On orbit braids

(Joint work with Hao Li and Fengling Li)

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Outline

- History of braid groups
- Orbit braid group
- Main Results
- Calculations of orbit braid groups
E. Artin first defined the braids and braid groups in 1925
History of braid groups

- **E. Artin** first defined the braids and braid groups in 1925

- **Roots of the notion** can be seen in the researches of the following
  Even in Gauss’s notebook
Development of the theory of braids

After the work of Artin, the subject has continued to further develop by extending ideas of braid groups or combining with various ideas and theories from other research areas.
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- Alternative description using the fundamental groups of configuration spaces (Fox and Neuwirth).
- Generalized braid groups by Brieskorn to all finite Coxeter groups.
- Applications in low-dimensional topology, especially in the study of links and knots. E.g., a vast family of link invariants were constructed using braids.
- Cohomology theory of braid groups (E.g., Arnol’d’s work)
- Connection with other theories, such as theory of free groups
- Connection with other areas, such as Chern-Simons perturbation theory in mathematical physics and Yang-Baxter equation in physics
Original viewpoint of Artin

Braids arise naturally as isotopy classes of a collection of \( n \) connected strings in three-dimensional space \( \mathbb{R}^2 \times I \).

Theorem (Artin)

\( Br_n \) is generated by \( \sigma_i, i = 1, \ldots, n - 1 \) with the relations

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1 \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}
\end{align*}
\]
In 1962, E. Fadell and L. Neuwirth introduced the notion of configuration spaces

**Definition**

For a topological space $X$, the **configuration space** of $X$ at $n$ points is defined as follows:

$$F(X, n) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \quad \forall i \neq j\}$$

with subspace topology where $n \geq 2$.

**Remark.** The notion of configuration space first appeared in physics in the 1940s.
Meanwhile, R. Fox and L. Neuwirth showed in 1962 that

**Theorem (Fox–Neuwirth)**

\[ Br_n \cong \pi_1(F(\mathbb{R}^2, n)/\Sigma_n) \]

**Remark** Recently, people often define the braid groups from the viewpoint of configuration spaces.

Let \( M \) be a connected top. manifold of dim \( > 1 \). Symmetric group \( \Sigma_n \rhd F(M, n) \) freely.
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- \( \pi_1(F(M, n)/\Sigma_n) \) is defined as the **braid group** on \( n \) strings in \( M \times I \), and is denoted by \( B_n(M) \).
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\[ 1 \longrightarrow P_n(M) \longrightarrow B_n(M) \longrightarrow \Sigma_n \longrightarrow 1 \]
An explanation in the general case

In the viewpoint of Artin,

Braids in $M \times I$ can be constructed by using paths in $F(M, n)$

- Given a path $\alpha = (\alpha_1, ..., \alpha_n) : I \rightarrow F(M, n)$ with $\alpha(0) = x$ and $\alpha(1) = x_\sigma \in \Sigma_n(x)$.
- $\alpha$ gives a braid $c(\alpha) = \{c(\alpha_1), ..., c(\alpha_n)\}$ of $n$ strings in $M \times I$, where each string $c(\alpha_i) = \{(\alpha_i(s), s) | s \in I\} \approx I$. 

Equivalence

Let $\alpha, \beta : I \rightarrow F(M, n)$ be two paths with the same endpoints. Then $\alpha \simeq \beta$ (rel $\partial I$) $\iff c(\alpha) \sim$ isotopy $c(\beta)$ in $M \times I$.

Theorem

$B_n(M) \simeq = \pi_1(F(M, n) / \Sigma_n)$
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**Equivalence**

Let $\alpha, \beta: I \to F(M, n)$ be two paths with the same endpoints. Then $\alpha \simeq \beta(\text{rel } \partial I) \iff c(\alpha) \sim_{\text{isotopy}} c(\beta)$ in $M \times I$. 
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**Theorem**

$$B_n(M) \cong \pi_1(F(M, n)/\Sigma_n)$$
Generalized braid groups

In 1970’s, Brieskorn generalized the concept of classical braid group from symmetric group to all finite Coxeter groups, which is called generalized braid group or Artin group.

