Almost Pogorelov polytopes

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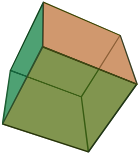
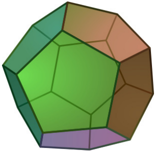
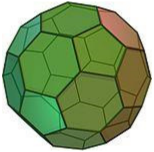
Polytopes

By a polytope P we mean a combinatorial convex 3-dimensional polytope.

A polytope P is simple, if any its vertex belongs to exactly 3 faces.

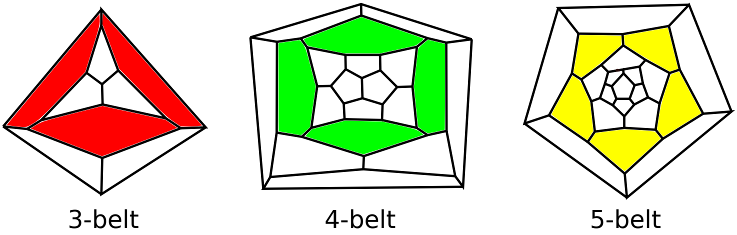






k-belts

A k-belt ($k \geqslant 3$) is a cyclic sequence of k faces such that faces are adjacent if and only if they follow each other and no three faces have a common vertex.



Proposition

Any simple 3-polytope $P \neq \Delta^3$ has a 3-, 4-, or 5-belt.

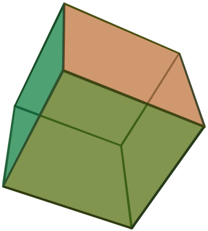
Flag polytopes

A simple polytope is called flag if any its set of pairwise intersecting faces has a nonempty intersection.

Proposition

A simple polytope P is flag iff $P \neq \Delta^3$ and P has no 3-belts;

A flag polytope with the smallest number of faces is the cube.



Pogorelov polytopes

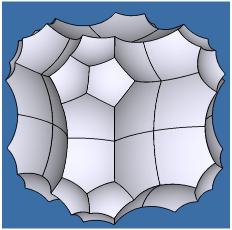
Problem (A.V. Pogorelov, 1967)

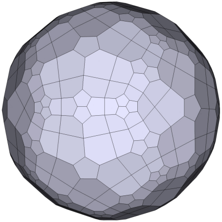
To characterize polytopes realizable in the Lobachevsky space $\mathbb{L}^3$ as bounded polytopes with right dihedral angles.

We call such polytopes Pogorelov polytopes.

Motivation

Such polytopes produce «regular» partitions of $\mathbb{L}^3$ into equal polytopes.





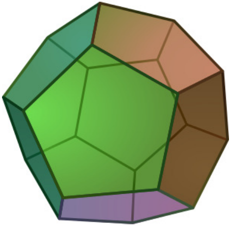
(figures by Ya.V. Kucherinenko)

Pogorelov polytopes

Theorem (A.V. Pogorelov, 1967, E.M. Andreev, 1970)

A polytope P is a Pogorelov polytope iff it is a flag polytope without 4-belts. The realization is unique up to isometries.

A *Pog*-polytope with the smallest number of faces is the dodecahedron.



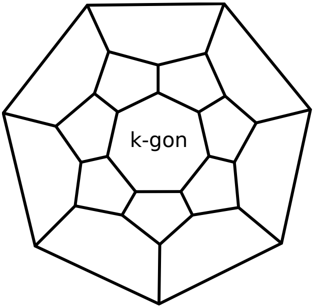
Cohomological rigidity

A family of manifolds is called cohomologically rigid over the ring R, if for any two manifolds from the family a graded isomorphism of cohomology rings over R implies a diffeomorphism of manifolds.

Pogorelov polytopes give rise to cohomologically rigid families:

- $(m+3)$-dimensional moment-angle manifolds $\mathcal{Z}_P$ over $\mathbb{Z}$, where m is the number of faces of P (F. Fan, J. Ma, X. Wang, 2015);
- 6-dimensional quasitoric manifolds $M(P,\Lambda)$ over $\mathbb{Z}$, and 3-dimensional hyperbolic manifolds $R(P,\Lambda_2)$ over $\mathbb{Z}_2$ (V. M. Buchstaber, N. Yu. Erokhovets, M. Masuda, T. E. Panov, S. Park, 2017)

Example of Pogorelov polytopes: *k*-barrels



A *k*-barrel is a Pogorelov polytope for $k \geqslant 5$;

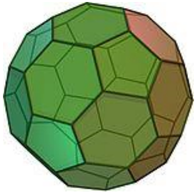
In 1931 F. Löbell glued 8 copies of the 6-barrel to construct the first example of a closed three-dimensional hyperbolic manifold.

