# Some new insights into $T^n$-action on the Grassmannians $G_{n,2}$


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# Complex Grassmann manifolds $G_{n,2} = G_{n,2}(\mathbb{C})$

$\mathbb{C}^n$ — $n$-dimensional complex vector space with fixed basis.
$G_{n,2}$ – 2-dimensional complex subspaces in $\mathbb{C}^n$,
$G_{n,2} = U(n)/U(2) \times U(n-2)$

The coordinate-wise $\mathbb{T}^n$ - action on $\mathbb{C}^n$ induces $\mathbb{T}^n$ - action on $G_{n,2}$.

This action is not effective — $T^{n-1} = \mathbb{T}^n/\Delta$ acts effectively.

$\dim G_{n,2} = 4(n-2)$, $d = 2(n-2) - (n-1) = n-3$ - complexity of $T^{n-1}$-action;

$d \geq 2$ for $n \geq 5$.

$\mathbb{T}^n$-action extends to coordinate-wise $(\mathbb{C}^*)^n$ -action on $G_{n,2}$

# Plücker embedding

The Plücker embedding $G_{n,2} \to \mathbb{C}P^{N-1}$, $N = \binom{n}{2}$, is given by

$$L \to P(L) = \big(P_I(A_L),\ I \subset \{1, \dots n\},\ |I| = 2\big),$$

$P_I(A_L)$ - Plücker coordinates of $L$ in a fixed basis.
Consider the representation

$$\rho_{n,2} : \mathbb{T}^n \to \mathbb{T}^N, \quad N = \binom{n}{2},$$

given by the second exterior power

$$(t_1, \dots, t_n) \to (t_1 t_2, \dots, t_{n-1} t_n).$$

$\rho_{n,2}$ defines the action $\mathbb{T}^n$ on $\mathbb{C}P^{N-1}$.

The Plücker embedding is equivariant for the representation $\rho_{n,2}$:

$$\mathbb{T}^n \curvearrowright G_{n,2} \to \mathbb{C}P^{N-1} \curvearrowleft \mathbb{T}^n$$

# Moment map

The weight vectors of the representation $\rho_{n,2}$ are:

$$\Lambda_I \in \mathbb{R}^n, \quad (\Lambda_I)_j = 1 \text{ for } j \in I, \quad (\Lambda_I)_j = 0 \text{ for } j \notin I,$$

where $I \subset \{1, \dots, n\}$, $|I| = 2$ and $\mathbb{R}^n$ is with a fixed basis.

$\Lambda_I$ has 1 at 2 places and it has 0 at the other $(n-2)$ places.

**The moment map** $\mu : G_{n,2} \to \mathbb{R}^n$ is defined by

$$\mu(L) = \frac{1}{|P(L)|^2} \sum |P_I(A_L)|^2 \Lambda_I, \quad |P(L)|^2 = \sum |P_I(A_L)|^2,$$

where the sum goes over the subsets $I \subset \{1, \dots, n\}$, $|I| = 2$.

- $\mu$ is $\mathbb{T}^n$-invariant
- $\mathrm{Im}\mu = \mathrm{convexhull}(\Lambda_I) = \Delta_{n,2}$ – hypersimplex.
- $\Delta_{n,k}$ is in the hyperplane $x_1 + \cdots + x_n = 2$ in $\mathbb{R}^n$, $\dim \Delta_{n,2} = n - 1$.

# Strata on $G_{n,2}$

Let $M_{ij} = \{L \in G_{n,2} \mid P_{ij}(L) \neq 0\}, \quad i,j \in \{1,\dots,n\},\ i<j.$

- $M_{ij}$ is an open and dense set in $G_{n,2}$ and $G_{n,2} = \bigcup M_{ij}$.
- $M_{ij}$ contains exactly one fixed point $x_{ij}$
- Set $Y_{ij} = G_{n,2} \setminus M_{ij}$.

Let $\sigma \subset \{\{i,j\},\ i,j \in \{1,\dots,n\},\ i \neq j\}$ and define the stratum $W_\sigma$ by

$W_\sigma = (\cap_{\{i,j\}\in\sigma} M_{ij}) \cap (\cap_{\{i,j\}\notin\sigma} Y_{ij})$ if this intersection is nonempty.

The main stratum is $W = \cap_{\{i,j\}\in\{\binom{n}{2}\}} M_I$ - an open and dense set in $G_{n,2}$.