Let
\[ W = \langle w_1, \ldots, w_k \mid w_i^2 = e, (w_i w_j)^{m_{ij}} = e \rangle \]
be a finite Coxeter group where \( m_{ij} = m_{ji} \).
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Definition of generalized braid group

The **generalized braid group** \( Br(W) \) of \( W \) is defined as the group with generators \( w_i \) and relations

\[ \text{prod}(m_{ij}; w_i, w_j) = \text{prod}(m_{ji}; w_j, w_i) \]

where the symbol \( \text{prod}(m; x, y) \) stands for the product \( xyxy \cdots \) with \( m \) factors.
Geometric realization of generalized braid groups

First

- $V$: an $n$-dim real vector space
- $W$: considered as a finite subgroup of $GL(V)$ generated by reflections
- $\mathcal{M}$: the set of hyperplanes such that $W$ is generated by the orthogonal reflections in the $M \in \mathcal{M}$, and assume that $w(M) \in \mathcal{M}$ for any $w \in W$ and any $M \in \mathcal{M}$.
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Next

- consider the complexification $V_\mathbb{C}$ of $V$ and the complexification $M_\mathbb{C}$ of $M \in \mathcal{M}$.
- Set $Y_W = V_\mathbb{C} - \bigcup_{M \in \mathcal{M}} M_\mathbb{C}$
- $W$ acts freely on $Y_W$, so we have the quotient $X_W = Y_W/W$.
- $1 \rightarrow \pi_1(Y_W) \rightarrow \pi_1(X_W) \rightarrow W \rightarrow 1.$
Geometric realization of generalized braid groups

Theorem (Brieskorn–Deligne)

(1) $\pi_1(X_W) \cong Br(W)$;
(2) The universal covering of $X_W$ is contractible, and hence $X_W$ is a space of $K(\pi; 1)$. 
Theorem (Brieskorn–Deligne)

(1) \( \pi_1(\mathcal{X}_W) \cong Br(W) \);
(2) The universal covering of \( \mathcal{X}_W \) is contractible, and hence \( \mathcal{X}_W \) is a space of \( K(\pi; 1) \).

Remark

- Generalized braid group \( Br(W) \) is realized by the fundamental group \( \pi_1(\mathcal{X}_W) \)
- The fundamental group \( \pi_1(\mathcal{Y}_W) \) is called the pure braid group, also denoted by \( P(W) \).
- \( 1 \rightarrow P(W) \rightarrow Br(W) \rightarrow W \rightarrow 1 \).
Definition in a general way

Compatible with various points of view, the notion of braid groups was uniformly defined by Vershinin in a general way as follows:
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Compatible with various points of view, the notion of braid groups was uniformly defined by Vershinin in a general way as follows:

- Choose a connected topological manifold $\mathcal{M}$ admitting an action of a finite group $G$. 

The fundamental group $\pi_1(\mathcal{M})$ is called the braid group of the action of $G$ on $\mathcal{M}$, denoted by $\text{Br}(\mathcal{M}, G)$, and the fundamental group $\pi_1(\mathcal{Y}_G)$ is called the pure braid group of the action of $G$ on $\mathcal{M}$, denoted by $\text{P}(\mathcal{M}, G)$. 
Compatible with various points of view, the notion of braid groups was uniformly defined by Vershinin in a general way as follows:

- Choose a connected topological manifold $\mathbb{M}$ admitting an action of a finite group $G$.
- Let $Y_G$ be formed by all points of free orbit type in $\mathbb{M}$. So the action of $G$ restricted to $Y_G$ is free. Assume that $Y_G$ is connected. Then there is a fibration $Y_G \to X_G$ with fiber $G$, which gives a short exact sequence:

$$1 \to \pi_1(Y_G) \to \pi_1(X_G) \to G \to 1$$
Definition in a general way

Compatible with various points of view, the notion of braid groups was uniformly defined by Vershinin in a general way as follows:

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$$1 \longrightarrow \pi_1(Y_G) \longrightarrow \pi_1(X_G) \longrightarrow G \longrightarrow 1$$

- The fundamental group $\pi_1(X_G)$ is called the braid group of the action of $G$ on $\mathbb{M}$, denoted by $Br(\mathbb{M}, G)$, and the fundamental group $\pi_1(Y_G)$ is called the pure braid group of the action of $G$ on $\mathbb{M}$, denoted by $P(\mathbb{M}, G)$. 
Motivation and Aim

To upbuild the theoretical framework of orbit braids.
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To upbuild the theoretical framework of orbit braids.

Our strategy

Our strategy to do this is to mix the original idea of Artin and the theory of transformation groups together by making use of the construction of orbit configuration spaces.
Orbit configuration space

Definition (M. A. Xicoténcatl, Thesis (Ph.D.)-University of Rochester. 1997)

Given a topological group $G$ and a topological space $X$ with an effective $G$-action. Then the orbit configuration space of the $G$-space $X$ is defined by

$$F_G(X, n) = \{(x_1, \ldots, x_n) \in X^n \mid G(x_i) \cap G(x_j) = \emptyset \text{ for } i \neq j\}$$

with subspace topology, where $n \geq 2$ and $G(x)$ denotes the orbit of $x$. 
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with subspace topology, where $n \geq 2$ and $G(x)$ denotes the orbit of $x$.

