Adiabatic limits, Theta functions, and Geometric Quantization

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Purpose & Main Theorems

Geometric quantization

Geometric quantization $\cdots$ a procedure to construct a Hilbert space (and a representation of $(C^\infty(M), \{\ ,\ \})$) from the given symplectic manifold (M, ω) in the geometric way

Classical mechanics		**Quantum mechanics**
(M, ω)	$\longrightarrow$	$Q(M, \omega)$: Hilbert space
$f \in C^\infty(M)$	$\longrightarrow$	$Q(f)$: operator on $Q(M, \omega)$

Q satisfies $Q(\{f, g\}) = \frac{2\pi\sqrt{-1}}{h}\{Q(f)Q(g) - Q(g)Q(f)\}$

Example (Canonical quantization)

$$\left(\mathbb{R}^{2n}, \omega_0 := \sum_{i=1}^{n} dp_i \wedge dq_i\right) \longrightarrow Q(\mathbb{R}^{2n}, \omega_0) := L^2(\mathbb{R}^n_q)$$

$$p_i, q_i \in C^\infty(\mathbb{R}^{2n}) \longrightarrow \begin{cases} Q(p_i) := \frac{h}{2\pi\sqrt{-1}}\frac{\partial}{\partial q_i} \\ Q(q_i) := q_i \times \end{cases}$$

(M, ω) closed symplectic manifold

(L, ∇^L) prequantum line bundle $\overset{\text{def}}{\Leftrightarrow} \begin{cases} L \to M \text{ Hermitian line bundle} \\ \nabla^L \text{ connection of } L \text{ with } \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega \end{cases}$

In the Kostant-Souriau theory, to obtain the quantum Hilbert space $Q(M, \omega)$, we need a polarization.

Definition

A polarization $\mathcal{P}$ is an integrable Lagrangian distribution of $TM \otimes \mathbb{C}$.

- Let $\mathcal{S}$ be the sheaf of germs of covariant constant sections of L along $\mathcal{P}$.

When a polarization $\mathcal{P}$ is given, $Q(M, \omega)$ is naively defined by

Definition

$$Q(M, \omega) := H^0(M; \mathcal{S})$$

Example (Kähler quantization)

(M, ω, J) closed Kähler manifold

(L, h, ∇^L) holomorphic Hermitian line bundle with Chern connection

$\Rightarrow$ $T^{0,1}M$ can be taken to be a polarization $\mathcal{P}$.

Definition

$$Q_{Kähler}(M, \omega) := H^0(M; \mathcal{O}_L)$$

- When the Kodaira vanishing holds, $\dim Q_{Kähler}(M, \omega) =$ index of the Dolbeault operator with coefficients in L.

Example (Real quantization)

$$\pi\colon (M^{2n},\omega) \to B^n \text{ Lagrangian fibration} \overset{\text{def}}{\Leftrightarrow} \begin{cases} \pi\colon \text{fiber bundle} \\ \omega|_{\text{fiber}} \equiv 0 \\ \dim\ \text{fiber} = \frac{1}{2}\dim M \end{cases}$$

Example

$$\pi_0\colon (\mathbb{R}^n \times T^n, \omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i) \to \mathbb{R}^n, \quad \pi_0(x,y) = x$$

Theorem (Arnold-Liouville)

Any Lagrangian fibration with compact, path-connected fibers is locally isomorphic to $\pi_0\colon (\mathbb{R}^n \times T^n, \omega_0) \to \mathbb{R}^n$.

- We assume a fiber is compact and path-connected. $\Rightarrow$ the fiber is T^n.
- B admits an integral affine structure (i.e., an atlas with integral affine transition maps)

Example (Real quantization) continued

$(L, \nabla^L) \to (M, \omega) \xrightarrow{\pi} B$ Lagrangian fibration with prequantum line bundle

- $(L, \nabla^L)|_{\pi^{-1}(b)}$ is a flat bundle for $\forall b \in B$.

