Higher Whitehead products in moment-angle complexes

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Conference "Toric Topology 2019 in Okayama"

Okayama, 18–22 November 2019

Whitehead products

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The higher Whitehead product $[f_1, \dots, f_m] \subseteq \pi_p(X)$ is a set of homotopy classes of maps

$$\begin{array}{ccccc} S^{p-1} & \longrightarrow & (\underline{S})^{\partial\Delta^m} & \dashrightarrow & X \\ & & \cup\!| & \nearrow_{f_1\vee\cdots\vee f_m} & \\ S^{p_1} \vee \dots \vee S^{p_m} & = & (\underline{S})^{\bigsqcup_m pt} & & \end{array}.$$

Here the upper left arrow is the attaching map of p-cell in the product $(\underline{S})^{\Delta^m} = S^{p_1} \times \dots \times S^{p_m}$.

Whitehead products in moment-angle complexes

We will consider iterated higher Whitehead products of maps:

$$\mu_i\colon S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^\infty \xrightarrow{i\text{-th inclusion}} (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

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Theorem (Buchstaber-Panov)

The moment-angle complex $(D^2, S^1)^{\mathcal{K}}$ is the homotopy fibre of the canonical inclusion $(\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m$.

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Therefore, any iterated higher Whitehead product lifts to a map

$$\begin{array}{ccccc} (D^2, S^1)^{\mathcal{K}} & \longrightarrow & (\mathbb{C}P^\infty)^{\mathcal{K}} & \longrightarrow & (\mathbb{C}P^\infty)^m \\ & \nwarrow & \uparrow{\scriptstyle w} & & \\ & & S^{d(w)}. & & \end{array}$$

Main problem

There are classes of simplicial complexes for which corresponding moment-angle complexes are homotopy equivalent to a wedge of spheres, for example, shifted (Grbić-Theriault), totally fillable (Iriye-Kishimoto). Moreover, in these cases every sphere maps to $(\mathbb{C}P^\infty)^\mathcal{K}$ by (iterated higher) Whitehead product.
This leads us to the follwing problem.

Problem (Buchstaber-Panov)

Assume the $\mathcal{Z}_\mathcal{K}$ is homotopy equivalent to a wedge of spheres. Is it true that all wedge summands are represented by lifts $S^P \to \mathcal{Z}_\mathcal{K}$ of iterated higher Whitehead products of the canonical maps $\mu_i \colon S^2 \to (\mathbb{C}P^\infty)^\mathcal{K}$?

From the discussion above follows that for shifted, totally fillable and, in addition, flag complexes (Grbić-Panov-Theriault-Wu) the answer for the problem is positive.

An example of unrealizability

Theorem (A.)

Let $\mathcal{K}$ be the simplicial complex

$$(\partial\Delta^2(1,2,3) * \partial\Delta^2(4,5,6)) \cup \Delta^2(1,2,3) \cup \Delta^2(4,5,6).$$

Then

$$\mathcal{Z}_{\mathcal{K}} \simeq (S^7)^{\vee 6} \vee (S^8)^{\vee 6} \vee (S^9)^{\vee 2} \vee S^{10},$$

but S^{10} cannot be realized by any linear combination of iterated higher Whitehead products.

Sketch of proof

Let X be a (desired) wedge $(S^7)^{\vee 6} \vee (S^8)^{\vee 6} \vee (S^9)^{\vee 2} \vee S^{10}$. By Bahri-Bendersky-Cohen-Gitler theorem, we have spliting $f\colon \Sigma X \xrightarrow{\simeq} \Sigma \mathcal{Z}_{\mathcal{K}}$. And because of dimensional reasons this equivalence can be desuspended to $f'\colon X \to \mathcal{Z}_{\mathcal{K}}$. Thus, $\mathcal{Z}_{\mathcal{K}}$ is a desired wedge.
The only Whitehead products of dimension 10 are of the form

$$[\mu_{i_1}, [\mu_{i_2}, \mu_{i_3}, \mu_{i_4}, \mu_{i_5}, \mu_{i_6}]], \quad [\mu_{i_1}, \mu_{i_2}, [\mu_{i_3}, \mu_{i_4}, \mu_{i_5}, \mu_{i_6}]].$$

Both types are not defined because of combinatorics of $\mathcal{K}$.

Positive answer

But still class of simplicial complexes with positive answer is large enough.

Theorem (A.)

Let $\mathcal{K}$ be a simplicial complex from the class, i.e. $\mathcal{Z}_{\mathcal{K}}$ is wedge of spheres and each sphere is a lift of Whitehead product. Then for any $n \in \mathbb{Z}_{\geqslant 0}$ the complex $(\mathcal{K} * \partial\Delta^n) \cup \Delta^n$ is from the class too.