Higher Whitehead products in moment-angle complexes

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joint work with Kouyemon Iriye

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Contents

1. Object of study

Moment-angle complex

- Let K be a simplicial complex with vertex set $[m] = \{1, \dots, m\}$.
- For $\sigma \subset [m]$, let

$$Z_\sigma = X_1 \times \cdots \times X_m, \quad \text{where} \quad X_i = \begin{cases} D^2 & i \in \sigma \\ S^1 & i \notin \sigma. \end{cases}$$

Def The moment-angle complex for K is defined by

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Def The moment-angle complex for K is defined by

$$Z_K = \bigcup_{\sigma \in K} Z_\sigma.$$

Eg Let $K = \partial\Delta^{m-1}$. For $m = 2$,

$$Z_K = (D^2 \times S^1) \cup (S^1 \cup D^2) = \partial(D^2 \times D^2) = S^3.$$

For m general,

$$Z_K = \partial(D^2)^m = S^{2m-1}.$$

Davis-Januszkiewicz space

- For $\sigma \subset [m]$ and a pointed space X, let

$$DJ_\sigma(X) = X_1 \times \cdots \times X_m, \quad \text{where} \quad X_i = \begin{cases} X & i \in \sigma \\ * & i \notin \sigma. \end{cases}$$

Def Define

$$DJ_K(X) = \bigcup_{\sigma \in K} DJ_\sigma(X),$$

where $DJ_K = DJ_K(\mathbb{C}P^\infty)$ is called the Davis-Januszkiewicz space.

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Eg Let $K = \partial\Delta^{m-1}$. For $m = 2$,

$$DJ_K(X) = (X \times *) \cup (* \times X) = X \vee X.$$

For m general, $DJ_K(X)$ is the m-fold fat-wedge of X, i.e.

$$DJ_K(X) = \{(x_1, \ldots, x_m) \in X^m \mid x_i = * \text{ for some } i\}.$$

Whitehead product

- Let
$$\overline{w}\colon Z_K \to DJ_K(S^2)$$
be the map induced from the pinch map $(D^2, S^1) \to (S^2, *)$.

Def The Whitehead product of maps $\alpha_1, \alpha_2\colon S^2 \to X$, denoted by $[\alpha_1, \alpha_2]$, is the composite

$$S^3 = Z_{\partial\Delta^1} \xrightarrow{\overline{w}} DJ_{\partial\Delta^1}(S^2) = S^2 \vee S^2 \xrightarrow{\alpha_1+\alpha_2} X.$$

Rem Whitehead products of maps from suspensions are similarly defined.

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Prop The cofiber of $S^3 = Z_{\partial\Delta^1} \xrightarrow{\overline{w}} DJ_{\partial\Delta^1}(S^2) = S^2 \vee S^2$ is $S^2 \times S^2$.

Cor TFAE:

1. $[\alpha_1, \alpha_2] = 0$;
2. $\alpha_1 + \alpha_2\colon S^2 \vee S^2 \to X$ extends over $S^2 \times S^2$.

Higher Whitehead product

- Let $\alpha_i \colon S^2 \to X$ be maps for $i = 1, \dots, m$.
- Suppose $\alpha_1 + \cdots + \alpha_m \colon \underbrace{S^2 \vee \cdots \vee S^2}_{m} \to X$ extends to a map

$$\alpha \colon DJ_{\partial\Delta^{m-1}}(S^2) \to X,$$

where $DJ_{\partial\Delta^{m-1}}(S^2)$ is the m-fold fat-wedge of S^2.

Def The higher Whitehead product for α is the composite

$$S^{2m-1} = Z_{\partial\Delta^{m-1}} \xrightarrow{\overline{w}} DJ_{\partial\Delta^{m-1}}(S^2) \xrightarrow{\alpha} X.$$

If α is clear by the context, we write it simply by $[\alpha_1, \dots, \alpha_m]$.

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Rem Higher Whitehead products of maps from suspensions are similarly defined.

Prop TFAE:

1. A higher Whitehead product of $\alpha_1, \dots, \alpha_m$ can be defined to be trivial;
2. $\alpha_1 + \cdots + \alpha_m \colon \underbrace{S^2 \vee \cdots \vee S^2}_{m} \to X$ extends over $(S^2)^m$.

The map w

The m-torus T^m acts on Z_K such that

$$Z_K \times_{T^m} ET^m \simeq DJ_K.$$

Then there is a homotopy fibration

$$Z_K \to DJ_K \to BT^m,$$

where we denote the fiber inclusion $Z_K \to DJ_K$ by w.

