A monomial basis for the cohomology rings of regular nilpotent Hessenberg varieties

Tatsuya Horiguchi
joint work with
Megumi Harada, Satoshi Murai, Martha Precup,
and Julianna Tymoczko

Osaka University

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Schubert calculus

$$F\ell(\mathbb{C}^n) := \{(V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i,\ 1 \leq i \leq n\}$$

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$\Omega_w := \overline{\Omega_w^\circ}$:**(opposite) Schubert variety**

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Problem of Schubert calculus

Calculate the intersection number among "Schubert varieties".

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Problem of Schubert calculus

Calculate the structure constants c_{uv}^w.

Schubert polynomial

It is well known that

$$H^*(F\ell(\mathbb{C}^n);\mathbb{Z}) \cong \mathbb{Z}[x_1,\dots,x_n]/(e_1,\dots,e_n)$$

e_i: the i-th elementary symmetric polynomial

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Monomial basis

The Schubert polynomials $\mathfrak{S}_w$ are inductively defined by using divided difference operators.

$$\mathfrak{S}_w = \sum_{\substack{I=(i_1,i_2,\ldots,i_n) \\ 0 \le i_j \le n-j}} a_I \; x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \quad (w \in S_n)$$

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Hessenberg variety

$\mathrm{Hess}(X,h) \subset F\ell(\mathbb{C}^n)$: Hessenberg variety
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- Particular examples:

· Springer fiber $\cdots$ geometric representation of S_n
· Peterson variety $\cdots$ quantum cohomology of $F\ell(\mathbb{C}^n)$
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- More related topics:

· Stanley's chromatic symmetric function in graph theory
· logarithmic derivation module of hyperplane arrangements

Hessenberg variety

$$h : [n] \to [n] \text{ Hessenberg ft.} \quad :\Longleftrightarrow \quad \begin{array}{l} h(1) \le h(2) \le \ldots \le h(n) \\ h(j) \ge j \ (j = 1, 2, \ldots, n) \end{array}$$

$$([n] := \{1, 2, \ldots, n\})$$

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$X : \mathbb{C}^n \to \mathbb{C}^n$ linear map.
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Then, Hessenberg variety $\mathrm{Hess}(X, h)$ is defined as follows:

$$\{(V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid XV_j \subset V_{h(j)} \; (j = 1, 2, \ldots, n)\}$$

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$h(j) = n$ for all $j = 1, 2, \ldots, n \Rightarrow \mathrm{Hess}(X, h) = F\ell(\mathbb{C}^n)$.

regular nilpotent Hessenberg variety

In this talk, we take X as

$$N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad : \text{regular nilpotent}$$

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Properties

- $\mathrm{Hess}(N, h)$ is irreducible
- $\mathrm{Hess}(N, h)$ is singular in general
- $\dim_{\mathbb{C}} \mathrm{Hess}(N, h) = \sum_{j=1}^{n} (h(j) - j)$

Cohomology

We define a polynomial

$$f_{i,j} := \sum_{k=1}^{j} \Big(\prod_{\ell=j+1}^{i} (x_k - x_\ell) \Big) x_k \qquad 1 \leq j \leq i \leq n$$

where $\prod_{\ell=j+1}^{i}(x_k - x_\ell) = 1$ whenever $i = j$.

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Theorem [Abe-Harada-H-Masuda]

$H^*(\mathrm{Hess}(N,h);\mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_n]/(f_{h(1),1}, f_{h(2),2}, \dots, f_{h(n),n})$

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Corollary [Abe-Harada-H-Masuda]

$H^*(\mathrm{Hess}(N,h);\mathbb{Q})$ is a Poincaré duality algebra.

Main Theorem

There are two types of bases of the cohomology $H^*(F\ell(\mathbb{C}^n);\mathbb{Z})$:

(1) Schubert classes $\{\sigma_w \mid w \in S_n\}$ $(\sigma_w = [\overline{\Omega_w^\circ}])$

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$\{\sigma_w^h\}$ forms a basis of the cohomology $H^*(\mathrm{Hess}(N,h);\mathbb{Q})$
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The set $\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \mid 0 \leq i_j \leq h(j)-j\}$ forms a basis of $H^*(\mathrm{Hess}(N,h);\mathbb{Q})$.

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