- $W_\sigma \cap W_{\sigma'} = \emptyset$ for $\sigma \neq \sigma'$,
- $W_\sigma$ is $\mathbb{T}^n$ - invariant, $G_{n,2} = \cup_\sigma W_\sigma$

# Strata on $G_{n,2}$

**Lemma**

$$\mu(W_\sigma) = \overset{\circ}{P}_\sigma, \quad P_\sigma = \text{convhull}(\Lambda_{ij}, \{i,j\} \in \sigma)$$

$P_\sigma$ – an admissible polytope

$\{W_\sigma\}$ coincide with the strata as defined by Gel'fand-Serganova:

$$W_\sigma = \{L \in G_{n,2} : \mu(\overline{\mathbb{C}^* \cdot L}) = P_\sigma\}$$

### Theorem

*All points from $W_\sigma$ have the same stabilizer $T_\sigma$.*

Torus $T^\sigma = T^n/T_\sigma$ acts freely on $W_\sigma$.

Moment map decomposes as $\mu : W_\sigma \to W_\sigma/T^\sigma \xrightarrow{\hat{\mu}} \overset{\circ}{P}_\sigma$.

### Theorem

*$\hat{\mu} : W_\sigma/T^\sigma \to \overset{\circ}{P}_\sigma$ is a locally trivial fiber bundle with a fiber an open algebraic manifold $F_\sigma$. Thus,*

$$W_\sigma/T^\sigma \cong \overset{\circ}{P}_\sigma \times F_\sigma.$$

$F_\sigma$ – the space of parameter for $W_\sigma$;

# Admissible polytopes for $G_{n,2}$

- $\dim \Delta_{n,2} = n-1$
- $\partial \Delta_{n,2} = (\cup_n \Delta^{n-2}) \cup (\cup_n \Delta_{n-1,2})$
- Admissible polytope: $P_\sigma = \mu(\overline{\mathbb{C}^n \cdot L})$ for $L \in G_{n,2}$

**Proposition**

If $\dim P_\sigma \leq n-3$ then $P_\sigma \subset \partial \Delta_{n,2}$.

# Admissible polytypes in dimension $n-2$

Let $\dim P_\sigma = n-2$ and $P_\sigma \subset \partial\Delta_{n,2}$:

- $P_\sigma = \Delta^{n-2}$ or
- $P_\sigma \subseteq \Delta_{n-1,2}$ is an admissible polytope for $G_{n-1,2}$.

Let $\mu_j = pr_j \circ \mu$ and $pr_j : \mathbb{R}^n \to \mathbb{R}^1_j$ – projection.

Lemma. If $P_\sigma \subset \partial\Delta_{n,2}$ then $\mu_j(P_\sigma) = 0$ or $\mu_j(P_\sigma) = 1$ for some $1 \leq j \leq n$.

# Interior admissible $(n-2)$- polytopes

Let $P_\sigma \cap \overset{\circ}{\Delta}_{n,2} \neq \emptyset$ - interior admissible polytope

$\Pi_{ij}$ the set of planes of dimension $n-2$ such that

- the vertex $\Lambda_{ij} \in \alpha_{ij}$ for any $\alpha_{ij} \in \Pi_{ij}$,
- $\alpha_{ij}$ is paralel to $n-2$ edges of $\Delta_{n,2}$ which are incident to $\Lambda_{ij}$
- $\alpha_{ij} \cap \overset{\circ}{\Delta}_{n,2} \neq \emptyset$

Lemma. $\Pi_{ij}$ consists of the planes $\alpha_{ij,l}^{s_1,\ldots,s_l} = \Lambda_{ij} + F_{l,s_1,\ldots,s_l}$ whose directrix $F_{l,s_1,\ldots,s_l}$ is spanned by the vectors

$$e_{js_k} = \Lambda_{ij} - \Lambda_{is_k},\ 1 \leq k \leq l,$$

$$e_{is} = \Lambda_{ij} - \Lambda_{js},\ 1 \leq s \leq n,\ \ s \neq i, j,\ \ s \neq s_1, \ldots, s_l,$$

where $1 \leq l \leq n-3$, $1 \leq s_1 < \ldots < s_l \leq n$ and $s_k \neq i, j$, $1 \leq m \leq l$

# Interior admissible polytopes

**Proposition.** The admissible polytopes of dimension $n-2$ which are not on $\partial\Delta_{n,2}$ are obtained by intersecting $\Delta_{n,2}$ with the planes $\Pi_{ij}$, $1 \leq i < j \leq n$