Remark: Pay our attention on the case:

$G$: a finite group
$X$: a connected topological manifold $M$ of dim $> 1$ with an effective $G$-action.
So $F_G(M, n)$ is connected.
Fix a point $\mathbf{x} = (x_1, \ldots, x_n) \in F_G(X, n)$ as a base point where the orbit $G(x_i)$ at $x_i$ is of free type. Let $\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, $\sigma \in \Sigma_n$.

**Braids in $M \times I$ from paths in $F_G(M, n)$**

- Take a path $\alpha : I \rightarrow F_G(M, n)$ with $\alpha(0) = \mathbf{x}$ and $\alpha(1) = \mathbf{x}_\sigma$. 
Fix a point $\mathbf{x} = (x_1, ..., x_n) \in F_G(X, n)$ as a base point where the orbit $G(x_i)$ at $x_i$ is of free type.

Let $\mathbf{x}_\sigma = (x_{\sigma(1)}, ..., x_{\sigma(n)})$, $\sigma \in \Sigma_n$.

**Braids in $M \times I$ from paths in $F_G(M, n)$**

- Take a path $\alpha : I \to F_G(M, n)$ with $\alpha(0) = \mathbf{x}$ and $\alpha(1) = \mathbf{x}_\sigma$.

- Set $c(\alpha) = \{c(\alpha_1), ..., c(\alpha_n)\}$ where
  
  $c(\alpha_i) = \{(\alpha_i(s), s) | s \in I\}$

  which gives a braid of $n$ strings in $M \times I$. 

Remark. If we forget the action of $G$ on $M$, then $c(\alpha)$ becomes a braid in the sense of Artin. Otherwise, $c(\alpha)$ would be different from the classical one.
Fix a point \( x = (x_1, ..., x_n) \in F_G(X, n) \) as a base point where the orbit \( G(x_i) \) at \( x_i \) is of free type. Let \( x_\sigma = (x_{\sigma(1)}, ..., x_{\sigma(n)}) \), \( \sigma \in \Sigma_n \).

**Braids in** \( M \times I \) **from paths in** \( F_G(M, n) \)

- Take a path \( \alpha : I \rightarrow F_G(M, n) \) with \( \alpha(0) = x \) and \( \alpha(1) = x_\sigma \).

- Set \( c(\alpha) = \{ c(\alpha_1), ..., c(\alpha_n) \} \) where

  \[
  c(\alpha_i) = \{ (\alpha_i(s), s) | s \in I \}
  \]

  which gives a braid of \( n \) strings in \( M \times I \).

**Remark.** If we forget the action of \( G \) on \( M \), then \( c(\alpha) \) becomes a braid in the sense of Artin. Otherwise, \( c(\alpha) \) would be different from the classical one.
Example

Consider the orbit configuration space $F_{\mathbb{Z}_2}(\mathbb{C}, n)$ where the action of $\mathbb{Z}_2$ on $\mathbb{C}$ is given by $z \mapsto -z$, so this action is non-free and fixes only the origin of $\mathbb{C}$. In the case of $n = 2$, let us see two closed paths $\alpha, \beta : I \rightarrow F_{\mathbb{Z}_2}(\mathbb{C}, 2)$ at the point $x = (1, 2)$ such that their corresponding braids $c(\alpha)$ and $c(\beta)$ are as shown below:

If we forget the action of $\mathbb{Z}_2$ on $\mathbb{C}$, then clearly $c(\alpha)$ and $c(\beta)$ are isotopic relative to endpoints in $\mathbb{C} \times I$. However, under the condition that $\mathbb{C}$ admits the action of $\mathbb{Z}_2$, both $c(\alpha)$ and $c(\beta)$ are not isotopic since the first string of $c(\alpha)$ cannot go through the orbit of the second string of $c(\alpha)$, as we can see from the following left picture.
Orbit braids

**Definition**

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) : I \to F_G(M, n) \) be a path such that 
\( \alpha(0) = x \) and \( \alpha(1) = gx_\sigma \) for some \((g, \sigma) \in G^n \times \Sigma_n\). Then 

\[
\overline{c(\alpha)} = \{\overline{c(\alpha_1)}, \ldots, \overline{c(\alpha_n)}\}
\]

is called an **orbit braid** in \( M \times I \),
Orbit braids

Definition

Let $\alpha = (\alpha_1, \ldots, \alpha_n) : I \to F_G(M, n)$ be a path such that $\alpha(0) = x$ and $\alpha(1) = gx_\sigma$ for some $(g, \sigma) \in G^{\times n} \times \Sigma_n$. Then

$$\overline{c(\alpha)} = \{ \overline{c(\alpha_1)}, \ldots, \overline{c(\alpha_n)} \}$$

is called an orbit braid in $M \times I$, where each orbit string $\overline{c(\alpha_i)} = \{ hc(\alpha_i) \mid h \in G \}$ is the orbit of the string $c(\alpha_i)$.
Orbit braids

**Definition**

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) : I \rightarrow F_G(M, n) \) be a path such that \( \alpha(0) = x \) and \( \alpha(1) = g x_\sigma \) for some \( (g, \sigma) \in G^\times n \times \Sigma_n \). Then

\[
\widetilde{c}(\alpha) = \{ \widetilde{c}(\alpha_1), \ldots, \widetilde{c}(\alpha_n) \}
\]

is called an **orbit braid** in \( M \times I \), where each orbit string \( \widetilde{c}(\alpha_i) = \{ hc(\alpha_i) | h \in G \} \) is the orbit of the string \( c(\alpha_i) \).

