Equivariant cohomology rings of the real flag manifolds

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Toric Topology 2019
at Okayama University of Science

November 22, 2019

Flag manifolds

For $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let $n = n_1 + \cdots + n_k$.

The **flag manifold** $Fl(n_1, \ldots, n_k, \mathbb{D}^n)$ consists of the flags

$$V_1 \subset V_2 \cdots \subset V_{k-1} \subset V_k = \mathbb{D}^n, \qquad \dim_{\mathbb{D}} V_i = n_1 + \cdots + n_i$$

or equivalently, consists of the $\mathbb{D}$-orthogonal decompositions

$$W_1 \oplus \cdots \oplus W_k = \mathbb{D}^n \qquad \dim_{\mathbb{D}} W_i = n_i$$

Complex and quaternionic flag manifolds

As homogeneous spaces,

$$Fl(n_1, \ldots, n_k, \mathbb{C}^n) \cong \frac{U(n)}{U(n_1) \times \cdots \times U(n_k)}$$

$$Fl(n_1, \ldots, n_k, \mathbb{H}^n) \cong \frac{Sp(n)}{Sp(n_1) \times \cdots \times Sp(n_k)}$$

where $n = n_1 + \cdots + n_k$.

Generalized complex flag manifolds

Let G be a complex simply-connected semisimple Lie group, P a parabolic subgroup. The homogeneous space

$$G/P$$

is called a **generalized complex flag manifold**.

Theorem (Bruhat-Chevalley decomposition)

$$G/P = \bigcup_{\lambda \in W_G/W_P} U\lambda P/P \cong \bigcup_{\lambda \in W_G/W_P} \mathbb{C}^{\mathscr{L}(\lambda)}.$$

Schubert presentation

Let $\{\mathcal{S}_\lambda, \lambda \in W_G/W_P\}$ be the Schubert classes, then the Schubert presentation of the cohomology of G/P is

$$H^*(G/P, \mathbb{Z}) = \frac{\mathbb{Z}[\mathcal{S}_\lambda \mid_{\lambda \in W_G/W_P}]}{\langle \mathcal{S}_\lambda \cdot \mathcal{S}_\mu = \sum_\nu C^\nu_{\lambda,\mu} \mathcal{S}_\nu \rangle}$$

where $C^\nu_{\lambda,\mu} \in \mathbb{Z}$ are the generalized Littlewood-Richardson coefficients.

Leray-Borel presentation

Theorem (Leray-Borel presentation 1)

Let G be a compact connected Lie group, and H be a closed connected subgroup that contains the maximal torus T, then

$$H^*(G/H,\mathbb{Q}) \cong \frac{H^*(BH,\mathbb{Q})}{\langle H^+(BG,\mathbb{Q})\rangle} \cong \frac{H^*(BT,\mathbb{Q})^{W_H}}{\langle H^+(BT,\mathbb{Q})^{W_G}\rangle}.$$

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Let G be a compact connected Lie group, and H be a closed connected subgroup that contains the maximal torus T, then

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For $G = U(n)$, we have $W = S_n$, and

$$H^*(BU(n), \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_n]^{S_n} = \mathbb{Z}[c_1, \ldots, c_n].$$

Leray-Borel presentation of complex flag manifolds

$$H^*\left(\frac{U(n)}{U(k)\times U(n-k)},\mathbb{Z}\right)=\frac{\mathbb{Z}[c_1,\ldots,c_k;c'_1,\ldots,c'_{n-k}]}{\langle cc'=1\rangle}.$$

Generally, for $n=n_1+\cdots+n_k$,

$$H^*\left(\frac{U(n)}{U(n_1)\times\cdots\times U(n_k)},\mathbb{Z}\right)=\frac{\mathbb{Z}[c_i^{(j)}\ |_{1\leqslant j\leqslant k,1\leqslant i\leqslant n_j}]}{\langle\prod_{j=1}^k c^{(j)}=1\rangle}.$$

