Lagrangian Fibrations on Grassmannians and Cluster Transformations

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Introduction

For each triangulation of a convex n-gon, one can associate

(1) a completely integrable system on $\mathrm{Gr}(2,n) = \mathrm{Gr}(2,\mathbb{C}^n)$
(N.-Ueda)

(2) a toric degeneration of $\mathrm{Gr}(2,n)$
(Speyer-Sturmfels)

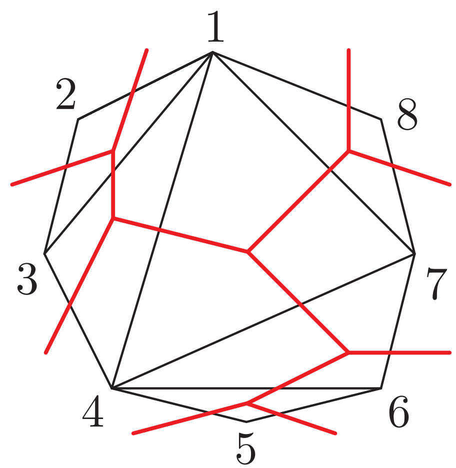
(3) a cluster chart on the Landau-Ginzburg mirror of $\mathrm{Gr}(2,n)$
(Fomin-Zelevinsky)

"Theorem". (1) and (3) are SYZ mirror.

Triangulations

For a triangulation of a convex n-gon

($\underset{\text{dual graph}}{\longleftrightarrow}$ a trivalent tree with n leaves), set



$$\Gamma^{\text{diag}} := \text{the set of diagonals } (i,j),$$
$$\Gamma^{\text{side}} := \{(i, i+1) \mid i = 1, \ldots, n\},$$
$$\Gamma := \Gamma^{\text{side}} \cup \Gamma^{\text{diag}},$$

where $(n, n+1) := (1, n)$.

Remark. $\#\Gamma^{\text{side}} = n$, $\#\Gamma^{\text{diag}} = n - 3$, and

$$\#\Gamma = (2n-4) + 1 = \dim_{\mathbb{C}} \mathrm{Gr}(2, n) + 1.$$

Lagrangian torus fibrations on the Grassmannian $\mathrm{Gr}(2,n)$

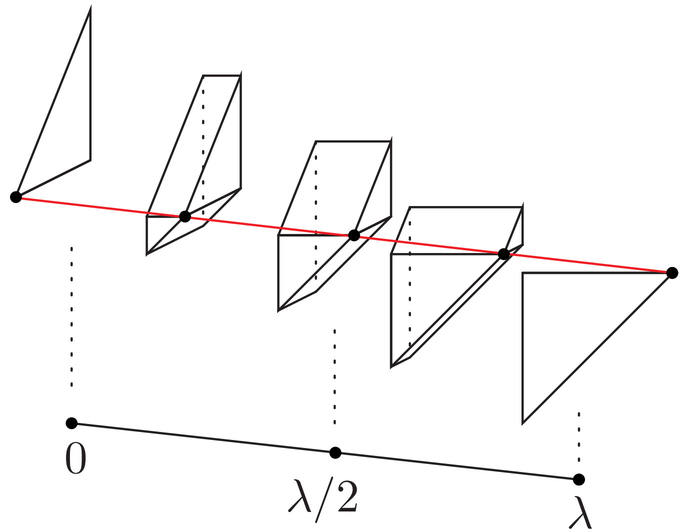
Theorem (N.-Ueda). *For each triangulation* Γ *of a convex* n*-gon, one can associate a completely integrable system*

$$\Phi_\Gamma = (\varphi_{ij})_{(i,j)\in\Gamma\setminus\{(1,n)\}} : \mathrm{Gr}(2,n) \to \mathbb{R}^N,$$

where $N = \dim_{\mathbb{C}} \mathrm{Gr}(2,n) = 2n-4$. *The image* $\Delta_\Gamma = \Phi_\Gamma(\mathrm{Gr}(2,n))$ *is an* N*-dimensional convex polytope.*