Fix \( \widetilde{c}(x) = \{ G(x_1), \ldots, G(x_n) \} \) as an **orbit base point**.

**Natural operation:**

\[
\widetilde{c}(\alpha) \circ \widetilde{c}(\beta) |_{s \in I} = \begin{cases} 
\widetilde{c}(\alpha) |_{2s \in I} & \text{if } s \in [0, \frac{1}{2}] \\
\widetilde{c}(\beta) |_{2s-1 \in I} & \text{if } s \in [\frac{1}{2}, 1]. 
\end{cases}
\]

but this operation is not associative.
Recall

**Equivalence in the theory of classical braids**

Let $\alpha, \beta : I \to F(M, n)$ be two paths with the same endpoints.

$$\alpha \simeq_{\text{rel } \partial I} \beta \iff c(\alpha) \sim_{\text{iso}} c(\beta).$$
Equivalence relation among ordinary braids

Recall

**Equivalence in the theory of classical braids**

Let $\alpha, \beta : I \to F(M, n)$ be two paths with the same endpoints.

$$\alpha \simeq_{\text{rel } \partial I} \beta \iff c(\alpha) \sim_{\text{iso}} c(\beta).$$

- In the theory of ordinary braids, isotopy is used as the equivalence relation among ordinary braids.
Recall

**Equivalence in the theory of classical braids**

Let $\alpha, \beta : I \longrightarrow F(M, n)$ be two paths with the same endpoints.

\[ \alpha \simeq_{\partial I} \beta \iff c(\alpha) \sim_{iso} c(\beta). \]

- In the theory of ordinary braids, isotopy is used as the equivalence relation among ordinary braids.
- However, equivariant isotopy is not sufficient enough to be used as the equivalence relation among orbit braids.
Example

Let the action of $\mathbb{Z}_2$ on $\mathbb{C}$ be the same as the above example. Consider the orbit configuration space $F_{\mathbb{Z}_2}(\mathbb{C}, n)$. In the case of $n = 2$, take two closed paths $\alpha, \beta : I \to F_{\mathbb{Z}_2}(\mathbb{C}, 2)$ at the base point $x = (1, 2)$ such that their corresponding ordinary braids $c(\alpha)$ and $c(\beta)$ are shown as follows:

Clearly, $c(\alpha) \sim_{iso}^{G} c(\beta)$. This means that orbit braids $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ as shown below are essentially the same in such a sense that the first string of $c(\alpha)$ can be deformed into the first string of $c(\beta)$ in $M \times I$ under the action of $G$. However, $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ are not equivariant isotopic since they are even not homeomorphic.
Equivalence relation among orbit braids

How to define equivalence relation among orbit braids?

Isotopy with respect to the $G$-action

Let $\alpha, \beta : I \to F_G(M, n)$ be two paths with the same endpoints. We say that $c(\alpha) \sim^G_{iso} c(\beta)$ (isotopic with respect to the $G$-action in $M \times I$) if there exist $n$ homotopy maps $\hat{h}_i : I \times I \to M \times I$ given by $\hat{h}_i(s, t) = (h_i(s, t), s)$, $i = 1, \ldots, n$, such that

1. $\bigsqcup_{i=1}^n \hat{h}_i(s, 0) = c(\alpha)$ and $\bigsqcup_{i=1}^n \hat{h}_i(s, 1) = c(\beta)$;
2. $\bigsqcup_{i=1}^n \hat{h}_i(0, t) = c(\alpha)|_{s=0} = c(\beta)|_{s=0}$ and $\bigsqcup_{i=1}^n \hat{h}_i(1, t) = c(\alpha)|_{s=1} = c(\beta)|_{s=1}$;
3. For any $(s, t) \in I \times I$, if $i \neq j$ then $G(h_i(s, t)) \cap G(h_j(s, t)) = \emptyset$. 
Equivalence relation among orbit braids

How to define equivalence relation among orbit braids?