- $S_n \hookrightarrow \{\Pi_{ij}, 1 \leq i < j \leq n\}$ with the stabilizer $S_2 \times S_{n-2}$.
- $S_2 \times S_{n-2} \hookrightarrow \Pi_{ij}$

**Proposition.** The irreducible representations for $S_{n-2}$-action on $\Pi_{ij}/S_2$ are in dimensions:

$$\text{for } n \text{ odd}: \binom{n-2}{k},\ 1 \leq k \leq [\frac{n-2}{2}]\ ,$$

$$\text{for } n \text{ even}: \binom{n-2}{k},\ 1 \leq k < [\frac{n-2}{2}] \text{ and } \frac{2}{n-2}\binom{n-2}{\frac{n-2}{2}}\ .$$

# Interior admissible $(n-2)$- polytopes

Corollary Those which are not on $\partial\Delta_{n,2}$ are, up to the $S_n$-action, obtained by intersecting $\Delta_{n,2}$ with the planes $\alpha_{12,l}^{3,\ldots,l+2}$, $1 \leq l \leq [\frac{n-2}{2}]$.

Corollary. An admissible polytope which is not on $\partial\Delta_{n,2}$ has $n_k$ vertices:

$$n_k = k(n-k), \text{ where } 2 \leq k \leq [\frac{n-2}{2}] + 1.$$

Moreover, the number of these polytopes which have $n_k$ vertices is

$$\text{for } n \text{ odd}: \quad p_k = 2\frac{\binom{n-2}{k-1}}{n_k}\binom{n}{2}, \; 2 \leq k \leq [\frac{n-2}{2}] + 1$$

$$\text{for } n \text{ even}: \quad p_k = 2\frac{\binom{n-2}{k-1}}{n_l}\binom{n}{2}, \; 1 \leq k < [\frac{n-2}{2}] + 1,$$

$$p_k = \frac{8(n-1)}{n(n-2)}\binom{n-2}{\frac{n-2}{2}}, \; k = \frac{n-2}{2} + 1.$$

Examples.

- $G_{4,2}$ – one generating admissible interior polytope of dimension 2, it has 4 vertices and there 3 interior polytopes.
- $G_{5,2}$ – one generating admissible interior polytope in dimension 3 , it has 6 vertices and the number of interior polytopes is 10.
- $G_{6,2}$ – 2 generating admissible interior polytopes (the representation for $S_2 \times S_4$- action on $\mathbb{C}^7$ has 2 irreducible summands of dimension 4 and 3), these polytopes have 8 and 9 vertices and their number is 15 and 10 respectively.

# The chamber decomposition for $\Delta_{n,2}$

Consider the hyperplane arrangement in
$\mathbb{R}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n, x_1 + \ldots + x_n = 2\}$:

$$\mathcal{A} = \{\Pi_{ij}, 1 \leq i < j \leq n\} \cup \{x_i = 0, 1 \leq i \leq n\} \cup \{x_i = 1, 1 \leq i \leq n\}$$

$\mathcal{C}(\Delta_{n,2})$ – chamber decompostion for $\Delta_{n,2}$ defined by $\mathcal{A}$.

Lemma. A chamber $C \in \mathcal{C}(\Delta_{n,2})$ is the intersection of all admissible polytopes which contain $C$

$$\mathcal{C} = \bigcap_{C \subset P_\sigma} P_\sigma$$

$L(\mathcal{A})$ – a face lattice for the arrangement $\mathcal{A}$ and $L(\Delta_{n,2}) = L(\mathcal{A}) \cap \Delta_{n,2}$.

$\mathcal{C}(S)$ – chamber decomposition for $S$ defined by $L(\Delta_{n,2})$ for $S \in L(\Delta_{n,2})$.

Lemma. Any $C \in \mathcal{C}(S)$ can be obtained as the intersection of all admissible polytopes which contain $S$.

# On regular points of the moment map

We proved:

$$\mathrm{rank}\, d\mu(L) = \dim P_\sigma, \quad P_\sigma = \mu(\overline{(\mathbb{C}^*)^n \cdot L}).$$

- If dim $P_\sigma = n-1$, then $d\mu(L)$ is an epimorphism,
- $M_x = \mu^{-1}(x)$ is a smooth submanifold of $G_{n,2}$ for $x \in \overset{\circ}{\Delta}_{n,2}$ such that $\dim P_\sigma = n-1$ for all $P_\sigma$ such that $x \in \overset{\circ}{P}_\sigma$ .
- $T^{n-1}$ acts freely on $M_x$ and $M_x/T^{n-1}$ is a smooth manifold.