In 1987 A. Yu. Vesnin constructed hyperbolic manifolds of «Löbell type» for all *k*-barrels, $k \geqslant 5$.

Fullerenes

A (mathematical) fullerene is a simple polytope with all faces pentagons and hexagons.

Buckminsterfullerene C_{60}



Truncated icosahedron

Theorem (T. Došlić, 1998, 2003)

Any fullerene is a Porogelov polytope.

Cyclic k-edge-connectivity (ck-connectivity)

Definition

- A simple polytope $P \neq \Delta^3$ is ck-connected, if it has no l-belts, $l < k$.
- A simple polytope $P \neq \Delta^3$ is strongly ck-connected (c^*k-connected), if it is ck-connected and any k-belt surrounds a face (is trivial).
- By definition Δ^3 is c^*3-connected, but not $c4$-connected.

Families of *ck*-connected polytopes

- Any simple polytope is $c3$-connected, but at most c^*5-connected.
- We obtain a chain of nested families of polytopes:

$$\mathcal{P}_s \supset \mathcal{P}_{aflag} \supset \mathcal{P}_{flag} \supset \mathcal{P}_{aPog} \supset \mathcal{P}_{Pog} \supset \mathcal{P}_{Pog^*}$$

- $c3$-connected $\mathcal{P}_s$ – all simple polytopes;
- c^*3-connected $\mathcal{P}_{aflag}$ – almost flag polytopes;
- $c4$-connected $\mathcal{P}_{flag}$ – flag polytopes;
- c^*4-connected $\mathcal{P}_{aPog}$ – almost Pogorelov polytopes ;
- $c5$-connected $\mathcal{P}_{Pog}$ – Pogorelov polytopes;
- c^*5-connected $\mathcal{P}_{Pog^*}$ – strongly Pogorelov polytopes.

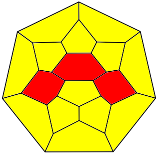
Theorem (G.D. Birkhoff, 1913)

The Four Colour Problem for planar graphs can be reduced only to Pog^* polytopes.

n-disk-fullerenes

Definition (M. Deza, M. Dutour Sikirić and M. I. Shtogrin, 2013)

An *n*-disk-fullerene is a simple polytope with marked *n*-gonal face such that all other faces are pentagons and hexagons.



A unique 7-disk-fullerene with the minimal number of faces

Theorem (V. M. Buchstaber, N. Yu. Erokhovets, 2015-2018)

- Any 3-disk fullerene is almost flag.
- Any 4-disk-fullerene is almost Pogorelov.
- Any 7-disk-fullerene is Pogorelov.
- For any $n \geqslant 8$ there exist n-disk-fullerenes P and Q, where P is not almost flag, and Q is Pog^*.

Andreev's theorem I

Theorem (E.M. Andreev, 1970)

A polytope $P \neq \Delta^3$ can be realized as a bounded polytope in $\mathbb{L}^3$ with dihedral angles $\varphi_{i,j} \in (0, \frac{\pi}{2}]$ at edges $F_i \cap F_j$ if and only if

- P is simple;
- $\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} > \pi$ for any vertex $F_i \cap F_j \cap F_k$;
- $\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} < \pi$ for any 3-belt (F_i, F_j, F_k);
- $\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,l} + \varphi_{l,i} < 2\pi$ for any 4-belt (F_i, F_j, F_k, F_l);
- if $P = \Delta^2 \times I$, then there is an edge at a base with the dihedral angle $< \frac{\pi}{2}$.

The realization is unique up to isometries.

Corollaries of Andreev's theorem I

Corollary 1

A simple polytope $P \neq \Delta^3$ can be realized in $\mathbb{L}^3$ as a bounded polytope with equal non-obtuse dihedral angles ($\in (\frac{\pi}{3}, \frac{\pi}{2}]$) $\Leftrightarrow$ P is flag.

