Loop homology of moment-angle-complexes and the Golod property of face rings (based on j.ws. with V.Buchstaber and T.Panov)

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Polyhedral products

Definition

Suppose $(\mathbf{X},\mathbf{A})=\{(X_i,A_i)\}_{i=1}^m$ is a set of topological pairs and K is a simplicial complex on the vertex set $[m]=\{1,\ldots,m\}$.

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A polyhedral product of $(\mathbf{X}, \mathbf{A})$ over K is a topological space

$$(\mathbf{X}, \mathbf{A})^K = \bigcup_{I \in K} \prod_{i \in I} Y_i,$$

where $Y_i = X_i$ if $i \in I$, and $Y_i = A_i$ if $i \notin I$.

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Moment-angle manifold

Suppose $K_P = \partial P^*$ is a nerve complex of a convex simple n-dimensional polytope P with m facets. A moment-angle manifold of P is a smooth closed $(m+n)$-dimensional 2-connected manifold $\mathcal{Z}_P$ homeomorphic to $\mathcal{Z}_{K_P}$.

Koszul homology of local rings

Koszul homology

Let $(A, \mathbf{m}, \mathbb{k})$ be a (commutative Noetherian) local ring, its unique maximal ideal $\mathbf{m}$ having a minimal set of generators $(x_1, \ldots, x_m)$ and its residue field being $\mathbb{k} = A/\mathbf{m}$.
Then Koszul complex of $(A, \mathbf{m}, \mathbb{k})$ is defined to be an exterior algebra $K_A = \Lambda A^m$, where A^m denotes the free A-module on $\{e_1, \ldots, e_m\}$, which is a d.g.a. with a differential d acting as follows:

$$d(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \sum_{r=1}^{k} (-1)^{r-1} x_{i_r} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_r}} \wedge \ldots \wedge e_{i_k}.$$

Poincaré series of local rings

Poincaré series

Let $(A, \mathbf{m}, \Bbbk)$ be a local ring. Then for an A-module M we define its Poincaré series to be formal power series of the type

$$P_A(M;t) = \sum_{i=0}^{\infty} \dim_{\Bbbk} \mathrm{Tor}_i^A(M, \Bbbk) t^i.$$

The $\Bbbk$-module $\mathrm{Tor}_i^A(M, \Bbbk)$ is defined to be the ith homology of a projective resolution for $\Bbbk$ (viewed as an A-module via the quotient map $A \to A/\mathbf{m} = \Bbbk$) tensored by M.
We call $P_A = P_A(\Bbbk; t)$ Poincaré series of a local ring A.

Using a spectral sequence associated with a presentation of a local ring as a quotient ring of a regular local ring, Serre showed that for any local ring A the following coefficient-wise inequality for its Poincaré series holds:

$$P_A \leq \frac{(1+t)^m}{1 - \sum b_i t^{i+1}},$$

where $b_i = \dim_{\mathbb{k}} H_i(K_A)$.

Golod rings

Using a spectral sequence associated with a presentation of a local ring as a quotient ring of a regular local ring, Serre showed that for any local ring A the following coefficient-wise inequality for its Poincaré series holds:

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Definition

A local ring A is called Golod if the Serre's inequality above turns into equality.

Theorem (E.S.Golod'62)

For a local ring A Serre's inequality turns into equality if and only if multiplication and all Massey products in $H_*(K_A)$ are vanishing.

Example

Let A be a free reduced nilpotent algebra, that is, a quotient ring $A_{n,r} = \frac{\Bbbk[x_1,\ldots,x_n]}{(x_1,\ldots,x_n)^r}$. Golod (1962) observed that multiplication and all Massey products are trivial in Koszul homology of $A_{n,r}$ and, furthermore, its Betti numbers are equal to $b_i = \binom{i+r-2}{r-1}\binom{n+r-1}{i+r-1}$. Therefore, $A_{n,r}$ is a Golod ring and

$$P_{A_{n,r}} = \frac{(1+t)^n}{1-\sum\limits_{i=1}^{n}\binom{i+r-2}{r-1}\binom{n+r-1}{i+r-1}t^{i+1}},$$

which generalizes computation of Poincaré series given by Kostrikin and Shafarevich (1957).