Example ($n = 4$). $\Phi_\Gamma : \mathrm{Gr}(2,4) \to \mathbb{R}^4$ coincides with the Gelfand-Cetlin system for each Γ (up to $U(4)$-action). Δ_Γ has an edge $\cong [0,\lambda]$ on which the fibers of Φ_Γ are non-torus Lagrangians:

$$L_t \cong \mathrm{U}(2) \cong S^1 \times S^3 \quad (0 < t < \lambda).$$



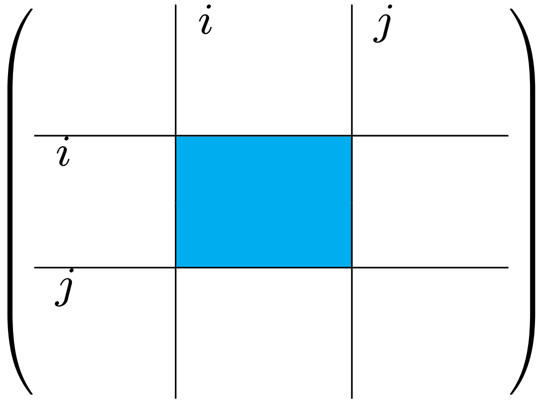
Construction of Φ_Γ: Gr$(2,n) \to \mathbb{R}^N$

Identify Gr$(2,n)$ with the adjoint orbit $\mathcal{O}_\lambda$ of diag$(\lambda,\lambda,0,\ldots,0)$ in $\sqrt{-1}\mathfrak{u}(n)$:

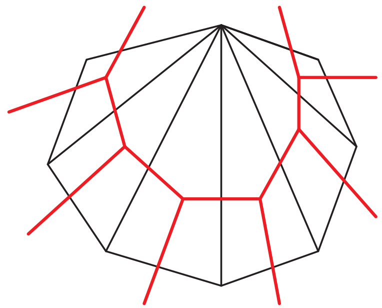
$$\mathcal{O}_\lambda = \{x \in \sqrt{-1}\mathfrak{u}(n) \mid \text{eigenval.} = \lambda,\lambda,0,\ldots,0\}.$$

For each $(i,j) \in \Gamma \setminus \{(1,n)\}$, define

$$\varphi_{ij}(x) = \text{max. eigenvalue of } (x_{kl})_{i \le k,l < j}.$$



Remark. If the triangulation Γ is "caterpillar", Φ_Γ is the Gelfand-Cetlin system (Guillemin-Sternberg).



Toric degenerations of Gr$(2, n)$

Theorem (Speyer-Sturmfels)**.**

$$\{\textit{toric degenerations of } \mathrm{Gr}(2,n)\}/\sim \;\overset{\textit{1-1}}{\longleftrightarrow}\; \{\textit{trivalent trees with } n \textit{ leaves } \Gamma\}.$$

Theorem (N.-Ueda)**.** *For each triangulation* Γ *of a convex* n*-gon, the central fiber* X_Γ *of the corresponding toric degeneration*

$$\begin{array}{ccc} \mathfrak{X}_\Gamma & \supset & X_\Gamma \\ \downarrow & & \downarrow \\ \mathbb{C} & \ni & 0 \end{array}$$

has moment polytope Δ_Γ*. Furthermore,* $\Phi_\Gamma \colon \mathrm{Gr}(2,n) \to \Delta_\Gamma$ *can be deformed into the toric moment map* $X_\Gamma \to \Delta_\Gamma$*.*

Mirror Symmetry for Fano Manifolds

X : N-dim. Fano $\longleftrightarrow$ $(X^\vee, W)$: Landau-Ginzburg model

$\begin{pmatrix} c_1(X) > 0 \\ \Leftrightarrow \text{Ric} > 0 \end{pmatrix}$ $\begin{pmatrix} X^\vee\text{: a non-cpt cpx mfd.} \\ W : X^\vee \to \mathbb{C} \text{ (or } \Lambda) \end{pmatrix}$

Symp. (resp. cpx.) geom. $\longleftrightarrow$ Cpx. (resp. symp.) geom.