Isotopy with respect to the $G$-action

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1. $\bigsqcup_{i=1}^n \hat{h}_i(s, 0) = c(\alpha)$ and $\bigsqcup_{i=1}^n \hat{h}_i(s, 1) = c(\beta)$;
2. $\bigsqcup_{i=1}^n \hat{h}_i(0, t) = c(\alpha)|_{s=0} = c(\beta)|_{s=0}$ and $\bigsqcup_{i=1}^n \hat{h}_i(1, t) = c(\alpha)|_{s=1} = c(\beta)|_{s=1}$;
3. For any $(s, t) \in I \times I$, if $i \neq j$ then $G(h_i(s, t)) \cap G(h_j(s, t)) = \emptyset$.

Proposition

Let $\alpha, \beta : I \to F_G(M, n)$ be two paths with the same endpoints. Then

$$\alpha \simeq \beta (rel \ \partial I) \iff c(\alpha) \sim^G_{iso} c(\beta).$$
Equivalence relation among orbit braids

We say that $\widetilde{c}(\alpha)$ and $\widetilde{c}(\beta)$ are equivalent, denoted by $\widetilde{c}(\alpha) \sim \widetilde{c}(\beta)$, if there are some $g$ and $h$ in $G^{\times n}$ such that $c(g\alpha) \sim_{iso}^G c(h\beta)$. 
Equivalence relation among orbit braids

We say that \( \sim \) and \( \tilde{c}(\beta) \) are equivalent, denoted by \( \sim \), if there are some \( g \) and \( h \) in \( G \times_n \) such that \( \sim \).

In terms of homotopy

**Proposition**

\[ \sim \) and \( \tilde{c}(\beta) \] there are two paths \( \alpha' \) and \( \beta' \) with \( \tilde{c}(\alpha') = \tilde{c}(\alpha) \) and \( \tilde{c}(\beta') = \tilde{c}(\beta) \), such that \( \alpha' \simeq \beta' \) rel \( \partial \).
Orbit braid groups

Let $B_{orb}^n(M, G)$ be the set consisting of the equivalence classes of all orbit braids at orbit base point $c(x)$ in $M \times I$. 
Orbit braid groups

Let $B_{orb}^n(M, G)$ be the set consisting of the equivalence classes of all orbit braids at orbit base point $\tilde{c}(x)$ in $M \times I$.

Define an operation $\ast$ on $B_{orb}^n(M, G)$ by

$$[\hat{c}(\alpha)] \ast [\hat{c}(\beta)] = [\hat{c}(\alpha) \circ \hat{c}(\beta)].$$
Orbit braid groups

- Let $\mathcal{B}_n^{\text{orb}}(M, G)$ be the set consisting of the equivalence classes of all orbit braids at orbit base point $\widetilde{c(x)}$ in $M \times I$.
- Define an operation $\ast$ on $\mathcal{B}_n^{\text{orb}}(M, G)$ by

$$[\widetilde{c(\alpha)}] \ast [\widetilde{c(\beta)}] = [\widetilde{c(\alpha) \circ c(\beta)}].$$

**Proposition**

$\mathcal{B}_n^{\text{orb}}(M, G)$ forms a group under the operation $\ast$, called the **orbit braid group** of the $G$-manifold $M$. 

Orbit braid groups

- Let $\mathcal{B}_n^{orb}(M, G)$ be the set consisting of the equivalence classes of all orbit braids at orbit base point $\tilde{c}(\mathbf{x})$ in $M \times I$.
- Define an operation $\ast$ on $\mathcal{B}_n^{orb}(M, G)$ by

\[
[\tilde{c}(\alpha)] \ast [\tilde{c}(\beta)] = [\tilde{c}(\alpha) \circ \tilde{c}(\beta)].
\]

Proposition

$\mathcal{B}_n^{orb}(M, G)$ forms a group under the operation $\ast$, called the \textbf{orbit braid group} of the $G$-manifold $M$.

Remark: Each class $[\tilde{c}(\alpha)]$ in $\mathcal{B}_n^{orb}(M, G)$ determines a unique pair $(g, \sigma) \in G^{\times n} \times \Sigma_n$. \textbf{Key Point!}
Subgroups of orbit braid groups

(1) Those classes $[\tilde{c}(\alpha)]$ with $\alpha(1) \in G^\times n(x)$ of $B_{n}^{orb}(M, G)$ form a subgroup of $B_{n}^{orb}(M, G)$, which is called the pure orbit braid group, denoted by $P_{n}^{orb}(M, G)$.

(2) Those classes $[\tilde{c}(\alpha)]$ with $\alpha(1) \in \Sigma_{n}(x) = \{x_{\sigma}|\sigma \in \Sigma_{n}\}$ of $B_{n}^{orb}(M, G)$ form a subgroup of $B_{n}^{orb}(M, G)$, which is called the braid group, denoted by $B_{n}(M, G)$.