Moment-angle-complexes and Koszul homology

Let $\mathbf{J} \in \mathbb{Z}_2^m$, $\mathrm{mdeg}(u_i) = (-1; 0, \ldots, 2, \ldots, 0)$, $\mathrm{mdeg}(v_i) = (0; 0, \ldots, 2, \ldots, 0)$ for $1 \leq i \leq m$.
A multigraded Tor-module of $\Bbbk[K]$ is a direct sum of $\Bbbk$-modules

$$\mathrm{Tor}_{\Bbbk[m]}^{-i,2\mathbf{J}}(\Bbbk[K], \Bbbk) = H^{-i,2\mathbf{J}}[\Bbbk[K] \otimes_{\Bbbk} \Lambda[u_1, \ldots, u_m], d] \cong \tilde{H}^{|J|-i-1}(K_J),$$

where $d(u_i) = v_i$ and $d(v_i) = 0$ for all $1 \leq i \leq m$.

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Theorem (V.Buchstaber, T.Panov'99)

A graded $\Bbbk$-algebra isomorphism holds:

$$H^*(\mathcal{Z}_K; \Bbbk) \cong \mathrm{Tor}_{\Bbbk[m]}^{*,*}(\Bbbk[K], \Bbbk) = H^{*,*}[\Bbbk[K] \otimes_{\Bbbk} \Lambda[u_1, \ldots, u_m], d].$$

- $\mathbb{k}[K]$ is a Golod ring if the following identity for its Poincaré series holds

$$P(\mathbb{k}[K];t) = Hilb(\mathrm{Ext}_{\mathbb{k}[K]}(\mathbb{k},\mathbb{k});t) = \frac{(1+t)^m}{1-\sum\limits_{i,j>0}\beta^{-i,2j}(\mathbb{k}[K])t^{-i+2j-1}},$$

where the bigraded Betti numbers are the dimensions of the Tor-components:

$$\beta^{-i,2j}(\mathbb{k}[K]) = \dim_{\mathbb{k}} \mathrm{Tor}^{-i,2j}_{\mathbb{k}[v_1,\ldots,v_m]}(\mathbb{k}[K],\mathbb{k});$$

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$$\beta^{-i,2j}(\Bbbk[K]) = \dim_{\Bbbk}\mathrm{Tor}^{-i,2j}_{\Bbbk[v_1,\ldots,v_m]}(\Bbbk[K],\Bbbk);$$

- $\Bbbk[K]$ is minimally non-Golod (A.Berglund, M.Jöllenbeck'07): $\Bbbk[K]$ is **not** Golod, but removing any vertex from K turns the face ring of a restricted complex into a Golod ring.

Golod complexes: examples

J.Grbić, T.E.Panov, S.Theriault, J.Wu'12: $\mathbb{R}P^2$ on 6 vertices

Suppose $K = \mathbb{R}P^2_6$. Then K is a Golod complex and $\mathcal{Z}_K$ has a homotopy type of a wedge:

$$\mathcal{Z}_K \simeq (S^5)^{\vee 10} \vee (S^6)^{\vee 15} \vee (S^7)^{\vee 6} \vee \Sigma^7 \mathbb{R}P^2.$$

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L.'15: $\mathbb{C}P^2$ on 9 vertices

Suppose $K = \mathbb{C}P^2_9$. Then K is a Golod complex, all its full subcomplexes K_J have free integral homology groups, but $\mathcal{Z}_K$ is not homotopy equivalent to a wedge of spheres:

$$\mathcal{Z}_K \simeq (S^7)^{\vee 36} \vee (S^8)^{\vee 90} \vee (S^9)^{\vee 84} \vee (S^{10})^{\vee 36} \vee (S^{11})^{\vee 9} \vee \Sigma^{10} \mathbb{C}P^2.$$

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K.Iriye, T.Yano'16: $cat(\mathcal{Z}_K)>1$

There exists a Golod simplicial complex K s.t. $\mathcal{Z}_K$ is not a co-H-space. That is, $cat(\mathcal{Z}_K) > cat_0(\mathcal{Z}_K) = \cup(\mathcal{Z}_K) = 1$.