# The chamber decomposition for $\Delta_{n,2}$

Let $C \in \mathcal{C}(\Delta_{n,2})$: then $C = \bigcap\limits_{C \subset P_\sigma} P_\sigma$, $\dim P_\sigma = n-1$.

- $M_C = \mu^{-1}(C)$ is a submanifold in $G_{n,2}$ and $T^{n-1}$ acts freely on $M_C$
- $M_C/T^{n-1}$ is a smooth manifold
- $\hat{\mu} : M_C/T^{n-1} \to C$ is a locally trivial smooth fibration.
- $M_x/T^{n-1}, M_y/T^{n-1}$ have the same diffeomoprhic type $F_C$ for $x, y \in C$.
- $M_C/T^{n-1} \cong F_C \times C$

# The chamber decomposition for $\Delta_{n,2}$

On the other hand:

- $M_C = \bigcap_{C \subset (P_\sigma} (W_\sigma \cap M_C)$.
- $M_C \subset W$ – the main stratum, $W \cap M_C$ - a dense set in $M_C$
- $W_\sigma / T^\sigma \cong F_\sigma \times \overset{\circ}{P}_\sigma$ for all $\sigma$:

Proposition. The manifold $F_C$ is a compactification of the space $F$. This compactification consists of the spaces $F_\sigma$ such that $C \subset P_\sigma$

$$F_C = \bigcup_{C \subseteq P_\sigma} F_\sigma .$$

# The chamber decomposition for $\Delta_{n,2}$

Let $S \in L(\Delta_{n,2})$ and consider the chamber decomposition $\mathcal{C}(S)$ of $S$:

$$\mathcal{C}(S) = S \setminus (S \cap (L(\Delta_{n,2}) \setminus S)).$$

Using the results of Goresky-MacPherson one can prove:

$\hat{\mu}^{-1}(x)$ is homeomorphic to $\hat{\mu}^{-1}(y)$ for any $x, y \in C_S$, $C_S \in \mathcal{C}(S)$.

Let $M_{C_S} = \mu^{-1}(C_S)$:

Lemma $\hat{\mu} : M_{C_S}/T^{n-1} \to C_S$ is a locally trivial fiber bundle with a fiber an open algebaric manifold $F_{C_S}$. Thus, $M_{C_S}/T^{n-1} \cong C_S \times F_{C_S}$.

Lemma. The space $F_{C_S}$ is a compactification of $F$. This compactification consists of the spaces $F_\sigma$ such that $C_S \subset \overset{\circ}{P}_\sigma$.

# Moment map and $F_C, F_{C_S}$

$S_n \hookrightarrow \mathcal{A}$ and $S_n \hookrightarrow \Delta_{n,2}$ by permuting the coordinates, so

$S_n$ permutes the elements of $\mathcal{C}(\Delta_{n,2})$, the elements of $L(\Delta_{n,k})$ and the elements of $\mathcal{C}(S)$ for any $S \in L(\Delta_{n,k})$.

On other hand $S_n \hookrightarrow G_{n,2}$ by permuting the coordinates and

Lemma.

1. $S_n$ action on $G_{n,2}$ is $T^n$-invariant and $\mu \circ S_n = S_n \circ \mu$.
2. $S_n$ is only such subgroup of $\mathrm{Aut}(G_{n,2})$.

Corollary. $\hat{\mu}^{-1}(\mathfrak{s}(x))$ are all homeomorphic for $\mathfrak{s} \in S^n$ and $x \in \Delta_{n,2}$.

Corollary. $F_C$, $F_{C_S}$ is homeomorphic to $\mathfrak{s}(F_C)$, $\mathfrak{s}(F_{C_S})$ for any $C \in \mathcal{C}(\Delta_{n,2})$ and any $C_S \in \mathcal{C}(S)$.