Definition (Bohr-Sommerfeld (BS) point)

$b \in B$ is *Bohr-Sommerfeld* $\overset{\text{def}}{\Leftrightarrow} \left\{ s \in \Gamma(L|_{\pi^{-1}(b)}) \mid \nabla^L s = 0 \right\} \neq \{0\}$

- BS points appear discretely.
- We denote by B_{BS} the set of BS points

Example (Local model)

$$\left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}\sum_{i=1}^{n} x_i dy_i \right) \to (\mathbb{R}^n \times T^n, \omega_0) \xrightarrow{\pi_0} \mathbb{R}^n \;\therefore\; \mathbb{R}^n_{BS} = \mathbb{Z}^n$$

Example (Real quantization) continued

$(L, \nabla^L) \to (M, \omega) \xrightarrow{\pi} B$ Lagrangian fibration with prequantum line bundle

$\Rightarrow$ The tangent bundle along the fiber $T_\pi M \otimes \mathbb{C}$ can be taken to be a polarization $\mathcal{P}$.

Assume (M, ω) is closed.

Theorem (Śniatycki)

$$H^q(M; \mathcal{S}) = \begin{cases} \oplus_{b \in B_{BS}} \left\{ s \in \Gamma(L|_{\pi^{-1}(b)}) \mid \nabla^L s = 0 \right\} & \textit{if } q = \frac{\dim_{\mathbb{R}} M}{2} \\ 0 & \textit{if } q : \textit{otherwise} \end{cases}$$

Definition (Real quantization)

$$Q_{real}(M, \omega) := \oplus_{b \in B_{BS}} \left\{ s \in \Gamma(L|_{\pi^{-1}(b)}) \mid \nabla^L s = 0 \right\}$$

Does $Q(M,\omega)$ depend on a choice of polarization?

Question

$$Q_{Kähler}(M,\omega) \cong Q_{real}(M,\omega) \ ?$$

- Several examples show it is true at least for dimension:
  - the moment map μ of a toric manifold (Danilov '78),

    $$\dim H^0(M;\mathcal{O}_L) = \#\mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}} = \#\text{BS pts}$$

  - the Gelfand-Cetlin system on the complex flag manifold (Guillemin-Sternberg '83)
  - the Goldman system on the moduli space of flat $SU(2)$-bundles on a Riemann surface (Jeffrey-Weitsman '92)

$Q_{Kähler} \cong Q_{real}$ as a limit of deformation of complex structures

Theorem (Baier-Florentino-Muorão-Nunes ’11)

When M is a toric manifold, they give one-parameter families of

- $\{J^t\}_{t>0}$ *complex structures of M*
- $\{\sigma_m^t\}_{m\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}}$ *bases of holomorphic sections of* $L \to (M, J^t)$

such that for $\forall m \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}$, σ_m^t *converges to a delta-function section supported on* $\mu^{-1}(m)$ *as* $t \to \infty$ *in the following sense, for any section s of L,*

$$\lim_{t\to\infty} \int_M \left\langle s, \frac{\sigma_m^t}{\|\sigma_m^t\|_{L^1}} \right\rangle_L \frac{\omega^n}{n!} = \int_{\mu^{-1}(m)} \langle s, \delta_m \rangle_L \, d\theta_m.$$

- Similar results have been obtained (but only for non-singular fibers):
  - the Gelfand-Cetlin system on the complex flag manifold (Hamilton-Konno ’14)
  - smooth irreducible complex algebraic variety with certain assumptions (Hamilton-Harada-Kaveh ’16)

How about the non-Kähler case?

For a non-integrable J, we have several generalizations of the Kähler quantization. Among these is the Spin^c quantization.

Purpose

To generalize BFMN apporach to the Spin^c quantization.