Corollary 2

A simple polytope $P \neq \Delta^3$ can be realized in $\mathbb{L}^3$ as a bounded polytope with right dihedral angles $\Leftrightarrow$ P is flag and has no 4-belts.

Idea (T.E. Panov, 2018)

Andreev's result imply that almost Pogorelov polytopes $\approx$ right-angled polytopes of finite volume in $\mathbb{L}^3$.

Andreev's theorem II (1970)

A polytope $P \neq \Delta^3$ can be realized as a polytope of finite volume in $\mathbb{L}^3$ with dihedral angles $\varphi_{i,j} \in (0, \frac{\pi}{2}]$ if and only if

- P has vertices of valency of 3 and 4;
- $\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} \geqslant \pi$ for any 3-valent vertex $F_i \cap F_j \cap F_k$;
- $\varphi_{i,j} = \frac{\pi}{2}$ for each edge at a 4-valent vertex;
- $\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} < \pi$ for any 3-belt (F_i, F_j, F_k);
- $\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,l} + \varphi_{l,i} < 2\pi$ for any 4-belt (F_i, F_j, F_k, F_l);
- if $P = \Delta^2 \times I$, then there is an edge at a base with the dihedral angle $< \frac{\pi}{2}$;
- $\varphi_{j,k} + \varphi_{k,i} < \pi$, if faces F_i and F_j intersect at a 4-valent vertex and F_k is adjacent to both of them and does not contain their common vertex.

The intersection with the absolute consists of the 4-valent vertices and the 3-valent vertices with the sum of dihedral angles equal to π.

Corollaries of Andreev's theorem

There is nothing about a uniqueness of the realization.

A polytope P can be realized as a polytope of finite volume in $\mathbb{L}^3$ with right dihedral angles $\Leftrightarrow$ P

- has vertices of valency 3 and 4;
- has no 3- and 4-belts;
- has no pair of faces intersecting at a 4-valent vertex and adjacent simultaneously to a face not containing it.

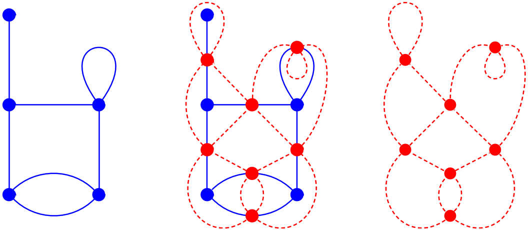
The intersection with the absolute consists of 4-valent vertices.

Strong (Mostow) rigidity $\Rightarrow$ uniqueness of realization.

Theorem (N.Yu. Erokhovets, 2018)

Cutting off 4-valent vertices gives a bijection between right-angled polytopes of finite volume in $\mathbb{L}^3$ and almost Pogorelov polytopes different from the cube I^3 and the pentagonal prism $M_5 \times I$.

Ideal right-angled polytopes

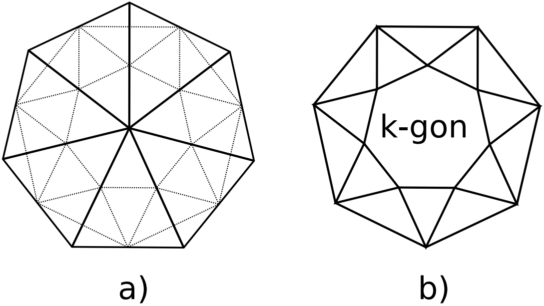


A medial graph of a plane graph G is another graph $M(G)$ that represents the adjacencies between edges in the faces of G.

- For any polytope P its medial graph $G(P)$ is the graph of an ideal right-angled polytope;
- The graph of any ideal right-angled polytope is the medial graph for exactly two (possibly equal) polytopes. Moreover, these polytopes are dual to each other.

k-antiprisms

- The graph of the ideal octahedron is the medial graph of the tetrahedron.
- The medial graph of a k-gonal pyramid is the graph of a k-antiprism.



a) a k-gonal pyramid and its medial graph; b) a k-antiprism

The Koebe-Andreev-Thurston theorem

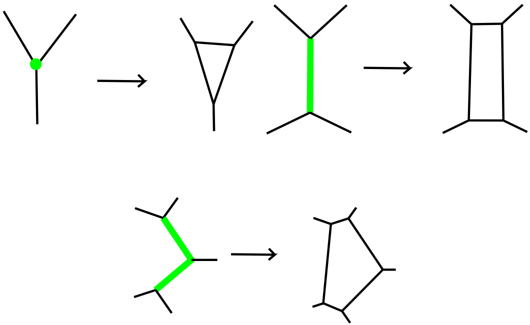
The correspondence between ideal right-angled polytopes and medial graphs plays a fundamental role in the well-known theorem.

