On the quasitoric manifolds over a simple polytope with one vertex cut

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§1. Introduction

Notations

- $T^n := (S^1)^n$
- The **standard action** $T^n \curvearrowright \mathbb{C}^n$:

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) := (t_1 z_1, \ldots, t_n z_n).$$

- $T^n \curvearrowright X, Y$

  $f : X \to Y$ is **weakly equivariant** $\overset{\text{def}}{\Longleftrightarrow}$
  $\exists \psi \in \mathrm{Aut}(T^n),\ \forall t \in T^n,\ \forall x \in X,\ f(t \cdot x) = \psi(t) \cdot f(x).$

Definition of a quasitoric manifold

- M : $2n$–dim smooth manifold
- $T^n \curvearrowright M$: smooth
- P : simple n–polytope (e.g. Δ^n, I^n)

Definition (Davis–Januszkiewicz 1991)

M is a **quasitoric manifold over** P if

(i) $[T^n \curvearrowright M] \overset{\text{local}}{\cong} [T^n \curvearrowright \mathbb{C}^n]$: weakly equivariant diffeo,

(ii) $M/T^n \cong P$: homeo as manifolds with corners.

A simple example is $\mathbb{C}P^2$ with the standard T^2–action. (following pages)

example: $\mathbb{C}P^2$ (1)

$T^2 \curvearrowright \mathbb{C}P^2$ by $(t_1, t_2) \cdot [z_0 : z_1 : z_2] := [z_0 : t_1 z_1 : t_2 z_2]$

Condition (i)

$[T^2 \curvearrowright \mathbb{C}P^2] \overset{\text{local}}{\cong} [T^2 \curvearrowright \mathbb{C}^2]$: weakly equivariant diffeo.

- $t = (t_1, t_2) \in T^2$
- $z = [z_0 : z_1 : z_2] \in \mathbb{C}P^2$
- $U_i = (z_i \neq 0)$, $\varphi_i \colon U_i \to \mathbb{C}^2$: i–th standard chart of $\mathbb{C}P^2$

For $\varphi_0 \colon U_0 \cong \mathbb{C}^2$ and $z = [1 : z_1 : z_2] \in U_0$,

$$\varphi_0(t \cdot z) = (t_1 z_1, t_2 z_2) = t \cdot \varphi_0(z).$$

example: $\mathbb{C}P^2$ (2)

For $\varphi_1\colon U_1 \cong \mathbb{C}^2$ and $z = [z_0 : 1 : z_2] \in U_1$,

$$\begin{aligned}\varphi_1(t \cdot z) &= \varphi_1([z_0 : t_1 : t_2 z_2]) \\ &= \varphi_1([t_1^{-1} z_0 : 1 : t_1^{-1} t_2 z_2]) = \psi_1(t) \cdot \varphi_1(z)\end{aligned}$$

where $\psi_1(t_1, t_2) = (t_1^{-1}, t_1^{-1} t_2)$, an automorphism of T^2.

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where $\psi_1(t_1, t_2) = (t_1^{-1}, t_1^{-1} t_2)$, an automorphism of T^2.

Similarly, for $z = [z_0 : z_1 : 1] \in U_2$,

$$\varphi_2(t \cdot z) = \psi_2(t) \cdot \varphi_2(z)$$

where $\psi_2(t_1, t_2) = (t_2^{-1}, t_1 t_2^{-1})$, an automorphism of T^2.

example: $\mathbb{C}P^2$ (3)

Condition (ii)

$\mathbb{C}P^2/T^2 \cong \Delta^2$: homeo as manifolds with corners.

$\mathbb{C}P^2/T^2 = S^5/T^3$. $(S^5 \subseteq \mathbb{C}^3)$

Then the moment map $(z_0, z_1, z_2) \mapsto (|z_0|, |z_1|, |z_2|)$ descends to a homeo $S^5/T^3 \to S^2 \cap (\mathbb{R}_{\geq 0})^3 \cong \Delta^2$.