(3) Those classes $[\tilde{c}(\alpha)]$ with $\alpha(1) = x$ of $B_{n}^{orb}(M, G)$ form a subgroup of $B_{n}^{orb}(M, G)$, which is called the pure braid group, denoted by $P_{n}(M, G)$. 
Let $\pi_1^E(F_G(M, n), x, x^{orb})$ be the set consisting of the homotopy classes relative to $\partial I$ of all paths $\alpha : I \to F_G(M, n)$ with $\alpha(0) = x$ and $\alpha(1) \in x^{orb}$, where $x^{orb} = \{gx_\sigma | g \in G^{\times n}, \sigma \in \Sigma_n\}$ is the orbit set at $x$ under two actions of $G^{\times n}$ and $\Sigma_n$. 
Let $\pi_1^E(F_G(M, n), x, x^{orb})$ be the set consisting of the homotopy classes relative to $\partial I$ of all paths $\alpha : I \longrightarrow F_G(M, n)$ with $\alpha(0) = x$ and $\alpha(1) \in x^{orb}$, where $x^{orb} = \{gx_\sigma | g \in G^{\times n}, \sigma \in \Sigma_n\}$ is the orbit set at $x$ under two actions of $G^{\times n}$ and $\Sigma_n$.

we can endow an operation $\bullet$ on $\pi_1^E(F_G(M, n), x, x^{orb})$ defined by

$$[\alpha] \bullet [\beta] = [\alpha \circ (g_\beta \sigma)]$$

where $(g, \sigma) \in G^{\times n} \times \Sigma_n$ is the unique pair determined by $\tilde{c(\alpha)}$. 
Theorem

\[ \pi_1^E(F_G(M, n), x, x_{orb}) \text{ forms a group under the operation } \bullet. \]
Furthermore, the map

\[ \Lambda : \pi_1^E(F_G(M, n), x, x_{orb}) \to B_{n}^{orb}(M, G) \]

given by \([\alpha] \mapsto \widetilde{c(\alpha)}\) is an isomorphism.

\[ \pi_1^E(F_G(M, n), x, x_{orb}) \] is called the \textbf{extended fundamental group} of \(F_G(M, n)\) at \(x_{orb}\).
Corollary

(1) $\mathcal{P}^{\text{orb}}_n(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, G^{\times n}(\mathbf{x}));$

(2) $\mathcal{B}_n(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, \Sigma_n(\mathbf{x}));$

(3) $\mathcal{P}_n(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, \mathbf{x}) = \pi_1(F_G(M, n), \mathbf{x}).$
Homotopy description of subgroups

**Corollary**

1. \( \mathcal{P}_n^{\text{orb}}(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, G^n(\mathbf{x})) \);
2. \( \mathcal{B}_n(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, \Sigma_n(\mathbf{x})) \);
3. \( \mathcal{P}_n(M, G) \cong \pi_1^E(F_G(M, n), \mathbf{x}, \mathbf{x}) = \pi_1(F_G(M, n), \mathbf{x}) \).

**Remark:** The above viewpoint can also be used in the theory of ordinary braids. Consider the case in which \( G = \{ e \} \). Then \( \mathcal{B}_n^{\text{orb}}(M, G) \) degenerates into the ordinary braid group \( \mathcal{B}_n(M) \), which is isomorphic to the extended fundamental group \( \pi_1^E(F(M, n), \mathbf{x}, \Sigma_n(\mathbf{x})) \) of \( F(M, n) \) at \( \Sigma_n(\mathbf{x}) \). There is the following short exact sequence

\[
1 \longrightarrow \pi_1(F(M, n), \mathbf{x}) \longrightarrow \pi_1^E(F(M, n), \mathbf{x}, \Sigma_n(\mathbf{x})) \longrightarrow \Sigma_n \longrightarrow 1
\]

from which we see that \( \pi_1^E(F(M, n), \mathbf{x}, \Sigma_n(\mathbf{x})) \) is actually the fundamental group of the unordered configuration space \( F(M, n)/\Sigma_n \). However, the case of \( G \neq \{ e \} \) will be quite different.
Five short exact sequences

Theorem

\[\begin{array}{ccc}
\mathcal{P}_n^\text{orb}(M, G) & \xrightarrow{\Phi_G} & G \times_n \\
\downarrow & & \downarrow \\
\mathcal{P}_n(M, G) & \xrightarrow{\Phi} & G \times_n \rtimes_{\varphi} \Sigma_n \\
\downarrow & & \downarrow \\
\mathcal{B}_n(M, G) & \xrightarrow{\Phi_\Sigma} & \Sigma_n \\
\downarrow & & \downarrow \\
\mathcal{B}_n(M, G) & \xrightarrow{\Phi_\Sigma} & \Sigma_n \\
\downarrow & & \downarrow \\
\mathcal{B}_n(M, G) & \xrightarrow{\Phi_\Sigma} & \Sigma_n \\
\downarrow & & \downarrow \\
\mathcal{P}_n(M, G) & \xrightarrow{\Phi} & G \times_n \rtimes_{\varphi} \Sigma_n \\
\downarrow & & \downarrow \\
1 & \xrightarrow{1} & 1
\end{array}\]
Main point of proof