Real and oriented flag manifolds

Let's focus on

$$Fl(n_1, \dots, n_k, \mathbb{R}^n) \cong \frac{O(n)}{O(n_1) \times \cdots \times O(n_k)}$$
$$Fl^o(n_1, \dots, n_k, \mathbb{R}^n) \cong \frac{SO(n)}{SO(n_1) \times \cdots \times SO(n_k)}$$

where $n \geqslant n_1 + \cdots + n_k$.

$$V_k(\mathbb{R}^n) = \frac{O(n)}{O(n-k)} \cong \frac{SO(n)}{SO(n-k)}$$

is the real Stiefel manifold consisting of the orthonormal k-frames in $\mathbb{R}^n$.

Canonical torus actions

$$SO(2)^{[\frac{n_1}{2}]+\cdots+[\frac{n_k}{2}]} \curvearrowright \frac{SO(n)}{SO(n_1)\times\cdots\times SO(n_k)}$$

$$O(1)^{n_1+\cdots+n_k} \curvearrowright \frac{O(n)}{O(n_1)\times\cdots\times O(n_k)}$$

Canonical torus actions

$$\mathbb{Z}_2^n \curvearrowright \frac{O(n+1)}{O(n)} = S^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\right\}$$

$$(\epsilon_1, \ldots, \epsilon_n) \cdot (x_0, \ldots, x_n) = (x_0, \epsilon_1 x_1, \ldots, \epsilon_n x_n)$$

$$T^n \curvearrowright \frac{O(2n+2)}{O(2n+1)} = S^{2n+1} = \left\{(z_0, z_1 \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} ||z_i||^2 = 1\right\}$$

$$(t_1, \ldots, t_n) \cdot (z_0, z_1 \ldots, z_n) = (z_0, t_1 z_1 \ldots, t_n z_n)$$

Equivariant cohomology

Definition

Let $G \curvearrowright M$. The **equivariant cohomology** of M is defined as

$$H^*_G(M,\mathbb{Q}) = H^*((EG \times M)/G,\mathbb{Q})$$

$$H^*_{T^n}(pt,\mathbb{Q}) = H^*(BT^n,\mathbb{Q}) = H^*((\mathbb{C}P^\infty)^n,\mathbb{Q}) = \mathbb{Q}[\alpha_1,\dots,\alpha_n]$$

$$H^*_{\mathbb{Z}_2^n}(pt,\mathbb{F}_2) = H^*(B\mathbb{Z}_2^n,\mathbb{F}_2) = H^*((\mathbb{R}P^\infty)^n,\mathbb{F}_2) = \mathbb{F}_2[\beta_1,\dots,\beta_n]$$

where $\deg\alpha_i = 2$, $\deg\beta_i = 1$.

Equivariant characteristic classes

If a vector bundle $\xi \to X$ is G-equivariant, then

$$C^G(\xi) \triangleq C((EG \times \xi)/G) \in H^*_G(X).$$

Lemma

View

$$\mathbb{Z}_2 \curvearrowright \mathbb{R} \qquad\qquad S^1 \curvearrowright \mathbb{R}^2$$

as $\mathbb{Z}_2$ and S^1-equivariant bundles over a single point. Write $H^*_{\mathbb{Z}_2}(pt, \mathbb{F}_2) = \mathbb{F}_2[\beta]$ and $H^*_{S^1}(pt, \mathbb{Z}) = \mathbb{Z}[\alpha]$. We have the equivariant characteristic classes

$$e^{\mathbb{Z}_2}(\tilde{\mathbb{R}}) = \beta \qquad\qquad e^{S^1}(\tilde{\mathbb{R}}^2) = \alpha$$

$$w^{\mathbb{Z}_2}(\tilde{\mathbb{R}}) = 1 + \beta \qquad\qquad p^{S^1}(\tilde{\mathbb{R}}^2) = 1 + \alpha^2$$