Golod complexes: flag case

Theorem (J.Grbić, T.Panov, S.Theriault, J.Wu'12; T.Panov, Ya.Veryovkin'16)

If K is **flag**, then the following statements are equivalent:

- $\mathrm{sk}^1(K)$ is a chordal graph;
- $\cup(\mathcal{Z}_K) = 1$, i.e. multiplication in $H^+(\mathcal{Z}_K; \mathbf{k})$ is trivial;
- $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres;
- $\Bbbk[K]$ is a Golod ring;
- Commutator subgroup $\pi_1(\mathcal{R}_K) = RC_K'$ of the right-angled Coxeter group RC_K is a free group;
- Associated graded Lie algebra $\mathrm{gr}(RC_K')$ is free.

Golod complexes: main result

Nonflag case: remark–example (L.Katthän'15)

(a) If $\dim K \leq 3$, then $\cup(\mathcal{Z}_K) = 1 \Longleftrightarrow K$ is a Golod complex;

(b) There exists a 4-dimensional simplicial complex K s.t.

- $\cup(\mathcal{Z}_K) = 1$;
- There is a nontrivial triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^*(\mathcal{Z}_K)$; therefore, K is **not** a Golod complex.

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Theorem (J.Grbić, T.Panov, S.Theriault, J.Wu'12; A.Berglund'10; L., T.E.Panov'19)

Let $\Bbbk$ be a field. The following are equivalent.

(1) $H_*(\Omega\mathcal{Z}_K; \Bbbk)$ is a graded free associative algebra;

(2) Multiplication and all Massey products in $H^+(\mathcal{Z}_K; \Bbbk)$ are trivial;

(3) $\Bbbk[K]$ is a Golod ring;

Golod complexes: main result

Theorem (J.Grbić, T.Panov, S.Theriault, J.Wu'12; A.Berglund'10; L., T.E.Panov)

(4) The following identity for the Hilbert series holds:

$$Hilb(H_*(\Omega\mathcal{Z}_K;\Bbbk);t)=\frac{1}{1-Hilb(\Sigma^{-1}\tilde{H}^*(\mathcal{Z}_K;\Bbbk);t)}.$$

Let $\Bbbk=\mathbb{Q}$. The following conditions are equivalent to (1)-(4).

(5) $L_{\mathcal{Z}_K}=\pi_*(\Omega\mathcal{Z}_K)\otimes\mathbb{Q}$ is a free graded Lie algebra;

(6) $\mathcal{Z}_K\simeq_{\mathbb{Q}}\vee S^i$, that is, $\mathrm{cat}_0(\mathcal{Z}_K)=1$.

Idea of the proof

We apply two spectral sequences associated with path–loop fibration for $\mathcal{Z}_K$:

- Milnor–Moore (bar) spectral sequence, which has $E_2^b = \mathrm{Tor}_{H_*(\Omega\mathcal{Z}_K)(\Bbbk,\Bbbk)}$ and converges to $\Sigma^{-1}\tilde{H}_*(\mathcal{Z}_K;\Bbbk)$;
- Adams (cobar) spectral sequence, which has $E_2^c = \mathrm{Cotor}_{H_*(\mathcal{Z}_K)}(\Bbbk;\Bbbk)$ and converges to $H_*(\Omega\mathcal{Z}_K;\Bbbk)$.

Key Lemma (L., T.Panov'19)

Conditions (1) and (4) above are equivalent to

- Both Adams (cobar) s.s. and Milnor–Moore (bar) s.s. collapse in E_2-terms.

(2) $\Longleftrightarrow$ (3) by graded version of Golod's theorem;
(3) $\Longleftrightarrow$ (4), since

$$\Omega(\mathbb{C}P^\infty)^K \simeq \Omega\mathcal{Z}_K \times \mathbb{T}^m \text{ and } P(\Bbbk[K];t) = Hilb(H_*(\Omega(\mathbb{C}P^\infty)^K;\Bbbk);t).$$

Minimally non-Golod face rings: flag case

Theorem (J.Grbić, T.Panov, S.Theriault, J.Wu'16)

Let $\Bbbk$ be a field and K be a flag simplicial complex. Then the following conditions are equivalent:

- $K = K_m$ for $m \geq 4$;
- $\mathcal{Z}_K \cong \# S^i \times S^j$;
- $\Bbbk[K]$ is minimally non-Golod.