"Classical" Mirror Symmetry

$$QH^*(X;\Lambda) \cong \text{Jac}(W) \text{``}=\text{''} \Lambda[y_1^{\pm 1}, \ldots, y_N^{\pm 1}] \Big/ \left(\frac{\partial W}{\partial y_1}, \ldots, \frac{\partial W}{\partial y_N} \right),$$

where $\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lambda_i \to \infty \right\}$ is the Novikov field.

Example. $(\mathbb{P}^1, \lambda\omega_{\text{FS}}) \overset{\text{mirror}}{\longleftrightarrow} \left(X^\vee = \mathbb{C}^\times, W(y) = y + \frac{Q}{y} \right)$, $\quad Q = T^\lambda$.

$$\text{Jac}(W) = \Lambda[y^{\pm}] \Big/ \left(1 - Q/y^2 \right) = \Lambda[y]/(y^2 - Q) \cong QH^*(\mathbb{P}^1;\Lambda)$$

cf. $H^*(\mathbb{P}^1;\Lambda) \cong \Lambda[y]/(y^2 = 0)$.

Strominger-Yau-Zaslow Conjecture

X and $X^\vee$ admit dual (special) Lagrangian torus fibrations

$$\begin{array}{ccc} X & & X^\vee \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle\Phi^\vee} & \\ & B & \end{array},$$

i.e., $(\Phi^\vee)^{-1}(\boldsymbol{u})$ = moduli of flat U(1)-connections on $L(\boldsymbol{u}) = \Phi^{-1}(\boldsymbol{u})$.

$$X^\vee = \text{moduli space of "Lagrangian branes"} \ (L(\boldsymbol{u}), \nabla).$$

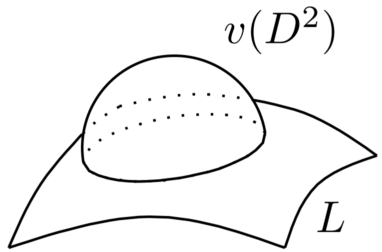
The superpotential W is given by the disk potential

$$\mathfrak{PO}(L,\nabla) = \sum_{\beta \in \pi_2(X,L)} n(\beta) z_\beta(L,\nabla),$$

$$z_\beta(L,\nabla) = T^{\omega(\beta)} \, \mathrm{hol}_\nabla(\partial\beta),$$

where

$$n(\beta) = \text{"\#"} \{ v \colon (D^2, \partial D^2) \overset{\text{hol.}}{\to} (X,L) \mid [v] = \beta \}$$
$$\mathrm{hol}_\nabla(\partial\beta) = \text{holonomy of } \nabla \text{ along } v(\partial D^2) \subset L.$$

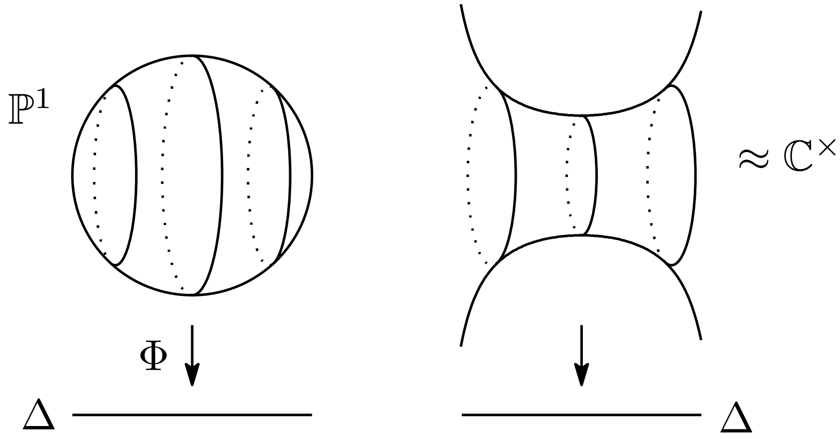


Example: SYZ mirror of $\mathbb{P}^1$

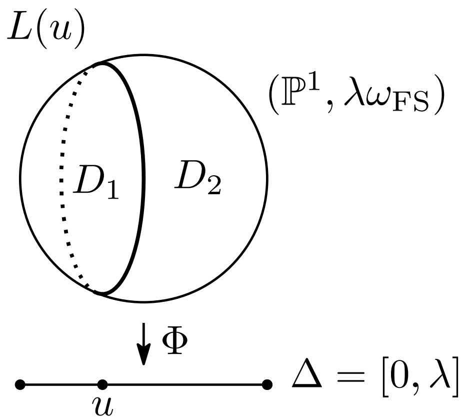
For a Lagrangian fiber $L(u) \subset \mathbb{P}^1$ of the moment map