Any polytope P has a geometric realization in $\mathbb{R}^3$ such that all its edges are tangent to a sphere.

Construction of simple polytopes

Theorem (V. Eberhard, 1891)

A polytope P is simple iff it is can be obtained from the simplex Δ^3 by a sequence of operations of cutting off a vertex, an edge, or two adjacent edges ($(2,k)$**-truncations**) by one hyperplane.



Construction of almost flag polytopes

Proposition (N.Yu. Erokhovets, 2018)

A polytope P is almost flag if and only if one of the equivalent conditions holds

- P can be obtained from $P = \Delta^3$ with at most two vertices cut by a sequence of operations of cutting off a vertex, an edge, or a pair of adjacent edges not equivalent to a cutting off a vertex of a triangle.
- P is obtained by a simultaneous cutting off a disjoint set of vertices of Δ^3 or a flag polytope.

Construction of flag polytopes

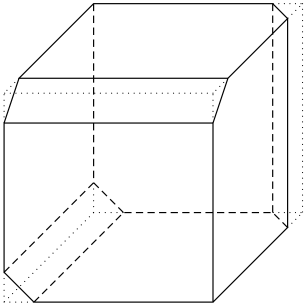
Theorem (A. Kotzig, 1967; V. Volodin, 2012+V.M. Buchstaber, N.Yu. Erokhovets, 2015)

A polytope is flag iff it can be obtained from the cube I^3 by a sequence of edge-truncations and $(\mathbf{2}, k)$-truncations, $k \geqslant 6$.

Construction of almost Pogorelov polytopes

Theorem (follows from the paper by D. Barnette, 1974)

A simple polytope P is almost Pogorelov iff either P is the cube, or the 5-gonal prism, or it can be obtained from the 3-dimensional associahedron (Stasheff polytope) by cuttings off edges not lying in 4-gons, and $(2, k)$-truncations, $k \geqslant 6$.

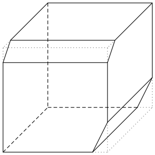


«Resolution of singularities»

- For any quadrangle of a flag polytope P there is a flag polytope Q such that P is obtained from Q by cutting off an edge producing the prescribed quadrangle.
- For almost Pogorelov polytopes analogous fact is not true.

Theorem (N.Yu. Erokhovets, 2018)

Any almost Pogorelov polytope $P \neq I^3, M_5 \times I$ is obtained by cutting off a disjoint set of edges (a matching) of an almost Pogorelov polytope Q or the polytope P_8, producing all its quadrangles.



Polytope P_8

Proposition

Any ideal right-angled polytope is obtained by a contraction of edges of a perfect matching of an almost Pogorelov polytope or the polytope P_8 containing exactly one edge of each quadrangle.

Problem

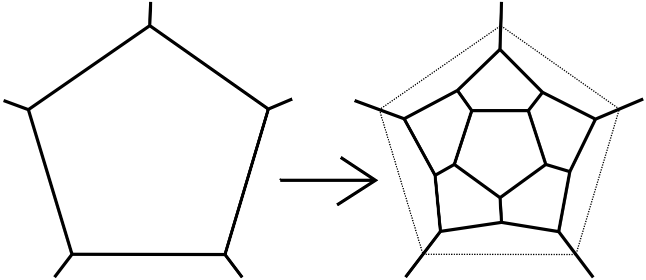
To characterize almost Pogorelov polytopes obtained by cutting off matchings of Pogorelov polytopes.

Necessary condition

Each quadrangle is adjacent by a pair of opposite edges to faces with at least six sides.

Connected sum along k-gonal faces

A connected sum of two simple polytopes P and Q along k-gonal faces F and G is a combinatorial analog of glueing of two polytopes along congruent faces orthogonal to adjacent faces.