Theorem (M.Ilyasova'19)

Let K be a flag simplicial complex. The following conditions are equivalent:

- RC_K' is a one-relator group;
- $H_2(\mathcal{R}_K) \cong \mathbb{Z}$;
- $K = K_p * \Delta^q$, where K_p is a boundary of an p-gon for $p \geq 4, q \geq -1$.

Conjecture (L., T.Panov'19)

Let $\mathbb{k} = \mathbb{Q}$ and K be a flag simplicial complex. Then the following conditions are equivalent:

(1) $H_*(\Omega\mathcal{Z}_K; \mathbb{k})$ is a graded associative algebra with one relation;

(2) the rational homotopy Lie algebra $L_{\mathcal{Z}_K}$ is a one-relation algebra;

(3) $\mathcal{Z}_K \simeq \# S^i \times S^j$;

(4) RC'_K is a one-relator group;

(5) $\mathrm{gr}(\mathrm{RC}'_K) \otimes \mathbb{Q}$ is a one-relation algebra;

(6) $K = K_p * \Delta^q$ for $p \geq 4, q \geq -1$;

(7) the following identity for the Poincaré series of $\mathbb{k}[K]$ holds:

$$P(\mathbb{k}[K]; t) = \frac{(1+t)^m}{1 - \sum_{i,j>0} \beta^{-i,2j}(\mathbb{k}[K]) t^{-i+2j-1} + (-1)^n t^m}.$$

Notation I

Consider a set of induced subcomplexes K_{I_j} on pairwisely disjoint subsets of vertices $\{I_j\}$ for $1 \leq j \leq k$ and their cohomology classes $\alpha_j \in \tilde{H}^{d(j)}(K_{I_j}) \subset H^{m(j)}(\mathcal{Z}_K), 1 \leq j \leq k$, where $m(j) = d(j) + |I_j| + 1$.

Notation II

If an s-fold Massey product ($s \leq k$) of consecutive classes $\langle \alpha_{i+1}, \ldots, \alpha_{i+s} \rangle$ for $1 \leq i+1 < i+s \leq k$ is defined, then

$$\langle \alpha_{i+1}, \ldots, \alpha_{i+s} \rangle \subset \tilde{H}^{d(i+1,i+s)}(K_{I_{i+1} \sqcup \ldots \sqcup I_{i+s}}) \subset H^{m(i+1,i+s)}(\mathcal{Z}_K),$$

where $d(i+1, i+s) = d(i+1) + \ldots + d(i+s) + 1$ and $m(i+1, i+s) = m(i+1) + \ldots + m(i+s) - s + 2$.

Key Lemma (L.'17)

Suppose $k \geq 3$ and

(1) $\tilde{H}^{d(s,r+s)-1}(K_{I_s \sqcup \ldots \sqcup I_{r+s}}) = 0, 1 \leq s \leq k-r, 1 \leq r \leq k-2$;

(2) Any of the following two conditions holds:

  (a) The k-fold Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ is defined, or

  (b) $\tilde{H}^{d(s,r+s)}(K_{I_s \sqcup \ldots \sqcup I_{r+s}}) = 0, 1 \leq s \leq k-r, 1 \leq r \leq k-2$.

Then the k-fold Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ is strictly defined.

The family $\mathcal{Q}$ of 2-truncated cubes

Construction (L.'16)

Suppose I^n is an n-dimensional cube with facets $F_1, \dots, F_{2n}$, such that F_i and F_{n+i}, $1 \leq i \leq n$ do not intersect. Then we define Q^n as a result of a consecutive cut of faces of codimension 2 from I^n, having the following Stanley-Reisner ideal:

$$I_{SR}(Q^n) = (v_k v_{n+k+i}, 0 \leq i \leq n-2, 1 \leq k \leq n-i, \dots),$$

where v_i correspond to F_i, $1 \leq i \leq 2n$ and in the last dots are the monomials corresponding to the new facets.

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We denote a family of these 2-truncated cubes by $\mathcal{Q}$.