$$\Phi : (\mathbb{P}^1, \lambda\omega_{\mathsf{FS}}) \to \Delta = [0, \lambda]$$

and a flat $U(1)$-connection
$\nabla = d + \sqrt{-1}x d\theta, \quad \theta \in \mathbb{R}/\mathbb{Z} \cong L(u)$,



$$\begin{aligned}\mathfrak{PO}(L(u), \nabla) &= \sum_{i=1}^{2} T^{\omega(D_i)} \,\mathrm{hol}_\nabla(\partial D_i) \\ &= T^u e^{\sqrt{-1}x} + T^{\lambda-u} e^{-\sqrt{-1}x} \\ &= y + \frac{Q}{y},\end{aligned}$$



where $y = T^u e^{\sqrt{-1}x}$, $Q = T^\lambda$.

Potential function for Lagrangian torus fibers $L(\boldsymbol{u})$

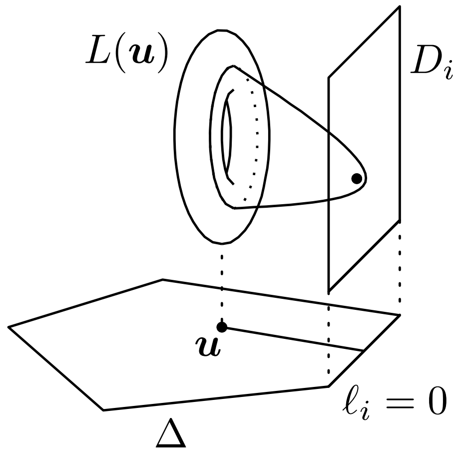
Theorem (Cho-Oh, Fukaya-Oh-Ohta-Ono). *Let X be a toric Fano manifold, and write its moment polytope as*

$$\Delta = \{\boldsymbol{u} \in \mathbb{R}^N \mid \ell_j(\boldsymbol{u}) = \langle \boldsymbol{v}_j, \boldsymbol{u} \rangle - \tau_j \geq 0, j = 1, \ldots, m\}$$

for primitive $\boldsymbol{v}_i \in \mathbb{Z}^N$. Then for a Lagrangian torus fibers $L(\boldsymbol{u})$ and a flat $U(1)$-connection $\nabla = d + \sqrt{-1}\sum_i x_i d\theta_i$ on $L(\boldsymbol{u})$,

$$\mathfrak{PO}(\boldsymbol{u}, \boldsymbol{x}) = \mathfrak{PO}(L(\boldsymbol{u}), \nabla) = \sum_{j=1}^{m} z_{\beta_j}(\boldsymbol{u}, \boldsymbol{x}),$$

$$z_{\beta_j}(\boldsymbol{u}, \boldsymbol{x}) = T^{\ell_j(\boldsymbol{u})} e^{\sqrt{-1}\langle \boldsymbol{v}_j, \boldsymbol{x} \rangle} = T^{-\tau_j} \boldsymbol{y}^{\boldsymbol{v}_j}.$$



Furthermore, $\mathfrak{PO}$ coincides with the superpotential of the LG mirror of X.

Landau-Ginzburg mirror of Grassmannians

Marsh-Rietsch: The LG mirror of $\mathrm{Gr}(k,n) = \mathrm{Gr}(k,\mathbb{C}^n)$ is given by

$$(X^\vee = \mathrm{Gr}(n-k,(\mathbb{C}^n)^*) \setminus D, W),$$

where D is an anti-canonical divisor of $\mathrm{Gr}(n-k,(\mathbb{C}^n)^*)$.