Let $\varphi : \Sigma_n \rightarrow \text{Aut}(G^{\times n})$ be a homomorphism defined by

$$\varphi(\sigma)(g) = g_{\sigma} = (g_{\sigma(1)}, \ldots, g_{\sigma(n)})$$

where $\sigma \in \Sigma_n$ and $g = (g_1, \ldots, g_n) \in G^{\times n}$. Then $\varphi$ gives a semidirect product $G^{\times n} \rtimes_{\varphi} \Sigma_n$, where the operation $\cdot$ on $G^{\times n} \rtimes_{\varphi} \Sigma_n$ is given by

$$(g, \sigma) \cdot (h, \tau) = (gh_{\sigma}, \sigma \tau)$$

for $(g, \sigma), (h, \tau) \in G^{\times n} \rtimes_{\varphi} \Sigma_n$. 
Main point of proof (continued)

Define a homomorphism

\[ \Phi : \mathcal{B}_n^{orb}(M, G) \rightarrow G^{\times n} \rtimes \varphi \Sigma_n \]

by \( \Phi([\widetilde{c(\alpha)}]) = (g, \sigma) \), where \((g, \sigma)\) is the unique pair determined by \([c(\alpha)]\).

**Lemma**

The homomorphism \( \Phi : \mathcal{B}_n^{orb}(M, G) \rightarrow G^{\times n} \rtimes \varphi \Sigma_n \) is an epimorphism.
Two typical actions on $\mathbb{C}$

The geometric presentation of classical braid group $B_n(\mathbb{R}^2)$ in $\mathbb{R}^2 \times I$ gives us much more insights to the case of orbit braid group. Thus we begin with our work from the case of $\mathbb{C} \approx \mathbb{R}^2$ with the following two typical actions:

(I) $\mathbb{Z}_p \bowtie \phi_1 \mathbb{C}$ defined by $(e^{\frac{2k\pi i}{p}}, z) \mapsto e^{\frac{2k\pi i}{p}} z$, which is non-free and fixes only the origin of $\mathbb{C}$, where $p$ is a prime, and $\mathbb{Z}_p$ is regarded as the group $\{e^{\frac{2k\pi i}{p}} | 0 \leq k < p\}$. If the action $\phi_1$ is restricted to $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, then the action $\mathbb{Z}_p \bowtie \phi_1 \mathbb{C}^\times$ is free.

(II) $(\mathbb{Z}_2)^2 \bowtie \phi_2 \mathbb{C}$ defined by

$$
\begin{align*}
&\begin{cases}
  z \mapsto \bar{z} \\
  z \mapsto -\bar{z}.
\end{cases}
\end{align*}
$$
Orbit braid group $\mathcal{B}_{n}^{\text{orb}}(\mathbb{C}, \mathbb{Z}_p)$ of $F_{\mathbb{Z}_p}(\mathbb{C}, n)$

**Proposition**

$\mathcal{B}_{n}^{\text{orb}}(\mathbb{C}, \mathbb{Z}_p)$ is generated by $b_k$ ($1 \leq k \leq n - 1$) and $b$, with relations

1. $b^p = e$;
2. $(bb_1)^p = (b_1b)^p$;
3. $b_kb = bb_k$ ($k > 1$);
4. $b_kb_{k+1}b_k = b_{k+1}b_kb_{k+1}$;
5. $b_kb_l = b_lb_k$ ($|k - l| > 1$).

where $b_k = [\hat{c}(\alpha^{(k)})]$ for $1 \leq k \leq n - 1$ and $b = [\hat{c}(\beta)]$ given by

$$\alpha^{(k)}(s) = (1 + i, \ldots, k + (k + 1)i + e^{\frac{s}{2}i(1-s)}, (k + 1) + ki + ie^{\frac{s}{2}i}, \ldots, n + ni)$$

$$\beta(s) = ((1 + i)e^{\frac{2\pi is}{p}}, 2 + 2i, \ldots, n + ni).$$
Orbit braid group $B_{n}^{\text{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ of $F_{\mathbb{Z}_p}(\mathbb{C}^\times, n)$

**Proposition**

$B_{n}^{\text{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ is generated by $b_k$ ($1 \leq k \leq n - 1$) and $b'$, with relations:

1. $(b'b_1)^p = (b_1b')^p$;
2. $b_kb' = b'b_k$ ($k > 1$);
3. $b_kb_{k+1}b_k = b_{k+1}b_kb_{k+1}$;
4. $b_kb_l = b_lb_k$ ($|k - l| > 1$).