# Weighted lattice for $G_{n,2}$

$$\mathcal{W}L(\Delta_{n,2}) = \bigcup_{S\in L(\Delta_{n,k})} (C_S \times F_{C_S}) - \text{weighted face lattice for} \Delta_{n,2}$$

$$S_n \hookrightarrow \mathcal{W}L(\Delta_{n,2}),\ \ \mathfrak{s}(C_S \times F_{C_S}) = \mathfrak{s}(C_S) \times \mathfrak{s}(F_{C_S})$$

$$M_{C_S} = \mu^{-1}(C_S)/T^{n-1} \cong C_S \times F_{C_S}$$

Remark

- For $G_{4,2}$ it holds $F_C \cong F_{C_S} \cong \mathbb{C}P^1$
- In general they are not all homeomorphic: easy to verify for $G_{5,2}$

# Atlas on $G_{n,2}$ and $(\mathbb{C}^*)^n$-action

$M_I$ is equipped with the coordinates: let $I = \{1, 2\}$ and $L \in M_I$. Then

$$A_L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_{11} & z_{12} \\ \vdots & \vdots \\ z_{n-2,1} & z_{n-2,2} \end{pmatrix}, \quad t \cdot A_L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{t_3}{t_1} z_{11} & \frac{t_3}{t_2} z_{12} \\ \vdots & \vdots \\ \frac{t_n}{t_1} z_{n-2,1} & \frac{t_n}{t_2} z_{n-2,2} \end{pmatrix}$$

$u_I : M_I \to (\mathbb{C}^*)^{2(n-2)}, \ u_I(L) = (z_{11}, z_{12}, \ldots, z_{n-2,1}, \ldots, z_{n-2,2})$

Conisder the representation $r_{n,2} : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{2(n-2)}$ given by

$$(t_1, \ldots, t_n) \to (\frac{t_3}{t_1}, \frac{t_3}{t_2}, \ldots, \frac{t_n}{t_1}, \frac{t_n}{t_2}).$$

The induced action $(\mathbb{C}^*)^n \hookrightarrow \mathbb{C}^{2(n-2)}$ is the composition of $r_{n,2}$ and the standard action of $(\mathbb{C}^*)^{2(n-2)}$ on $\mathbb{C}^{2(n-2)}$

# Atlas on $G_{n,2}$ and $(\mathbb{C}^*)^n$-action

To obtain an effective action of $(\mathbb{C}^*)^{n-1}$ on $\mathbb{C}^{2(n-2)}$ we put

$$\tau_i = \frac{t_3}{t_i},\ i = 1, 2, \quad \tau_{i+2} = \frac{t_{i+3}}{t_1},\ 1 \leq i \leq n-3.$$

$$\frac{t_p}{t_s} = \frac{\tau_{p-1}\tau_s}{\tau_1} \text{ for } 3 \leq p \leq n.\ s = 1, 2.$$

It is obtained the representation of $(\mathbb{C}^*)^{n-1}$ in $(\mathbb{C}^*)^{2(n-2)}$:

$$(\tau_1, \ldots, \tau_{n-1}) \to (\tau_1, \tau_2, \tau_3, \frac{\tau_3\tau_2}{\tau_1}, \tau_4, \frac{\tau_4\tau_2}{\tau_1}, \ldots, \tau_{n-1}, \frac{\tau_{n-1}\tau_2}{\tau_1}). \qquad (1)$$

The induced effective action $(\mathbb{C}^*)^{n-1} \hookrightarrow \mathbb{C}^{2(n-2)}$ is the composition of (1) and the standard action of $(\mathbb{C}^*)^{2(n-2)}$ on $\mathbb{C}^{2(n-2)}$.

# Strata in a chart

The $(\mathbb{C}^*)^n$ - orbits in the main stratum $W$ are given by:

$$c_{ij}^{'} z_{i1} z_{j2} = c_{ij} z_{j1} z_{i2}, \;\; 1 \leq i < j \leq n-2, \tag{2}$$

$(c_{ij}^{'} : c_{ij}) \in \mathbb{C}P^1$ and $c_{ij}, c_{ij}^{'} \neq 0$ and $c_{ij} \neq c_{ij}^{'}$, $2 \leq i < j \leq n-2$.

The parameters $(c_{ij} : c_{ij}^{'})$ satisfy the relations:

$$c_{ki}^{'} c_{kj} c ij^{'} = c_{ki} c_{kj}^{'} c_{ij}, \;\; 1 \leq k < i < j \leq n-2. \tag{3}$$

$F = W/(\mathbb{C}^*)^n$ - the space of parameters for $W$

$F$ is embedded in $(\mathbb{C}P^1)^N$ by (2), (3), where $N = \frac{(n-3)(n-2)}{2}$.

The compactification of $F$ in $(\mathbb{C}P^1)^N$ is given by the intersection of the cubic hypersurfaces (3).