Universal vector bundles on real flag manifolds

Recall the real flag manifold $Fl(n_1, \ldots, n_k, \mathbb{R}^n)$, where $n_1 + \cdots + n_k = n$, consists of the orthogonal decompositions

$$W_1 \oplus \cdots \oplus W_k = \mathbb{R}^n \qquad \dim_{\mathbb{R}} W_i = n_i.$$

Universal vector bundles on real flag manifolds

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The *j*-th universal vector bundle ($1 \leqslant j \leqslant k$):

$$\mathcal{W}_j \triangleq \Big\{(W_\bullet; v) \in Fl(n_1, \ldots, n_k, \mathbb{R}^n) \times \mathbb{R}^n \mid v \in W_j\Big\}$$

satisfies

$$\oplus_{j=1}^k \mathcal{W}_j = \mathbb{R}^n.$$

Equivariant characteristic classes of real flag manifolds

If $n_1 + \cdots + n_k = N \leqslant n$, then

$$\left(\oplus_{j=1}^{k} \mathcal{W}_j \right) \oplus \left(\oplus_{i=1}^{n-N} \mathcal{L}_i \right) = \mathbb{R}^n$$

where $\mathcal{L}_i$ are trivial line bundles. Equivariant!

Take the total equivariant Stiefel-Whitney and Pontryagin classes, we have the algebraic equations:

$$\prod_{j=1}^{k} \tilde{w}^{(j)} = \prod_{l=1}^{N} (1 + \beta_l), \qquad \prod_{j=1}^{k} \tilde{p}^{(j)} = \prod_{l=1}^{M} (1 + \alpha_l^2)$$

$$(\tilde{e}^{(j)})^2 = \tilde{p}^{(j)}_{[n_j/2]}$$

$$\prod_{j=1}^{k} \tilde{e}^{(j)} = \tilde{e}\left(\oplus_{j=1}^{k} \mathcal{W}_j \right).$$

Mod-2 cohomology of real Stiefel manifolds

Theorem (Borel, Miller 53)

$$H^*\left(\frac{O(n)}{O(k)}, \mathbb{F}_2\right) = \frac{\mathbb{F}_2[h_i \mid_{k\leqslant i\leqslant n-1}]}{\langle h_i^2 = h_{2i} \mid_{k\leqslant i\leqslant n-1} \rangle}$$

where $\deg h_i = i$*, and* $h_{2i} = 0$ *if* $2i > n - 1$*.*

$\mathbb{Z}_2^N$-equivariant cohomology of real Stiefel manifolds

Lemma

The $\mathbb{Z}_2^N$-action on $\frac{O(n)}{O(N)}$ has the fixed-point set $O(n-N)$.

$\mathbb{Z}_2^N$-equivariant cohomology of real Stiefel manifolds

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Observation

1. $\dim H^*(O(n)/O(N), \mathbb{F}_2) = \dim H^*(O(n-N), \mathbb{F}_2) = 2^{n-N}$

$\mathbb{Z}_2^N$-equivariant cohomology of real Stiefel manifolds

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Observation

1. $\dim H^*(O(n)/O(N), \mathbb{F}_2) = \dim H^*(O(n-N), \mathbb{F}_2) = 2^{n-N}$ *implies the group isomorphism*

$$H^*_{\mathbb{Z}_2^N}\left(\frac{O(n)}{O(N)}, \mathbb{F}_2\right) \cong \mathbb{F}_2[\beta_1, \ldots, \beta_N] \otimes_{\mathbb{F}_2} H^*\left(\frac{O(n)}{O(N)}, \mathbb{F}_2\right).$$

2. *The embedding $\iota_{n,N} : O(n-N) \hookrightarrow \frac{O(n)}{O(N)}$ induces an injective ring homomorphism*

$$\iota^*_{n,N} : H^*_{\mathbb{Z}_2^N}\left(\frac{O(n)}{O(N)}, \mathbb{F}_2\right) \longrightarrow H^*_{\mathbb{Z}_2^N}(O(n-N), \mathbb{F}_2).$$

$\mathbb{Z}_2^N$-equivariant cohomology of real Stiefel manifolds

Theorem (H.)