Spin^c quantization – a generalization of the Kähler quantization

$(L, \nabla^L) \to (M, \omega)$ closed symplectic manifold with prequantum line bundle

$\Rightarrow$ By taking a compatible almost complex structure J, we can obtain the Spin^c Dirac operator

$$D \colon \Gamma\left(\wedge^\bullet (T^*M)^{0,1} \otimes L\right) \to \Gamma\left(\wedge^\bullet (T^*M)^{0,1} \otimes L\right).$$

- D is a 1^{st} order, formally self-adjoint, elliptic differential operator.

Definition (Spin^c quantization)

$$Q_{Spin^c}(M, \omega) := \ker(D|_{\wedge^{0,even}}) - \ker(D|_{\wedge^{0,odd}}) \in K(pt) \cong \mathbb{Z}$$

- $\dim Q_{Spin^c}(M, \omega) = \operatorname{ind} D$ depends only on ω and does not depend on the choice of J and ∇^L.
- If (M, ω, J) is Kähler (hence, (L, ∇^L) is holomorphic with Chern connection), then $D = \sqrt{2}(\bar{\partial} \otimes L + \bar{\partial}^* \otimes L)$ and

$$\operatorname{ind} D = \sum_{q \geq 0} (-1)^q \dim H^q(M, \mathcal{O}_L).$$

Deformation of almost complex structure

$\pi \colon (M, \omega) \to B$: Lagrangian fibration

J: compatible almost complex structure of (M, ω)

$\Rightarrow \; TM = JT_\pi M \oplus T_\pi M \quad (T_\pi M$: tangent bundle along the fiber of $\pi)$

Definition

For each $t > 0$, define J^t by

$$J^t v := \begin{cases} \frac{1}{t} Jv & \text{if } v \in T_\pi M \\ tJv & \text{if } v \in JT_\pi M. \end{cases}$$

- J^t is still a compatible almost complex structure of (M, ω).
- Assume J is invariant along the fiber of π. Then,

$$J\text{: integrable} \; \Leftrightarrow \; J^t\text{: integrable } \forall t > 0$$

- As $t \to +\infty$, $T_\pi M$ becomes smaller and $JT_\pi M$ becomes larger with respect to $g^t := \omega(\cdot, J^t \cdot)$. (adiabatic-type limit)
- For each $t > 0$, we denote by D^t the Dirac operator with respect to J^t.

Main Theorem

$(L, \nabla^L) \to (M, \omega) \xrightarrow{\pi} B$: Lagrangian fibration with prequantum line bundle

J: compatible almost complex structure of (M, ω) invariant along the fiber of π

$\{J^t\}_{t>0}$: the deformation of J defined as above

Theorem (Y '19)

Assume M is closed and B is complete (i.e., $\tilde{B} \cong \mathbb{R}^n$). For the given data and for each $t > 0$, we give orthogonal sections $\{\vartheta^t_m\}_{m \in B_{BS}}$ on L indexed by B_{BS} such that

1. *each ϑ^t_m converges to a delta-function section supported on $\pi^{-1}(m)$ as $t \to \infty$ in the following sense, for any section s of L,*

$$\lim_{t\to\infty} \int_M \left\langle s, \frac{\vartheta^t_m}{\|\vartheta^t_m\|_{L^1}} \right\rangle_L \frac{\omega^n}{n!} = \int_{\pi^{-1}(m)} \langle s, \delta_m \rangle_L \, |dy|.$$

2. $\lim_{t\to\infty} \|D^t \vartheta^t_m\|_{L^2} = 0.$

Moreover, if J is integrable, then, with a technical assumption, we can take $\{\vartheta^t_m\}_{m \in B_{BS}}$ to be an orthogonal basis of holomorphic sections of $L \to (M, \omega, J^t)$.