Connected sum with the dodecahedron along 5-gons.

Construction of Pogorelov and Pogolelov* polytopes

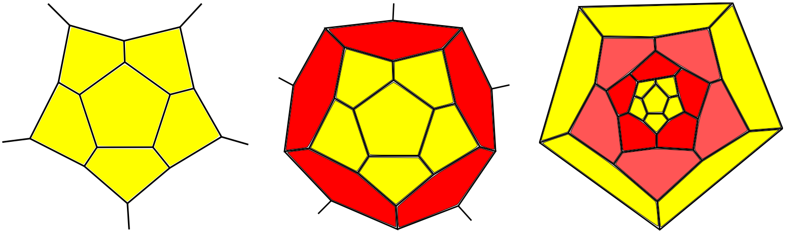
Theorem (D. Barnette, 1977+V.M. Buchstaber-E., 2017)

A polytope P is Pog iff either P is a q-barrel, $q \geqslant 5$, or it can be constructed from the 5- or the 6-barrel by a sequence of $(\mathbf{2}, k)$-truncations, $k \geqslant 6$, and connected sums with the 5-barrel.

Theorem (D. Barnette+V. M. Buchstaber, N. Yu. Erokhovets)

A polytope P is Pog* iff either P is a q-barrel, $q \geqslant 5$, or it can be constructed from the 6-barrel by a sequence of $(\mathbf{2}, k)$-truncations, $k \geqslant 6$.

Non-Pog^* fullerenes=$(5, 0)$-nanotubes



1. Take patch C of the dodecahedron drawn on the left;
2. add $k \geqslant 0$ five-belts of hexagons;
3. glue up by the patch C again to obtain the fullerene D_{5k}.

Proposition

A fullerene has the form D_{5k} iff it contains a patch C.

Th. (F. Kardoš, R. Škrekovski vs K. Kutnar, D. Marušič, 2008)

A fullerene is not Pog^* if and only if it is D_{5k}, $k \geqslant 1$.

Construction of fullerenes

Theorem (V.M. Buchstaber, N.Yu. Erokhovets, 2017)

Any Pogorelov fullerene either is the dodecahedron or can be obtained from the 6-barrel by a sequence of $(2,6)$- and $(2,7)$-truncations such that intermediate polytopes are fullerenes or 7-disk-fullerenes with the heptagon adjacent to a pentagon.*

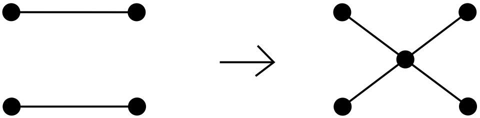
Construction of ideal right-angled polytopes

In the survey [*Right-angled polyhedra and hyperbolic 3-manifolds*, Russian Math. Surveys, 72:2 (2017), 335–374] A.Yu. Vesnin comparing results by

- I. Rivin (1996) on ideal polytopes, and
- G. Brinkmann, S. Greenberg, C. Greenhill, B.D. McKay, R. Thomas, and P. Wollan (2005) on graph theory

formulated theorem

> Any ideal right-angled polytope can be obtained from some k-antiprism, $k \geqslant 3$, by operations of edge-twist.

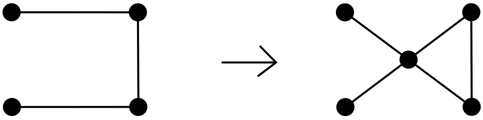


Edge-twist. The edges belong to one face and are not adjacent.

Construction of ideal right-angled polytopes

Theorem (N.Yu. Erokhovets, 2019)

A polytope P is ideal right-angled if and only if either P is a k-antiprism, $k \geqslant 3$, or P can be obtained from the 4-antiprism by a sequence of restricted edge-twists.



Restricted edge-twist. Edges are adjacent to the same edge.

Rigid properties

Definition

A property is rigid for the family of manifolds, if any isomorphism of graded rings $\varphi\colon H^*(M_1) \to H^*(M_2)$, $M_1, M_2 \in \mathcal{F}$ implies that both manifolds either have or do not have this property.

We say that a property is rigid for the class of polytopes, if it rigid for the corresponding family of moment-angle manifolds.