$H^*_{\mathbb{Z}_2^N}(O(n)/O(N), \mathbb{F}_2)$ can be identified, via the injective homomorphism ι^*, as an $\mathbb{F}_2[\beta_1, \dots, \beta_N]$-subalgebra of

$$H^*_{\mathbb{Z}_2^N}(O(n-N), \mathbb{F}_2) \cong \mathbb{F}_2[\beta_1, \dots, \beta_N] \otimes_{\mathbb{F}_2} \frac{\mathbb{F}_2[h_i \mid_{0 \leqslant i \leqslant n-N-1}]}{\langle h_i^2 = h_{2i} \mid_{0 \leqslant i \leqslant n-N-1} \rangle}$$

generated on $\tilde{h}_i$'s for $N \leqslant i \leqslant n-1$ defined by the formula:

$$\iota^*\Big(\sum_{i=N}^{n-1} \tilde{h}_i\Big) = \Psi_N\Big(\prod_{j=1}^{N}(1+\beta_j) \cdot \sum_{i=0}^{n-N-1} h_i\Big).$$

Rational cohomology of real Stiefel manifolds

Theorem (Borel, Miller 53')

$$H^*\left(\frac{SO(2m)}{SO(2k+1)}, \mathbb{Q}\right) = \Lambda_{\mathbb{Q}}[y_{k+1}, \ldots, y_{m-1}, x_m]$$

$$H^*\left(\frac{SO(2m+1)}{SO(2k+1)}, \mathbb{Q}\right) = \Lambda_{\mathbb{Q}}[y_{k+1}, \ldots, y_{m-1}, y_m]$$

$$H^*\left(\frac{SO(2m)}{SO(2k)}, \mathbb{Q}\right) = \Lambda_{\mathbb{Q}}[e_k, y_{k+1}, \ldots, \ldots, y_{m-1}, x_m]$$

$$H^*\left(\frac{SO(2m+1)}{SO(2k)}, \mathbb{Q}\right) = \Lambda_{\mathbb{Q}}[e_k, y_{k+1}, \ldots, \ldots, y_{m-1}, y_m]$$

where $\deg x_m = 2m-1$, $\deg y_i = 4i-1$, $\deg e_k = 2k$, *and* e_k *is the Euler class of the universal bundle* $\mathcal{V}$.

T-equivariant cohomology of real Stiefel manifolds

Theorem (H.)

For the canonical T^k-actions on the real Stiefel manifolds,

$$H^*_{T^k}\left(\frac{SO(2m)}{SO(2k+1)}, \mathbb{Q}\right) = \mathbb{Q}[\alpha_1, \ldots, \alpha_k] \otimes \Lambda_{\mathbb{Q}}[\tilde{y}_{k+1}, \ldots, \tilde{y}_{m-1}, \tilde{x}_m]$$

$$H^*_{T^k}\left(\frac{SO(2m+1)}{SO(2k+1)}, \mathbb{Q}\right) = \mathbb{Q}[\alpha_1, \ldots, \alpha_k] \otimes \Lambda_{\mathbb{Q}}[\tilde{y}_{k+1}, \ldots, \tilde{y}_m]$$

$$H^*_{T^k}\left(\frac{SO(2m)}{SO(2k)}, \mathbb{Q}\right) = \frac{\mathbb{Q}[\alpha_1, \ldots, \alpha_k; \tilde{e}_k]}{\langle \tilde{e}_k^2 = \alpha_1^2 \cdots \alpha_k^2 \rangle} \otimes \Lambda_{\mathbb{Q}}[\tilde{y}_{k+1}, \ldots, \tilde{y}_{m-1}, \tilde{x}_m]$$

$$H^*_{T^k}\left(\frac{SO(2m+1)}{SO(2k)}, \mathbb{Q}\right) = \frac{\mathbb{Q}[\alpha_1, \ldots, \alpha_k; \tilde{e}_k]}{\langle \tilde{e}_k^2 = \alpha_1^2 \cdots \alpha_k^2 \rangle} \otimes \Lambda_{\mathbb{Q}}[\tilde{y}_{k+1}, \ldots, \tilde{y}_m].$$