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Eg If $K = \partial\Delta^1$, then the above homotopy fibration is nothing but the (external) Ganea fibration

$$S^3 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^2.$$

Problem

Prop The map $w\colon Z_K \to DJ_K$ factors as the composite

$$Z_K \xrightarrow{\overline{w}} DJ_K(S^2) \to DJ_K$$

where the second map is induced from the inclusion $S^2 \to \mathbb{C}P^\infty$.

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- Let $a_i\colon S^2 \to DJ_K$ be the inclusion into the i-th $\mathbb{C}P^\infty$ in DJ_K.

Eg

1. The fiber inclusion $S^3 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty$ of the Ganea fibration is the Whitehead product $[a_1, a_2]$.
2. More generally, if $K = \partial\Delta^{m-1}$, then the map $w\colon Z_K \to DJ_K$ is the higher Whitehead product $[a_1, \dots, a_m]$.

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2. More generally, if $K = \partial\Delta^{m-1}$, then the map $w\colon Z_K \to DJ_K$ is the higher Whitehead product $[a_1, \dots, a_m]$.

Problem For which K is the map w described in terms of (higher) Whitehead products of $a_1, \dots, a_m$?

Previous work

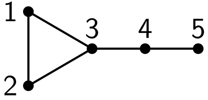
- Grbić and Theriault studied the problem for a small class of simplicial complexes by computing the rational loop homology of DJ_K.
- Unfortunately, the proof of their main result includes an unfixable mistake. But our result recovers their main result.
- There are several example calculations of the map w by others.
- All techniques used so far are not comprehensive.
- We will use the fat-wedge filtration technology which is the only one comprehensive technique to investigate the homotopy type of Z_K.

2. Result

Minimal non-face

- A subset $\sigma \subset [m]$ is called a minimal non-face of K if $\sigma \notin K$ and any proper subset of σ is a simplex of K.

Eg Minimal non-faces of a simplicial complex



are $123, 14, 15, 24, 25, 35$.

Minimal non-face

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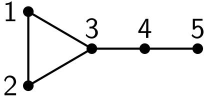
Eg Minimal non-faces of a simplicial complex



are $123, 14, 15, 24, 25, 35$.

- For $I \subset [m]$,

$$K_I = \{\sigma \in K \mid \sigma \subset I\}$$

is called the full subcomplex on I.

Eg $\sigma \subset [m]$ is a minimal non-face if and only if $K_\sigma = \partial\sigma$.

Proposition For any $\emptyset \neq I \subset [m]$, Z_{K_I} is a retract of Z_K.

Fillable complex

Recall that (higher) Whitehead products are defined by using boundaries of simplices.

Then if the homotopy type of Z_K is controlled by minimal non-faces of K, the map w might be described in terms of (higher) Whitehead products.

Def 1. K is called fillable if there are minimal non-faces $\sigma_1, \ldots, \sigma_r$ such that $|K \cup \sigma_1 \cup \cdots \cup \sigma_r|$ is contractible.

2. K is called totally fillable if all full subcomplexes are fillable.

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2. K is called totally fillable if all full subcomplexes are fillable.

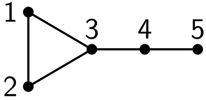
- The above $\{\sigma_1, \ldots, \sigma_r\}$ is called a filling, where there are possibly several fillings.
- We will assume that a fillable complex K is equipped with a specific filling $\mathcal{F}(K)$.

Eg Any skeleton of a simplex is totally fillable.

Proposition Dual shellable complexes are totally fillable.

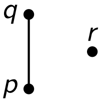
Example

The following simplicial complex K is totally fillable.

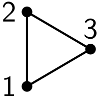


Indeed, its non-contractible full subcomplexes are K itself and

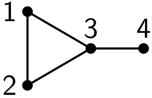
$K_{\{i,j\}}$



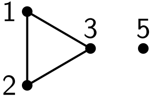
$K_{\{p,q,r\}}$



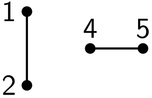
$K_{\{1,2,3\}}$



$K_{\{1,2,3,4\}}$



$K_{\{1,2,3,5\}}$



$K_{\{1,2,4,5\}}$

all of which are fillable, where $(i,j) = (1,4),(1,5),(2,4),(2,5),(3,5)$ and $(p,q,r) = (1,2,4),(1,2,5),(1,3,5),(2,3,5),(4,5,1),(4,5,2)$.