Proposition (F. Fan, J. Ma, X. Wang, 2015)

A property to be a flag polytope is rigid in the class of simple 3-polytopes.

Proof: The polytope $P \neq \Delta^3$ is flag if and only if

$$H^{m-2}(\mathcal{Z}_P) \subset (\widetilde{H}^*(\mathcal{Z}_P))^2.$$

Rigid properties

Proposition (F. Fan, J. Ma, X. Wang, 2015)

A property to be Pogorelov polytope is rigid in the class of simple 3-polytopes.

Proof: The flag polytope P is Pogorelov if and only if the multiplication

$$H^3(\mathcal{Z}_P) \otimes H^3(\mathcal{Z}_P) \to H^6(\mathcal{Z}_P)$$

is trivial.

Conjecture

The property to be almost Pogorelov polytope is rigid it the class of simple 3-polytopes.

Rigid sets

Let P be a simple 3-polytope. Then the cohomology rings $H^*(\mathcal{Z}_P)$ and $H^*(M(P,\Lambda))$ have no torsion.

Assume that for any manifold M from a family $\mathcal{F}$ a set $S_M \subset H^*(M)$ is given.

Definition

A set S_M is rigid for the family $\mathcal{F}$ if $\varphi(S_{M_1}) = S_{M_2}$ for any isomorphism of graded rings $\varphi\colon H^*(M_1) \to H^*(M_2)$, $M_1, M_2 \in \mathcal{F}$.

The group $H^3(\mathcal{Z}_P)$ is a group with the basis $\{a_{i,j}\}$ corresponding to pairs of non-adjacent faces F_i and F_j.

Lemma (F. Fan, J. Ma, X. Wang, 2015)

The set $\{\pm a_{i,j}\}$ is rigid for the class of Pogorelov polytopes.

Rigidity for belts (F. Fan, J. Ma, X. Wang, 2015)

Each k-belt corresponds to an element $H^{k+2}(\mathcal{Z}_P)$.

The free abelian subgroup in $H^{k+2}(\mathcal{Z}_P)$ with the basis corresponding to k-belts is rigid for the class of all simple 3-polytopes.

The subset in $H^{k+2}(\mathcal{Z}_P)$ of $\pm$elements corresponding to k-belts is rigid for the class of Pogorelov polytopes.

The subset in $H^{k+2}(\mathcal{Z}_P)$ of $\pm$elements corresponding to k-belts around faces is rigid for the class of Pogorelov polytopes.

Thus, any isomorphism of graded rings $\varphi\colon H^*(\mathcal{Z}_P) \to H^*(\mathcal{Z}_Q)$ for Pogorelov polytopes P and Q defines a bijection between sets of faces.

This bijection sends adjacent faces to adjacent faces.

Rigidity for quasitoric manifolds (V.M. Buchstaber, N.Yu. Erokhovets, M. Masuda, T.E. Panov, S. Park, 2016)

Each face F_i corresponds to the element v_i in $H^2(M(P,\Lambda))$.

The set of elements $\{\pm v_i \colon F_i \text{ is a face}\}$ is rigid for the class of Pogorelov polytopes.

Thus, any isomorphism of graded rings $\varphi\colon H^*(M(P,\Lambda_P)) \to H^*(M(Q,\Lambda_Q))$ for Pogorelov polytopes P and Q defines a bijection between sets of faces.

This bijection sends adjacent faces to adjacent faces.

Toric topology of almost Pogorelov polytopes

The image of $H^3(\mathcal{Z}_P) \otimes H^3(\mathcal{Z}_P) \to H^6(\mathcal{Z}_P)$ is the subgroup with the basis consisting of elements corresponding to 4-belts.

Proposition

The subset in $H^6(\mathcal{Z}_P)$ of $\pm$ elements corresponding to 4-belts is rigid for the class of almost Pogorelov polytopes different from the cube I^3 and the pentagonal prism $M_5 \times I$.

Problem

Is the set of

1. $\pm a_{i,j} \subset H^3(\mathcal{Z}_P)$;
2. $\pm$ elements corresponding to belts;
3. $\pm$ elements corresponding to belts around faces;
4. $\pm v_i$ in $H^2(M(P, \Lambda))$

rigid for the class of almost Pogorelov polytopes $\neq I^3, M_5 \times I$?

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Thank You for the Attention!