# Strata in a chart

Further

$$(c_{ij} : c_{ij}^{'}) = (c_{1i}^{'}c_{1j} : c_{1i}c_{1j}^{'}),\ \ 2 \leq i < j \leq n-2.$$

$$(c_{1i} : c_{1i}^{'}) \neq (c_{1j} : c_{1j}^{'}),\ \ 2 \leq i < j \leq n-2.$$

It follows that

$$F = (\mathbb{C}P^1_A)^{n-3} \setminus \Delta,$$

$$A = \{(0:1),(1:0),(1:1)\} \text{ and } \Delta = \bigcup_{2\leq i<j\leq n-2} \Delta_{ij}$$

for the diagonals

$$\Delta_{ij} = \{((c_{12} : c_{12}^{'}), \ldots, (c_{n-3,n-2}, c_{n-3,n-2}^{'})) \in (\mathbb{C}P^1_A)^{n-3} |$$

$$(c_{1i} : c_{1i}^{'}) \neq (c_{1j} : c_{1j}^{'})\}.$$

# Strata in a chart

$W_\sigma \subset M_{12}$ – defined by $P^{1i_1} = 0, P^{2j_2} = 0$ and $P^{ij} = 0$,
$3 \leq i_1, j_1, i, j \leq n$ , $i \neq j$.

In the local coordinates: $z_{2i_1} = z_{1j_2} = 0$ and $z_{1i}z_{2j} = z_{1j}z_{2i}$.

Any $W_\sigma \subset M_{12}$ is obtained by restricting the surfaces (2) to some $\mathbb{C}^J$, where $J \subset \{(1,1), \dots, (2, n-2)\}$ and $|J| = l$ for some $0 \leq l \leq N$.

Proposition. The manifold $F_C$ (a space $F_{C_S}$) is the compactifications of $F$ given by the spaces $F_\sigma$, $C$(or $C_S$) $\subset P_\sigma$. Any $F_\sigma$ is a point or it is homeomporhic to the space obtained by restricting the hypersurfaces (3) to some $(\mathbb{C}P^1_B)^q \subset (\mathbb{C}P^1_A)^N$, $B = \{(1:0), (0:1)\}$ and $0 \leq q \leq l$, $n-1 \leq l \leq N$.

# Strata in a chart

Let $W_\sigma \subset M_{12}$ such that $P_\sigma$ - interior polytope

Lemma. In the local coordinates $W_\sigma$ is given by

$$z_{i_1,1} = \ldots = z_{i_p,1} = 0,\ z_{i,1} \neq 0,$$

$$z_{i_1,2} = \ldots = z_{i_p,2} \neq 0,\ z_{i,2} = 0$$

$i \neq i_1, \ldots i_p,\ 3 \leq i_1 < \ldots i_p \leq n,\ p \geq 1.$

Corollary The space of parameters $F_\sigma$ for $W_\sigma$ is a point.

# A universal space of parameters $\mathcal{F}$

Find an ambient space in which all compactification $F_C$ that is $F_{C_S}$ happen.

- $W_\sigma \subset M_{12}$: $z_{2,i_1} = z_{1,j_2} = 0$ and $z_{i1}z_{2j} = z_{1j}z_{2i}$
- $W$ given by (3) is a dense set in $G_{n,2}$.

Assign the new space of parameters $\tilde{F}_{\sigma,12}$ to $W_\sigma$ in $M_{12}$.

In which ambient space $\mathcal{F} = \overline{F}$ this assignment is to be done?

Determined by: $\sigma \to \tilde{F}_{\sigma,ij}$ must not depend on the fixed chart $M_{ij}$.

1. $\mathcal{F}$ contains the compactification of $F$ in $(\mathbb{C}P^1)^N$, which is the intersection of hypersurfaces (3).
2. The cooordinate change $g_{ij,kl} : M_{ij} \to M_{kl}$ gives the homeomorphism $f_{ij,kl} : F_{ij} \to F_{kl}$ . It should extend to homeomorphism $\overline{f}_{ij,kl} : \mathcal{F}_{ij} \to \mathcal{F}_{kl}$.