Main theorem

- For a totally fillable complex K, we put

$$W_K = \bigvee_{\emptyset \neq I \subset [m]} \bigvee_{\sigma \in \mathcal{F}(K_I)} S^{|\sigma|+|I|-1}.$$

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$$W_K = \bigvee_{\emptyset \neq I \subset [m]} \bigvee_{\sigma \in \mathcal{F}(K_I)} S^{|\sigma|+|I|-1}.$$

Theorem If K is totally fillable, then there is a homotopy equivalence

$$\epsilon_K \colon Z_K \xrightarrow{\simeq} W_K$$

such that for $\emptyset \neq I \subset [m]$ and $\sigma \in \mathcal{F}(K_I)$, the composite

$$S^{|\sigma|+|I|-1} \to W_K \xrightarrow{\epsilon_K^{-1}} Z_K \xrightarrow{w} DJ_K$$

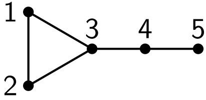
is the iterated Whitehead product

$$[[\cdots[[[a_{i_1}, \dots, a_{i_k}], a_{j_1}], a_{j_2}]\cdots], a_{j_{|I|-k}}]$$

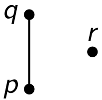
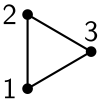
for some ordering $I - \sigma = \{j_1 < \cdots < j_{|I|-k}\}$, where $\sigma = \{i_1, \dots, i_k\}$.

Example

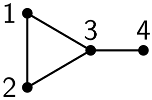
Let us apply the main theorem to the following fillable complex K.



Here is the list of full subcomplexes and the corresponding spheres and Whitehead products.

$K_{\{i,j\}}$	$K_{\{p,q,r\}}$	$K_{\{1,2,3\}}$
		
$\mathcal{F}(K_{\{i,j\}}) = \{ij\}$	$\mathcal{F}(K_{\{p,q,r\}}) = \{qr\}$	$\mathcal{F}(K_{\{1,2,3\}}) = \{123\}$
$I - \sigma = \emptyset$	$I - \sigma = \{p\}$	$I - \sigma = \emptyset$
S^3	S^4	S^5
$[a_i, a_j]$	$[[a_q, a_r], a_p]$	$[a_1, a_2, a_3]$

$K_{\{1,2,3,4\}}$



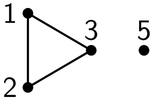
$\mathcal{F}(K_{\{1,2,3,4\}}) = \{123\}$

$I - \sigma = \{4\}$

S^6

$[[a_1, a_2, a_3], a_4]$

$K_{\{1,2,3,5\}}$



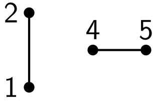
$\mathcal{F}(K_{\{1,2,3,5\}}) = \{123, 35\}$

$I - \sigma = \{5\}, \{1 < 2\}$

$S^6 \vee S^5$

$[[a_1, a_2, a_3], a_5] \vee [[[a_3, a_5], a_1], a_2]$

$K_{\{1,2,4,5\}}$



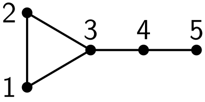
$\mathcal{F}(K_{\{1,2,4,5\}}) = \{24\}$

$I - \sigma = \{1 < 5\}$

S^5

$[[[a_2, a_4], a_1]a_5]$

$K_{\{1,2,3,4,5\}}$



$\mathcal{F}(K_{\{1,2,3,4,5\}}) = \{123\}$

$I - \sigma = \{4 < 5\}$

S^7

$[[[a_1, a_2, a_3], a_4,], a_5]$

3. Proof

Fat wedge filtration

Def Put

$$Z_K^i = \{(x_1, \dots, x_m) \in Z_K \mid \text{at least } m - i \text{ of } x_i \text{ are the basepoint}\}.$$

Then we get a filtration

$$* = Z_K^0 \subset Z_K^1 \subset \cdots \subset Z_K^m = Z_K$$

which we call the fat-wedge filtration.

Fat wedge filtration

Def Put

$$Z_K^i = \{(x_1, \ldots, x_m) \in Z_K \mid \text{at least } m - i \text{ of } x_i \text{ are the basepoint}\}.$$

Then we get a filtration

$$* = Z_K^0 \subset Z_K^1 \subset \cdots \subset Z_K^m = Z_K$$

which we call the fat-wedge filtration.

- The fat wedge filtration was found in the attempt to understand (or desuspend) the known stable splitting of Z_K.
- It clarifies how combinatorics of K is connected to Z_K.
- It already has several applications in topology and combinatorial commutative algebra.

