma) \to \mathbb{Z}^n$ satisfying

$$\alpha(\nabla_e e') - \alpha(e') = \langle c(e), e'\rangle \cdot \alpha(e). \tag{3}$$

We call $\mathcal{A}'$ the group of label functions on (Γ, ∇, c).

Sheaves on GKM-graphs: examples-II

For $\alpha \in \mathcal{A}'$, there is $\mathrm{rk}\,\alpha$. Axial functions on (Γ, ∇) with inv. function c are those elements of $\mathcal{A}'$ of rank k which are 2-linearly independent.

Example

Define the local system $\mathcal{C}'$ on $(\mathrm{Top}\,\Gamma, V)$ by $\mathcal{C}'_u = \mathrm{Hom}(\mathbb{Z}E_u(\Gamma), \mathbb{Z}^n)$, and

$$\varphi_e : \ \mathcal{C}_{i(e)} \to \mathcal{C}_{t(e)}, \ x \mapsto \varphi_e(x),$$

$$\varphi_e(x)(\nabla_e e') = x(e') + \langle c(e), e' \rangle \cdot x(e), \ \ e' \in E_{i(e)}(\Gamma). \tag{4}$$

$\mathcal{C}'$ is well-defined and $\mathcal{A}' = \mathcal{C}'(\Gamma)$.

Theorem (—, '19)

Sheaf isom. $\mathcal{C}' \simeq \mathcal{C}^{\oplus n}$. *In particular, group isom.* $\mathcal{A}' \simeq \mathcal{A}^{\oplus n}$.

Global sections as monodromy invariants

Path connected top. space X, locally constant sheaf $\mathcal{G}$ on X with fiber G. There is the monodromy representation

$$\pi_1(X, x_0) \to \mathrm{Aut}\, G, \tag{5}$$

of the group $\pi_1(X, x_0)$.

Proposition

$\mathcal{G}$ is completely determined by (5). There is the group isomorphism

$$\mathcal{G}(X) \simeq G^{\pi_1(X, x_0)}.$$

Choosing generators $a_1, \dots, a_r$ of $\pi_1(X, x_0)$ one can compute $G^{\pi_1(X,x_0)} = \bigcap_{i=1}^r F_{\mathcal{G}}(a_i)$, where $F_{\mathcal{G}} = \mathrm{Ker}(A_i - Id) \subseteq G$, and $A_i \in \mathrm{Aut}\, G$ corresponds to a_i.

Gluing global sections from invariant loops

Open cover $(U_i)_{i\in I}$ of X.
Global sections $s \in \mathcal{G}(X) \overset{1:1}{\leftrightarrow}$ is a collection $s_i \in \mathcal{G}(U_i)$ s.t.

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}.$$

Consider (Γ, ∇).

If ∇ is 2-complete, then there exists open cover of $\operatorname{Top}\Gamma$ by invariant loops γ_i (i.e. invariant simple cycles), $i = 1, \ldots, r$.

Computation of $\mathcal{G}(X)$

1) compute $\mathcal{G}(\gamma_i)$;
2) glue sections together.

Loop shellings-I

Definition

We call $\{\gamma_i\}_{i=1}^r$ the loop shelling of Γ, if

$$(\bigcup_{i=1}^{t} \gamma_i) \cap \gamma_{t+1}, \tag{6}$$

is connected for any $t = 1, \dots, r-1$.

For a loop-shelling, choose vertices v_i in intersection (6).

Proposition

$s_i \in \mathcal{G}(\gamma_i)$ agree iff $(s_i)_{v_i} = (s_{i+1})_{v_i}$, $i = 1, \dots, r-1$.

Loop shellings-II

Definition ([BrMa])

For a pure k-dimensional polytopal complex $\mathcal{U}$ in $\mathbb{R}^n$, its facets $F_1, \dots, F_q$ are shelling, if either $k = 0$, or $(\bigcup_{i<a} F_i) \cap F_a$ is isomorphic to a triangulation of Δ^{k-1} for $a = 2, \dots, q$ ($\Rightarrow$ connected). Shellable.

Theorem (Bruggesser, Mani [BrMa])

For a convex polytope P^n, ∂P is shellable.

Let $T^k : M^{2n}$ be a GKM-action on a Hamiltonian mfd with (Γ, α, ∇).

Proposition

Let $k = 3$. Then there is a loop-shelling on Γ.

Is $\operatorname{sk}^2 P$ shellable? Implies the above for arbitrary $k \geqslant 4$.

Sections of $\mathcal{C}$ over invariant loops

Inv. edge loop $\gamma = (e_1, \dots, e_q)$ in (Γ, ∇). Monodromy operator ([Ta])

$$\Pi_\gamma : \ \mathbb{Z}E_{i(\gamma)}(\Gamma) \to \mathbb{Z}E_{t(\gamma)}(\Gamma), \ e \mapsto \nabla_{e_q} \circ \dots \circ \nabla_{e_1}(e).$$

Call an edge $e \in E_v(\Gamma)$ internal, if $e \in E_v(\Gamma')$, and external, otherwise. Introduce variables $s = \sum_{e \in E_u(\Gamma)} y_e \cdot e \in \mathcal{C}_v$, inner y_e span V_{int}. One has

$$\varphi_\gamma^{\mathcal{C}} s(u) - s(u) \equiv \Pi_\gamma(s(u)) - s(u) \mod V_{int}.$$

Corollary

If Π_γ acts trivially on ext. edges, then the subspace $F_{\mathcal{C}}(\gamma) \subseteq \mathcal{C}_v$ is given by s.l.e. on 2 internal variables.