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Remark

$\mathbb{C}P^n$ is a quasitoric manifold over Δ^n.

Fact

Moreover, any projective non-singular toric variety is a quasitoric manifold.

Fundamental correspondence

Fact

There is the following one-to-one correspondence:

$$\{\text{a toric variety}\} \longleftrightarrow \{\text{a fan}\}$$
$$\{\text{a quasitoric manifold}\} \longleftrightarrow \{\text{a characteristic pair}\}$$

A characteristic pair is (P, λ) where P is a simple polytope and λ is a **characteristic matrix on** P.

A characteristic matrix on P is an integer matrix satisfying a certain condition depending on P, which reflects the information of isotropy subgroups.

For example, the characteristic matrix of $\mathbb{C}P^2$ is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Construction of quasitoric manifold

λ: characteristic matrix on P

$\rightsquigarrow$ $M(\lambda) := (P \times T^n)/\sim_\lambda$, a qt mfd over P with ch mat λ

$\sim_\lambda$: isotropy information represented by λ

This construction gives the fundamental correspondence in the previous page.

Motivation

Up to weakly equivariant homeo, the classification of qt mfd's is a purely combinatoric matter through the fundamental correspondence.

However, the characteristic matrices give very little informations on the homeo's which are NOT weakly equivariant.

Aim

Develop a new method to find homeomorphisms between quasitoric manifolds which are not weakly equivariant.

We denote "not (necessarily) weakly equivariant" by "not eqv."

§2. Previous studies

Connected sum (1)

We regard the simple polytopes as a subclass of the manifolds with corners.

Fact

The connected sum of simple polytopes is a simple polytope.

Let M, N be quasitoric manifolds over P, Q.

Then the equivariant connected sum $M \sharp N$ is a quasitoric manifold over $P \sharp Q$.

On the other hand, we can easily determine whether a quasitoric manifold is decomposed into an equivariant connected sum of quasitoric manifolds.
(by using the characteristic matrix)

Connected sum (2)

We can get not eqv homeo's by using the connected sum.

For example, suppose that $P = Q \sharp \Delta^n$ and there are qt mfd's over P corresponding to the ch matrices λ, λ'.

Moreover, suppose that $\lambda = (A|B|\mathbf{v}), \lambda' = (A|B|\mathbf{v}')$ where $(A|B)$ is on Q and $(B|\mathbf{v}), (B|\mathbf{v}')$ are on Δ^n.

If $\det B = \pm 1$,

$$M(\lambda) \cong M(A|B) \sharp M(B|\mathbf{v}), \quad M(\lambda') \cong M(A|B) \sharp M(B|\mathbf{v}').$$

Since $M(B|\mathbf{v}) \cong M(B|\mathbf{v}') \cong \mathbb{C}P^n$ (the classification over Δ^n), we see that $M(\lambda) \cong M(\lambda')$.

Toric manifolds over $\mathrm{vc}(I^n)$

Let us consider the case $P = \mathrm{vc}(I^n) = I^n \sharp \Delta^n$.

On the toric manifolds over $\mathrm{vc}(I^n)$, we have the following. (They are a subclass of qt mfds over $\mathrm{vc}(I^n)$.)

Note that any ch mat can be denoted by $(-E_n|B|\mathbf{v})$ now.

Theorem (H–Kuwata–Masuda–Park 2018)

$\det B$ is invariant under cohomology isomorphism.

Theorem (H–Kuwata–Masuda–Park 2018)

Let $\mathcal{V}^n(q)$ denote the set of isomorphism classes of toric manifolds over $\mathrm{vc}(I^n)$ with $\det B = q$.

(1) If $q \neq 0, 1, 2$, then $\mathcal{V}^n(q)$ contains only 1 element.

(2) If $q = 0, 2$, then $\mathcal{V}^n(q)$ contains only 1 diffeo class.