Cone decomposition

Thm (Iriye & K '19) For each $\emptyset \neq I \subset [m]$, there is a map

$$\varphi_{K_I} \colon |K_I| * S^{|I|-1} \to Z_K^{|I|-1}$$

by which

$$Z_K^i = Z_K^{i-1} \bigcup_{I \subset [m],\, |I|=i} C(|K_I| * S^{|I|-1}).$$

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by which

$$Z_K^i = Z_K^{i-1} \bigcup_{I \subset [m],\, |I| = i} C(|K_I| * S^{|I|-1}).$$

Cor If all φ_{K_I} are null homotopic, then there is a homotopy equivalence

$$Z_K \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma^{|I|+1} K_I|$$

which is natural with respect to K and null homotopies of φ_{K_I}.

Homotopy decomposition

Prop (Iriye & K '19) The map φ_{K_I} factors through the inclusion

$$|K_I| * S^{|I|-1} \to |K_I \cup \{\text{minimal non-faces}\}| * S^{|I|-1}.$$

Cor If K is totally fillable, then $\varphi_{K_I} \simeq *$ for all $\emptyset \neq I \subset [m]$.

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Cor If K is totally fillable, then $\varphi_{K_I} \simeq *$ for all $\emptyset \neq I \subset [m]$.

Prop If K is fillable, then

$$|\Sigma K| \simeq \bigvee_{\sigma \in \mathcal{F}(K)} S^{|\sigma|-1}.$$

Proof $|\Sigma K| \simeq |K \bigcup_{\sigma \in \mathcal{F}(K)} \sigma|/|K| = \bigvee_{\sigma \in \mathcal{F}(K)} S^{|\sigma|-1}$. □

Thm If K is totally fillable, then there is a homotopy equivalence

$$\epsilon_K \colon Z_K \xrightarrow{\simeq} W_K$$

which is natucal with respect to K and fillings of its full subcomplexes.

Reduction

- Suppose K is totally fillable.
- Let $\sigma \in \mathcal{F}(K_I)$ for $\emptyset \neq I \subset [m]$.
- Let L be the simplicial complex which is the union of vertices in I and $\partial\sigma$.

Cor There is a homotopy commutative diagram

$$\begin{array}{ccccc} S^{|\sigma|+|I|-1} & \xrightarrow{\text{incl}} & W_L & \xrightarrow{\epsilon_L^{-1}} & Z_L \\ \Big\| \text{incl} & & \Big\downarrow & & \Big\downarrow \text{incl} \\ S^{|\sigma|+|I|-1} & \xrightarrow{\text{incl}} & W_K & \xrightarrow{\epsilon_K^{-1}} & Z_K \end{array}$$

whenever we choose an appropriate ordering of $I - \sigma$.

Then the proof reduces to the case of L, which is done by a homotopical observation.

4. Generalization

Polyhedral products

- Let $(\underline{X}, \underline{A}) = \{(X_i, A_I)\}_{i=1}^m$ be a collection of pairs of spaces.

One can associate to K and $(\underline{X}, \underline{A})$ the space

$$Z_K(\underline{X}, \underline{A})$$

called the polyhedral product.

Eg

1. If $(X_i, A_i) = (D^2, S^1)$ for all i, then $Z_K(\underline{X}, \underline{A}) = Z_K$.
2. If $(X_i, A_i) = (X, *)$ for all i, then $Z_K(\underline{X}, \underline{A}) = DJ_K(X)$.

- Let $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i=1}^m$ and $(\Sigma\underline{X}, *) = \{(\Sigma X_i, *)\}_{i=1}^m$.

The pinch maps $(CX_i, X_i) \to (\Sigma X_i, *)$ induce a map

$$\overline{w}\colon Z_K(C\underline{X}, \underline{X}) \to Z_K(\Sigma\underline{X}, *)$$

which specializes to $\overline{w}\colon Z_K \to DJ_K(S^2)$.

Generalization to polyhedral products

We can generalize our main theorem to the map

$$\overline{w}\colon Z_K(C\underline{X}, \underline{X}) \to Z_K(\Sigma\underline{X}, *)$$

whenever all X_i are suspensions because we have the following.

- Our main theorem is in fact a corollary of a similar result on the map

$$\overline{w}\colon Z_K \to DJ_K(S^2).$$

- We can define the fat-wedge filtration of $Z_K(C\underline{X}, \underline{X})$.
- The fat-wedge filtartion of $Z_K(C\underline{X}, \underline{X})$ is a cone decomposition whenever all X_i are suspensions.
- In this case, the attaching maps have the same properties as Z_K.