The homeomorphism $f_{12,13} : F_{12} \to F_{13}$ is given by

$$((c_{12} : c_{12}^{'}), \dots, (c_{n-3,n-2} : c_{n-3,n-2}^{'})) \to$$

$$((c_{12} : c_{12} - c_{12}^{'}), \dots, (c_{1n-2} : c_{1n-2} - c_{1n-2}^{'}),$$

$$(c_{13}^{'} c_{23}(c_{12} - c_{12}^{'}) : c_{12}^{'} c_{23}^{'}(c_{13} - c_{13}^{'})), \dots, (c_{1j}^{'} c_{ij}(c_{1i} - c_{1i}^{'}) : c_{1i}^{'} c_{ij}^{'}(c_{1j} - c_{1j}^{'})),$$

$$\dots, (c_{1n-2}^{'} c_{n-3,n-2}(c_{1n-3} - c_{1n-3}^{'}) : c_{1n-3}^{'} c_{n-3,n-2}^{'}(c_{1n-2} - c_{1n-2}^{'})).$$

- $f_{12,13}$ can not be continuously extended to the submanifolds in $\bar{F}_{12}$ given by $\partial F_{12,ij} = \{c_{1i} = c_{1i}^{'}, c_{1j} = c_{1j}^{'}, 2 \leq i < j \leq n-2\}$.
- $\bar{F} \subset (\mathbb{C}P^1)^N$ is not an appropriate compactification of $F$.

One needs to blow up $\bar{F}_{12}$ along the surfaces $\partial F_{12,ij}$, $2 \leq i < j \leq n-2$.

## Definition

A space $\mathcal{F}$ obtained by the blow ups of $F_{12}$ along the surfaces $\partial F_{12,ij}$, $2 \leq i < j \leq n-2$ is called a universal space of parameters.

- For $n = 5$ there is just $\partial F_{12,23} = ((1:1),(1:1),(1:1))$ and $\mathcal{F}$ is the blow up of $\bar{F} = \{(((c_{12}:c_{12}^{'}),(c_{13}:c_{13}^{'}),(c_{23}:c_{23}^{'})) \in (\mathbb{C}P^1)^3 | c_{12}^{'}c_{13}c_{23}^{'} = c_{12}c_{13}^{'}c_{23}\}$ at the point $((1:1),(1:1),(1:1))$.
- For $n \geq 6$ the spaces $\partial F_{12,ij}$ are not point and they intersect. The blow ups do now commute in general and the question of uniqueness arises.
- The compactification of $F = (\mathbb{C}P_A^1)^{n-3} \setminus \Delta$ provided by S. Keel is exactly done by an iterated blow ups . It gives a smooth, compact algebraic variety and it coincides with Chow quotient of $G_{n,2}$ by Kapranov and with Grotendick-Knudsen compactification $\overline{M_{0,n}}$ of the moduli space of smooth pointed curves of genus zero.

# Virtual spaces of parameters

Proposition. For any chart $M_{ij}$ and any stratum $W_\sigma$ there is a subspace $\tilde{F}_{\sigma,ij} \subset \mathcal{F}_{ij}$ whose homeomorphic type $\tilde{F}_{\sigma,ij}$ depends on the stratum $W_\sigma$ but it does not depend on the chart $M_{ij}$.

Definition The homeomorphic type $\tilde{F}_\sigma$ of the space $\tilde{F}_{\sigma,ij}$ is called the virtual space of parameters for the stratum $W_\sigma$.

Definition The virtual space of parameters $\tilde{F}_{C_S}$ for a chamber $C_S$ is defined by

$$\tilde{F}_{C_S} = \bigcup_{C_S \subset \overset{\circ}{P}_\sigma} \tilde{F}_\sigma \subset \mathcal{F}. \tag{4}$$

$\tilde{F}_{C_S}$ is a formal disjoint union, so it is defined the function $m : \tilde{F}_{C_S} \to \Sigma$ by $m(y) = \sigma$ if and only if $y \in \tilde{F}_\sigma$.

# Ilustration

Let $W_\sigma \subset M_{12}$, given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_{31} & z_{32} \\ z_{41} & z_{42} \\ z_{51} & 0 \\ \vdots & \vdots \\ z_{n-2,1} & 0 \end{pmatrix} \quad z_{ij} \neq 0 \ \text{ and } z_{31}z_{42} = z_{41}z_{32}$$

$$\tilde{F}_{\sigma,12} = ((c_{ij} : c_{ij}^{'})) \in \mathbb{C}P^N, N = \binom{n-2}{2},$$

$$(c_{1i} : c_{1i}^{'}) = (1:0), i = 2,3,4, \ \ (c_{2i} : c_{2i}^{'}) = (0:1), \ i \geq 3$$

$$(c_{34} : c_{34}^{'}) = (1:1), \ \ (c_{3i} : c_{3i}^{'}) = (c_{4i} : c_{4i}^{'}) = (1:0), \ i \geq 5$$

$$(c_{1i} : c_{1i}^{'}) \in \mathbb{C}P^1, \ i \geq 5, \ \ (c_{kl} : c_{kl}^{'}) \in \mathbb{C}P^1, \ k \geq 5$$