where $b_k = [\tilde{c}(\alpha^{(k)})]$ for $1 \leq k \leq n - 1$ and $b' = [\tilde{c}(\beta)]$ given by

$$
\alpha^{(k)}(s) = (1 + i, \ldots, k + (k + 1)i + e^{-\frac{2\pi i}{p}(1-s)}, (k + 1) + ki + ie^{\frac{2\pi i}{p}s}, \ldots, n + ni)
$$

$$
\beta(s) = ((1 + i)e^{\frac{2\pi is}{p}}, 2 + 2i, \ldots, n + ni).
$$
Proposition

$\mathcal{B}_n^{\text{orb}}(\mathbb{C}, \mathbb{Z}_2^2)$ is generated by $b_k$ ($1 \leq k \leq n - 1$), $b^x$ and $b^y$ with relations

1. $(b^x)^2 = (b^y)^2 = e$;
2. $b^x b^y = b^y b^x$;
3. $b^x b_1 b^x b_1 = b_1 b^x b_1 b^x$, $b^y b_1 b^y b_1 = b_1 b^y b_1 b^y$;
4. $b_k b^x = b^x b_k$, $b_k b^y = b^y b_k$ ($k > 1$);
5. $b_k b_{k+1} b_k = b_{k+1} b_k b_{k+1}$;
6. $b_k b_l = b_l b_k$ ($|k - l| > 1$).
Generators of orbit braid group $B_{n}^{\text{orb}}(\mathbb{C}, \mathbb{Z}_2^2)$

1. $b_k$ is chosen as $[c(\alpha^{(k)})]$ where
$$\alpha^{(k)}(s) = (1 + i, \ldots, k + (k + 1)i + e^{-\frac{\pi}{2}i(1-s)}, (k + 1) + ki + ie^\frac{\pi}{2}is, \ldots, n + ni);$$

2. $b^x$ is chosen as $[c(\alpha^x)]$ where $\alpha^x$ is the path given by
$$\alpha^x(s) = (1 + (1 - 2s)i, 2 + 2i, \ldots, n + ni)$$

such that $\alpha^x_1$ and $\overline{\alpha^x_1}$ intersect at $x$-axis $\times l$;

3. $b^y$ is chosen as $[c(\alpha^y)]$ where $\alpha^y$ is the path given by
$$\alpha^y(s) = ((1 - 2s) + i, 2 + 2i, \ldots, n + ni)$$

such that $\alpha^y_1$ and $-\overline{\alpha^y_1}$ intersect at $y$-axis $\times l$. 
Relation with generalized braid group

- It is known from Goryunov's work: two orbit configuration spaces $F_{\mathbb{Z}_2}(\mathbb{C}, n)$ and $F_{\mathbb{Z}_2}((\mathbb{C}^\times, n)$ are classifying spaces of two generalised pure braid groups $P(D_n)$ and $P(B_n)$.
- In the viewpoint of Brieskorn, $F_{\mathbb{Z}_2}(\mathbb{C}, n) = Y_{D_n}$ so
  $$1 \rightarrow P(D_n) \rightarrow Br(D_n) \rightarrow D_n \rightarrow 1$$
  and $F_{\mathbb{Z}_2}((\mathbb{C}^\times, n) = Y_{B_n}$ so
  $$1 \rightarrow P(B_n) \rightarrow Br(B_n) \rightarrow B_n \rightarrow 1$$
- In our viewpoint, there are
  $$1 \rightarrow \mathcal{P}_n(\mathbb{C}, \mathbb{Z}_2) \rightarrow \mathcal{B}_n^{orb}(\mathbb{C}, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2^n \rtimes_\varphi \Sigma_n \rightarrow 1$$
  $$1 \rightarrow \mathcal{P}_n((\mathbb{C}^\times, \mathbb{Z}_2) \rightarrow \mathcal{B}_n^{orb}((\mathbb{C}^\times, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2^n \rtimes_\varphi \Sigma_n \rightarrow 1$$
  It can be checked that $\mathbb{Z}_2^n \rtimes_\varphi \Sigma_n \cong B_n$
Relation with generalized braid group

For the case of $F_{\mathbb{Z}_2}(\mathbb{C}^\times, n)$, two viewpoints are identical. In this case, $F_{\mathbb{Z}_2}(\mathbb{C}^\times, n) = Y_{B_n}$, so that

$$\text{Br}(B_n) \cong B_n^{\text{orb}}(\mathbb{C}^\times, \mathbb{Z}_2)$$

For the case of $F_{\mathbb{Z}_2}(\mathbb{C}, n)$, two viewpoints are not the same. However,

$Br(D_n)$ is isomorphic to a subgroup of $B_n^{\text{orb}}(\mathbb{C}, \mathbb{Z}_2)$. 
Thank You!