Remark. $X^\vee$ is a partial compactification of $(\mathbb{C}^\times)^N$, $N = \dim_{\mathbb{C}} \mathrm{Gr}(k,n)$.

$k = 2$ **case**: Identify $\mathrm{Gr}(n-2,(\mathbb{C}^n)^*) \cong \mathrm{Gr}(2,\mathbb{C}^n) = \mathrm{Gr}(2,n)$.

$$D = \bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} \{p_{i,i+1} = 0\},$$

$$W = \sum_{i=1}^{n-2} \frac{p_{i,i+2}}{p_{i,i+1}} + Q\frac{p_{1,n-1}}{p_{n-1,n}} + \frac{p_{2,n}}{p_{1,n}},$$

where $[p_{ij}]_{1 \le i < j \le n}$ are the Plücker coordinates on $\mathrm{Gr}(2,n) \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^n)$.

The disk potential for $\Phi_\Gamma \colon \mathrm{Gr}(2,n) \to \Delta_\Gamma$

Theorem (N.-Ueda). *For each triangulation Γ, the disk potential $\mathfrak{PO}_\Gamma$ for the Lagrangian torus fibers of Φ_Γ is given by the same formula as in the toric Fano case:*

$$\mathfrak{PO}_\Gamma(\boldsymbol{y}) = \sum_{i=1}^{m} T^{-\tau_j} \boldsymbol{y}^{\boldsymbol{v}_j}, \qquad \Delta_\Gamma = \{\boldsymbol{u} \mid \langle \boldsymbol{v}_j, \boldsymbol{u} \rangle - \tau_j \geq 0, \ \ j = 1, \dots, m\}.$$

which gives a rational function

$$\mathfrak{PO}_\Gamma \colon (\mathbb{C}^\times)^N = (\mathbb{C}^\times)^{\Gamma \setminus \{(1,n)\}} \longrightarrow \mathbb{C}.$$

Moreover, there exists an embedding $\iota_\Gamma \colon (\mathbb{C}^\times)^N \hookrightarrow X^\vee$ such that

$$\iota_\Gamma^* W = \mathfrak{PO}_\Gamma.$$

Idea of the proof. Deform Φ_Γ into a toric moment map, and count holomorphic disks in a (singular) toric variety.

Cluster charts on Gr$(2, n)$

A cluster algebra is a commutative ring generated by a "system" of cluster variables. Two sets of cluster variables $\boldsymbol{x} = \{x_1, \ldots, x_N\}$ and $\boldsymbol{x}' = (\boldsymbol{x} \setminus \{x_k\}) \cup \{x'_k\}$ are related by an cluster mutation

$$x_k x'_k = \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i$$

Cluster algebra structure on $\mathbb{C}[\mathrm{Gr}(2, n)]$

cluster var. $\{p_{ij}\}_{(i,j)\in\Gamma} = \{\underbrace{(p_{ij})_{(i,j)\in\Gamma^{\mathrm{diag}}}}_{\text{mutable}}, \underbrace{(p_{ij})_{(i,j)\in\Gamma^{\mathrm{side}}}}_{\text{frozen}}\}$,

cluster mutation = Plücker rel. $p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk}$.

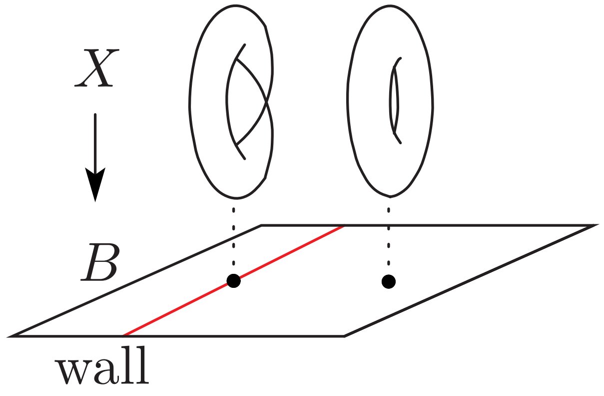
The "SYZ mirror" of $\Phi_\Gamma\, \mathrm{Gr}(2, n) \to \Delta_\Gamma$ gives a cluster chart

$$U_\Gamma := \iota_\Gamma((\mathbb{C}^\times)^N) = \{[p_{ij}] \in \mathrm{Gr}(2, n) \mid p_{ij} \neq 0,\ (i, j) \in \Gamma\}.$$