# Illustration

$W_\sigma \subset M_{13}$ is given by

$$\begin{pmatrix} 1 & 0 \\ w_{21} & w_{22} \\ 0 & 1 \\ 0 & w_{42} \\ w_{51} & 0 \\ \vdots & \vdots \\ w_{n-2,1} & 0 \end{pmatrix} \quad w_{ij} \neq 0 \quad (\text{note} : F_\sigma - \text{ point})$$

$$\tilde{F}_{\sigma,13} = ((d_{ij} : d_{ij}^{'})) \in \mathbb{C}P^N, N = \binom{n-2}{2},$$

$$(d_{1i} : d_{1i}^{'}) = (1:0), i = 2,3,4, \ (d_{1i} : d_{1i}^{'}) \in \mathbb{C}P^1, \ i \geq 5$$

$$(d_{2i} : d_{2i}^{'}) = (1:0), i = 3,4, \ (d_{2i} : d_{2i}^{'}) = (d_{4i} : d_{4i}^{'}) = (0:1), i \geq 5$$

$$(d_{34} : d_{34}^{'}) \in \mathbb{C}P^1, (d_{3i} : d_{3i}^{'}) = (1:0), i \geq 5, \ (d_{kl} : d_{kl}^{'}) \in \mathbb{C}P^1, \ k \geq 5$$

# Virtual and real spaces of parameters

There are two spaces of parameters for a stratum $W_\sigma$ in a chart $M_{ij}$:

- $F_{\sigma,ij}$ such that $W_{\sigma,ij}/T^\sigma \cong \overset{\circ}{P}_\sigma \times F_{\sigma,ij}$,
- $\tilde{F}_{\sigma,ij}$ - the virtual space of parameters defined by $W_{\sigma,ij}/T^\sigma \subset \partial(W_{ij}/T^{n-1}) \subset P^k \times \mathcal{F}_{ij}$

The spaces $F_{\sigma,ij}$ and $\tilde{F}_{\sigma,ij}$ do not coincide in generel (even for $G_{4,2}$). We prove:

**Theorem**

There exists the canonical projection $p_{\sigma,ij} : \tilde{F}_{\sigma,ij} \to F_{\sigma,ij}$ for any $\sigma$.

**Corollary**

There exists the canonical projection $p_{C_S,ij} : \tilde{F}_{C_S,ij} \to F_{C_S,ij}$ defined by $p_{C_S,ij}(y) = p_{m(y),ij}(y)$. where $y \in \tilde{F}_{\sigma,ij}$.

## The orbit space $G_{n,2}/T^n$

Let us consider the weighted lattice

$$\mathfrak{C} = \mathcal{W}L(\Delta_{n,2}) = \bigcup_{C_S}(C_S \times \tilde{F}_C), \qquad (5)$$

There is a canonical embedding

$$h : \mathfrak{C} \to \Delta_{n,2} \times \mathcal{F}, \quad h(x, f_{C_S}) = (x, i_{C_S}(f_{C_S})),$$

$i_{C_S} : \tilde{F}_{C_S} \to \mathcal{F}$ is given by the inclusion $\tilde{F}_{\sigma,ij} \to \mathcal{F}_{ij}$ in a fixed chart $M_{ij}$.

The map $h$ defines the topology on $\mathfrak{C}$:

$U \subset \mathfrak{C}$ is an open set if and only if $h(U)$ is an open set in $\Delta_{n,2} \times \mathcal{F}$.

On the other hand there is a homeomoprhism:

$$h_{C_S,ij} : C_S \times F_{C_S,ij} \to M_{C_S}/T^{n-1}$$

# The orbit space $G_{n,2}/T^n$

For any fixed chart $M_{ij}$ we define the map

$$G_{ij} : \mathfrak{C}_{ij} \to G_{n,2}/T^n, \;\; G_{ij}(x, y) = h_{C_S,ij}(x, p_{C_S,ij}(y)),$$

for $(x, y) \in C_S \times \tilde{F}_{C_S,ij}$

**Theorem**

*The map $G_{ij}$ is a continuous surjection and the orbit space $G_{n,2}/T^n$ is homeomorphic to the quotient of the space $\mathfrak{C}$ by the map $G_{ij}$.*