§3. Main part

Setting

$P = \mathrm{vc}(Q) = Q \sharp \Delta^n$

$\lambda = (A|B|\mathbf{v}), \lambda' = (A|B|\mathbf{v}')$: characteristic matrices on P

$M := M(\lambda), M' = M(\lambda')$

Consider whether an analogue of the previous theorem holds in this situation.

Problem

Are M and M' homeomorphic?

By an elementary calculation on the characteristic matrices, we obtain the following.

Lemma

If $|\det B| \geq 3$, then $\mathbf{v} = \mathbf{v}'$.

Idea

Denote by $\pi \colon M \to P, \pi' \colon M' \to P$ the projections.

Let F be the facet made by vertex cut, N be a tubular neighborhood of F and $S := \partial N$.

Then

$$M = \pi^{-1}(P \setminus \mathring{N}) \bigcup_{\pi^{-1}(S)} \pi^{-1}(N),$$

$$M' = \pi'^{-1}(P \setminus \mathring{N}) \bigcup_{\pi'^{-1}(S)} \pi'^{-1}(N).$$

From the construction of $M(\lambda)$ and $M(\lambda')$, we have

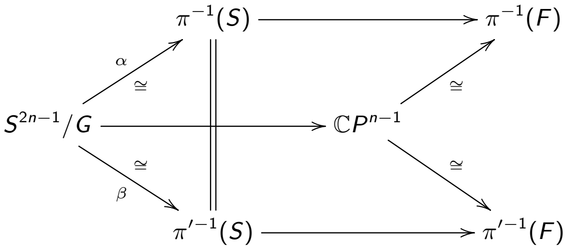
$$\pi^{-1}(P \setminus \mathring{N}) = \pi'^{-1}(P \setminus \mathring{N}), \quad \pi^{-1}(S) = \pi'^{-1}(S).$$

Moreover, $\pi^{-1}(N)$ is the disk bundle of a complex line bundle over $\pi^{-1}(F) \cong \mathbb{C}P^{n-1}$ classified by $\det B$.

The case $\det B \neq 0$

Suppose $\det B \neq 0$ and let $G := \mathbb{Z}/|\det B|$.

To compare the attaching diffeo, we consider the following diagram.



If $\det B = \pm 2$, we can take α and β specifically and show that $\beta^{-1} \circ \alpha$ is isotopic to the identity.

Proposition

If $\det B \neq 0$, then $M \cong M'$.

Remark on the case $\det B = 0$

Recall:

$P = \mathrm{vc}(Q) = Q \sharp \Delta^n$

$\lambda = (A|B|\mathbf{v}), \lambda' = (A|B|\mathbf{v}')$: characteristic matrices on P

$M := M(\lambda), M' = M(\lambda')$

Problem

Are M and M' homeomorphic?

For the case $\det B = 0$, we can find some counterexample for this problem in the known classification results.

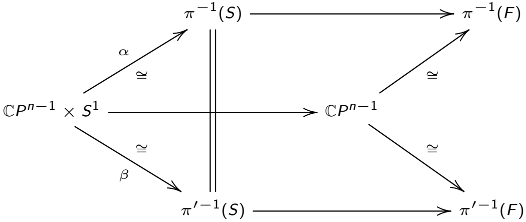
So we have to consider the following in this case.

Problem

Find some additional conditions for λ and λ' to make $M \cong M'$ hold.

The case $\det B = 0$ (on-going)

In this case, we consider the following diagram.



Taking specific α and β, we see $\beta^{-1} \circ \alpha(z, t) = (f(t) \cdot z, t)$ where $f\colon S^1 \to U(n)$, $f(t) = \mathrm{diag}(t^{b_1}, \ldots, t^{b_{n-1}}, 1)$ $(b_i \in \mathbb{Z})$.

We want to show $[f] = 0$ in $\pi_1(U(n)) = \mathbb{Z}$, which requires $b_1 + \cdots + b_{n-1} = 0$.

It seems good to consider whether the existence of good cohomology isomorphism leads to this equality.