Corollary

When $\pi = \mathrm{p}_1 : M = T^n \times T^n \to B = T^n$,

$$\vartheta_m(x, y) = e^{\pi\sqrt{-1}(-m\cdot\Omega m + x\cdot\Omega x)}\vartheta \begin{bmatrix} m \\ 0 \end{bmatrix} (-\Omega x + y, \Omega) \, .$$

Construction of ϑ_m^t

$(L, \nabla^L) \to (M, \omega) \overset{\pi}{\to} B$ Lagrangian fibration with prequantum line bundle

Key lemma1

If B is complete, then, the pull-back of $(L, \nabla^L) \to (M, \omega) \overset{\pi}{\to} B$ to $\tilde{B}$ is identified with

$$(\tilde{L}, \nabla^{\tilde{L}}) := \left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}\sum_{i=1}^{n} x_i dy_i \right) \to (\mathbb{R}^n \times T^n, \omega_0) \overset{\pi_0}{\to} \mathbb{R}^n.$$

In paticular, $(L, \nabla^L) \to (M, \omega) \overset{\pi}{\to} B$ is obtained as the quotient of this standard model by the $\pi_1(B)$-action.

Compatible almost complex structures

Let $\mathcal{S}_n$ be the Siegel upper half space

$$\mathcal{S}_n := \{Z = X + \sqrt{-1}Y \in M_n(\mathbb{C}) \mid X, Y \in M_n(\mathbb{R}), {}^tZ = Z, Y > 0\}.$$

Lemma

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n \times T^n, \mathcal{S}_n) & \overset{1:1}{\longleftrightarrow} & \{\textit{comp. almost cpx str. on } (\mathbb{R}^n \times T^n, \omega_0)\} \\ \Cup & & \Cup \\ Z = X + \sqrt{-1}Y & \longmapsto & \tilde{J} := \begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix} \end{array}$$

- J on $(M, \omega) \Leftrightarrow \pi_1(B)$-equiv. $\tilde{J}$ on $(\mathbb{R}^n \times T^n, \omega_0)$

Lemma

For any $\pi : (M, \omega) \to B$, there exists J of (M, ω) s.t. the pull-back of J to $(\mathbb{R}^n \times T^n, \omega_0)$ is invariant under the natural T^n-action.

- We assume such a condition on J. $\Rightarrow Z_{\tilde{J}} \in C^\infty(\mathbb{R}^n, \mathcal{S}_n)$.

Dirac operator on $(\mathbb{R}^n \times T^n, \omega_0)$

Let

$$\tilde{D}\colon \Gamma\left(\wedge^\bullet T^*(\mathbb{R}^n \times T^n)^{0,1} \otimes \tilde{L}\right) \to \Gamma\left(\wedge^\bullet T^*(\mathbb{R}^n \times T^n)^{0,1} \otimes \tilde{L}\right)$$

be the Spin^c Dirac operator associated with a $\pi_1(B)$-equivariant $\tilde{J}$ on $(\mathbb{R}^n \times T^n, \omega_0)$ corresponding to $Z = X + \sqrt{-1}Y$.

Lemma

For $s = \sum_{m\in\mathbb{Z}^n} a_m(x)e^{2\pi\sqrt{-1}m\cdot y} \in \Gamma\left((\mathbb{R}^n \times T^n \times \mathbb{C})\right)$,

$$0 = \tilde{D}s \iff 0 = \begin{pmatrix} \partial_{x_1} a_m \\ \vdots \\ \partial_{x_n} a_m \end{pmatrix} + 2\pi\sqrt{-1}a_m\Omega(m - x) \quad \forall m \in \mathbb{Z}^n, \tag{1}$$

where

$$\Omega := (Y + XY^{-1}X)^{-1}ZY^{-1} \in C^\infty(\mathbb{R}^n, \mathcal{S}_n).$$

Key lemma 2

$$0 = \begin{pmatrix} \partial_{x_1} a_m \\ \vdots \\ \partial_{x_n} a_m \end{pmatrix} + 2\pi\sqrt{-1} a_m \Omega(m - x) \quad \forall m \in \mathbb{Z}^n. \tag{1}$$

Key lemma2

The following conditions are equivalent:

- (1) *has a non-trivial solution* a_m *for* $\forall m \in \mathbb{Z}^n$.
- $\partial_{x_i} \Omega_{jk} = \partial_{x_j} \Omega_{ik} \ \forall i, j, k = 1, \dots, n$
- *J is integrable.*

Moreover, in this case, the solution of (1) *is*

$$a_m(x) = a_m(0) \exp \left\{ -2\pi\sqrt{-1} \sum_{i=1}^{n} \int_0^{x_i} \sum_{j=1}^{n} \Omega_{ij}(m_j - x_j) dx_i \Big|_{x_1 = \cdots = x_{i-1} = 0} \right\}.$$

Integrable case

When J is integrable, for $\forall m \in F \cap \mathbb{Z}^n \cong B_{BS}$, define $s_m \in \Gamma\left((\mathbb{R}^n \times T^n \times \mathbb{C})\right)$ by

$$s_m(x, y) := \exp 2\pi\sqrt{-1}\left\{-\sum_{i=1}^{n}\int_0^{x_i}\sum_{j=1}^{n}\Omega_{ij}(m_j - x_j)dx_i\Big|_{x_1=\cdots=x_{i-1}=0} + m \cdot y\right\}.$$

Definition

For $\forall m \in B_{BS}$, define $\vartheta_m \in \Gamma\left((\mathbb{R}^n \times T^n \times \mathbb{C})\right)^{\pi_1(B)} \cong \Gamma(L)$ by

$$\vartheta_m(x, y) := \sum_{\gamma \in \pi_1(B)} \tilde{\tilde{\rho}}_\gamma \circ s_m \circ \tilde{\rho}_{\gamma^{-1}}(x, y),$$

where $\tilde{\rho}$, $\tilde{\tilde{\rho}}$ are the $\pi_1(B)$-actions on $\mathbb{R}^n \times T^n$, $\mathbb{R}^n \times T^n \times \mathbb{C}$, respectively.

Theorem

1. *If $Y + XY^{-1}X$ is constant, then, all ϑ_m's converge absolutely and uniformly on M.*
2. *If all ϑ_m's converge absolutely and uniformly on M, $\{\vartheta_m\}_{m \in B_{BS}}$ is an orthogonal basis of the space of holomorphic sections of $L \to (M, \omega, J)$.*

Non integrable case

When J is not integrable,

$$0 = \tilde{D}s \iff 0 = \begin{pmatrix} \partial_{x_1} a_m \\ \vdots \\ \partial_{x_n} a_m \end{pmatrix} + 2\pi\sqrt{-1} a_m \Omega(m - x) \quad \forall m \in \mathbb{Z}^n \tag{1}$$

has no solution. But, for each $m \in \mathbb{Z}^n$, the approxiamation

$$0 = \begin{pmatrix} \partial_{x_1} a_m \\ \vdots \\ \partial_{x_n} a_m \end{pmatrix} + 2\pi\sqrt{-1} a_m \Omega(m)(m - x) \tag{2}$$

has the following solution

$$s'_m(x, y) := e^{2\pi\sqrt{-1}N\left\{\frac{1}{2}(x-m)\cdot\Omega(m)(x-m)+m\cdot y\right\}},$$

where Ω is replaced by $\Omega(m)$, the value of Ω at m.

Definition

For $\forall m \in B_{BS}$, define $\vartheta_m \in \Gamma\left((\mathbb{R}^n \times T^n \times \mathbb{C})\right)^{\pi_1(B)} \cong \Gamma(L)$ by

$$\vartheta_m(x, y) := \sum_{\gamma \in \pi_1(B)} \tilde{\tilde{\rho}}_\gamma \circ s'_m \circ \tilde{\rho}_{\gamma^{-1}}(x, y).$$

Non integrable continued

Proposition

1. ϑ_m *converges absolutely and uniformly on* M.
2. $\{\vartheta_m\}_{m \in B_{BS}}$ *is an orthogonal family of the sections of L.*

Thank you for your attention!