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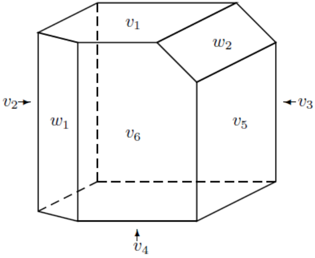
Theorem (L.'17)

Suppose $\alpha_i \in H^3(\mathcal{Z}_{Q^n})$ for $1 \leq i \leq n$ is represented by $v_i u_{n+i} \in K^{-1,4}_{\Bbbk[Q^n]}$ with $n \geq 2$. Then the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is strictly defined and nontrivial.

3-dimensional 2-truncated cube from the family $\mathcal{Q}$



$$I_{SR}(Q^3) = (v_1 v_4, v_2 v_5, v_3 v_6, v_1 v_5, v_2 v_6, w_1 v_3, w_1 v_5, w_2 v_2, w_2 v_4, w_1 w_2)$$

Let $\mathcal{F} = \{P^n \mid n \geq 0\}$ be a family of polytopes.

Definition (V.Buchstaber, L.'18)

A family $\mathcal{F}$ is called

- an Algebraic Direct Family of Polytopes (ADFP) if $\forall r, n > r$ $\exists i_r^n\colon F^r \hookrightarrow P^n$ s.t. $F^r = P^r$ and $\{P^r, i_r^n\}$ is a direct system;
- a Geometric Direct Family of Polytopes (GDFP) if it is algebraic **and** $\forall r, n > r$ $\exists J \subset [m(n)]$ s.t. $j_r^n\colon K_{P^r} \cong (K_{P^n})_J$ and $\{K_{P^r}, j_r^n\}$ is a direct system.

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A DFP $\mathcal{F}$ is called a direct family with nontrivial Massey products if $\exists\, 0\notin\langle\alpha_1,\ldots,\alpha_k\rangle\subset H^*(\mathcal{Z}_{P^n})$ for $k\to\infty$ as $n\to\infty$. Such a family $\mathcal{F}$ is called special if for any $n\geq 2$ there exists a nontrivial strictly defined k-fold Massey product in $H^*(\mathcal{Z}_{P^n})$ for all $2\leq k\leq n$.

Theorem (V.Buchstaber, L.'18)

- $\mathcal{Z}_{Q^n}$ is a submanifold and a retract of $\mathcal{Z}_{Q^{n+1}}$ for any $n \geq 1$;
- $\mathcal{Q}$ is a special geometric direct family of 2-truncated cubes with nontrivial Massey products.

Families of polytopes and LS-category: main result

Definition (LS-cat)

A covering of a space X is said to be categorical if every set in the covering is open and contractible in X. That is, the inclusion map of each set into X is nullhomotopic.
The Lusternik–Schnirelmann category (or simply LS-category) $\mathrm{cat}(X)$ of X is the smallest integer k s.t. X admits a categorical covering by $k+1$ open sets:

$$X = U_0 \cup \ldots \cup U_k.$$

Theorem (V.Buchstaber, L.'19)

Let Q^n be the n-dimensional 2-truncated cube from the family $\mathcal{Q}$ for $n \geq 2$. Then

$$\cup(\mathcal{Z}_{Q^n}) = \mathrm{cat}(\mathcal{Z}_{Q^n}) = n.$$

Application: special families and spectral sequences

Definition (V.Buchstaber, L.'19)

We say that a space X has *length* $\ell(X) \geq k$ w.r.t. a given spectral sequence if **there exists** a nontrivial differential $d_p, p \geq k$ in the spectral sequence of its path-loop fibration $\Omega X \to PX \to X$.

Corollary (V.Buchstaber, L.'19)

- Differentials d_r for $r \leq n-1$ in Eilenberg–Moore spectral sequence for $\mathcal{Z}_{Q^n}$ are nontrivial. In particular,

$$\ell_{EM}(\mathcal{Z}_{Q^n}) \geq n-1;$$

- Milnor–Moore spectral sequence for $\mathcal{Z}_{Q^n}$ degenerates in the E^{n+1}-term, and therefore,

$$\ell_{MM}(\mathcal{Z}_{Q^n}) \leq n.$$

THANK YOU FOR YOUR ATTENTION!