Wall-crossing formula for disk counting

If a Lagrangian torus fibration $\Phi\colon X \to B$ has a singular fiber, the weighted count $z_\beta(L, \nabla)$ of pseudo-holo. disks for Lag. fibers $L = \Phi^{-1}(\boldsymbol{u})$ changes when $\boldsymbol{u}$ crosses a wall

$$\{\boldsymbol{u} \in B \mid L \text{ bounds hol. disk of } \mu_L = 0\}.$$



Theorem (Auroux, Pascaleff-Tonkonog). *Assume that each fiber $L(\boldsymbol{u}_0)$ on the wall bounds a unique simple pseudo-holo. disk α of Maslov index 0. Then z_β's on chambers separated by the wall are related by*

$$z_\beta(L, \nabla) \mapsto z_\beta(L, \nabla)\Big(1 + \underbrace{\sum_{k \geq 1} a_k z_\alpha(L, \nabla)^k}_{\text{disk bubbles}}\Big)^{[\partial\beta]\cdot[\partial\alpha]}.$$

Wall-crossing formula and Plücker relations

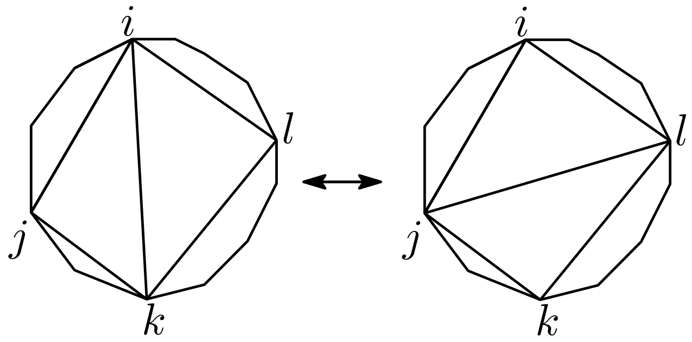
Theorem (N.-Ueda). *Suppose that* Γ *and* Γ' *are related by a flip. Then there exists a 1-parameter family* Φ_t $(0 \le t \le 1)$ *of Lagrangian fibrations on* $\mathrm{Gr}(2, n)$ *such that*

$$\Phi_0 = \Phi_\Gamma, \quad \Phi_1 = \Phi_{\Gamma'}.$$

For $0 < t < 1$*, the wall-crossing formula* $z_{\beta'} = z_\beta(1 + z_\alpha)$ *for* Φ_t *coincides with the Plücker relation*

$$\frac{p_{jl}}{p_{ij}} \cdot \frac{p_{i,i+1}}{p_{il}} = \frac{p_{kl}}{p_{ik}} \cdot \frac{p_{i,i+1}}{p_{il}}\left(1 + \underline{\underline{\frac{p_{il}p_{jk}}{p_{ij}p_{kl}}}}\right)$$

disk bubbles



Remarks

1. In general, $\bigcup_{\Gamma} U_\Gamma \subsetneq X^\vee$, and $X^\vee \setminus \bigcup_{\Gamma} U_\Gamma$ contains critical points of the superpotential W. The complement should corresponds to
   - non-torus Lagrangian fibers of Φ_Γ (N.-Ueda).
   - singular Lagrangian fibers of Φ_t (Hong-Kim-Lau).

2. [Scott] For $k \geq 3$, $\mathbb{C}[\mathrm{Gr}(k,n)]$ is a cluster algebra such that

$$\{\text{Plücker coordinates}\} \subsetneq \{\text{cluster variables}\}.$$

   - $^\exists$Lagrangian fibrations corresponding to cluster charts given by Plücker coord. (Castronovo).
   - wall-crossing on $\mathrm{Gr}(3,6)$ (In progress).