Proposition

If (Γ, ∇) admits 4-linearly independent axial function α, then Π_γ acts identically on external edges.

Invariant functions admitting axials

Let $\gamma = (e_1, \ldots, e_q)$ be an edge loop in Γ. Define integers

$$a_r := \langle c(e_r), \overline{e_{r-1}} \rangle, \; b_r := \langle c(e_r), \Pi_{\gamma_{r-1}}(e) \rangle.$$

a_i's are independent on the orientation of γ. Let

$$A_i := \begin{pmatrix} 0 & -1 \\ 1 & -a_i \end{pmatrix}.$$

Proposition

Suppose that (Γ, ∇) with 2-complete ∇ admits an axial function with invariant function c. Then $2q+2$ equations hold (in cyclic order):

$$A_{q+r} \cdots A_r = \mathrm{Id}, \; r = 1, \ldots, q, \tag{7}$$

$$b_1 A_1 + \cdots + b_q A_q \cdots A_1 = 0.$$

Similar to toric surfaces classification (weighted dual graphs, [Oda]).

Monodromy relations-I

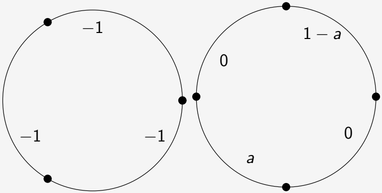


Fig.: 3-gons and 4-gons

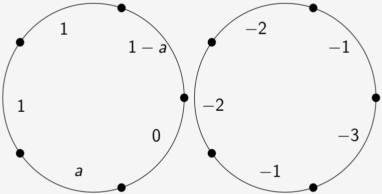
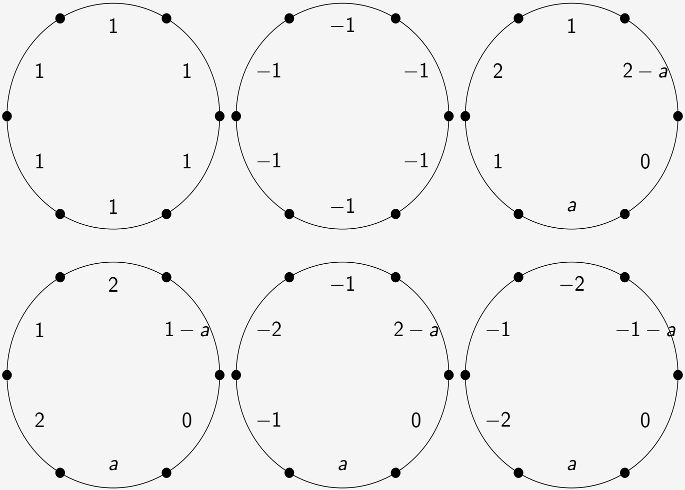


Fig.: 5-gons

Monodromy relations-II

Fig.: 6-gons



Finding axials

Corollary

Let γ be an invariant loop of length q admitting axial. If $c|_\gamma = 0$, then $q \equiv 0 \pmod 4$.

Proposition

Let ∇ be 2-complete. Then $\alpha \in \mathcal{C}'(\Gamma)$ is an axial iff $\operatorname{rk}_\gamma \alpha = 2$ for any invariant loop γ.

Joint vertices: Choose $\{w_i\}_{i \in I} \subseteq V(\Gamma)$ s.t. any invariant loop in Γ contains some w_i.

Corollary

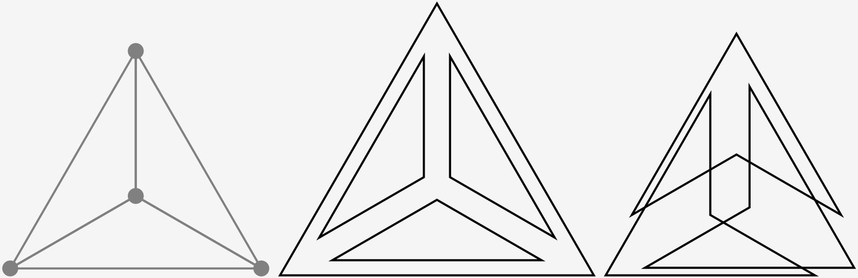
$\alpha \in \mathcal{C}'(\Gamma)$ is an axial iff $(\alpha)_{w_i}$ is 2-linearly independent, $i \in I$.

Invariant functions on Δ^3 admitting axials

Remark

2-complete $\nabla \overset{1:1}{\leftrightarrow}$ Invariant loops in Γ.

Fig.: 2-complete connections on the edge graph (gray) of Δ^3. All a_i are: left -1; right 0.



No loop-shelling on the right.

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Thank you!

Bonus: why invariant functions are important?

Let M^{2n} be a non-singular toric projective variety with T^n-action. Let P^n be a Delzant moment polytope of M^{2n}.

Proposition

Γ is the edge graph of P^n. Inv. function c and (Γ, ∇) determines α and P^n up to $GL_n(\mathbb{Z})$-action.

For an edge $e \in E(\Gamma)$, define the curvature of e by

$$\operatorname{curv} e = -\sum_{e' \in E_{i(e)}(\Gamma)} \langle c(e), e' \rangle.$$

Proposition ([Ay])

For $n = 3$, one has $\sum_{e \in E(\Gamma)} \operatorname{curv} e = 48.$