Leray-Borel presentation

Theorem (Leray-Borel presentation 2)

Let G be a compact connected Lie group, and $T \subset H \subset K$ be closed subgroups of the same rank, then the bundle

$$K/H \longrightarrow G/H \longrightarrow G/K$$

has collapsed Leray-Serre spectral sequence. Moreover, there are ring isomorphisms

$$H^*(G/H, \mathbb{Q}) \cong H^*(G/K, \mathbb{Q}) \otimes_{H^*(BK,\mathbb{Q})} H^*(BH, \mathbb{Q})$$
$$H^*_T(G/H, \mathbb{Q}) \cong H^*_T(G/K, \mathbb{Q}) \otimes_{H^*(BK,\mathbb{Q})} H^*(BH, \mathbb{Q}).$$

(It also works for 2-rank and in $\mathbb{F}_2$ coefficients)

$\mathbb{Z}_2^N$-equivariant cohomology of real flag manifolds

Given $n \geqslant n_1 + \cdots + n_k$, set $N = n_1 + \cdots + n_k$, the bundle

$$\frac{O(N)}{O(n_1) \times \cdots \times O(n_k)} \stackrel{\iota}{\hookrightarrow} \frac{O(n)}{O(n_1) \times \cdots \times O(n_k)} \stackrel{\pi}{\longrightarrow} \frac{O(n)}{O(N)}$$

is $\mathbb{Z}_2^N$-equivariant.

$\mathbb{Z}_2^N$-equivariant cohomology of real flag manifolds

By Leray-Borel presentation, we have

$$
\begin{aligned}
& H^*_{\mathbb{Z}_2^N}\Big(\frac{O(n)}{O(n_1)\times\cdots\times O(n_k)},\mathbb{F}_2\Big) \\
\cong\ & H^*_{\mathbb{Z}_2^N}\Big(\frac{O(n)}{O(N)},\mathbb{F}_2\Big)\otimes_{H^*(BO(N),\mathbb{F}_2)} H^*(BO(n_1)\times\cdots\times BO(n_k),\mathbb{F}_2)
\end{aligned}
$$

$\mathbb{Z}_2^N$-equivariant cohomology of real flag manifolds

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$$
\begin{aligned}
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\cong\ & H^*_{\mathbb{Z}_2^N}\Big(\frac{O(n)}{O(N)},\mathbb{F}_2\Big)\otimes_{H^*(BO(N),\mathbb{F}_2)} H^*(BO(n_1)\times\cdots\times BO(n_k),\mathbb{F}_2) \\
\cong\ & H^*_{\mathbb{Z}_2^N}\Big(\frac{O(n)}{O(N)},\mathbb{F}_2\Big)\otimes_{\mathbb{F}_2[\beta_1,\ldots,\beta_N]} \frac{\mathbb{F}_2[\tilde{w}_i^{(j)}\ |_{1\leqslant j\leqslant k,1\leqslant i\leqslant n_j}]}{\langle \prod_{j=1}^k \tilde{w}^{(j)} = \prod_{l=1}^N(1+\beta_l)\rangle}.
\end{aligned}
$$

T-equivariant cohomology of pure even oriented flag manifolds

$$T^n \curvearrowright \frac{SO(2n)}{SO(2n_1)\times\cdots\times SO(2n_k)}, \qquad n = n_1 + \cdots + n_k$$

has the rational equivariant cohomology ring:

$$\frac{\mathbb{Q}[\alpha_l \mid_{1\leqslant l\leqslant n}][\tilde{p}_i^{(j)} \mid_{1\leqslant j\leqslant k, 1\leqslant i\leqslant n_j}][\tilde{e}^{(j)} \mid_{1\leqslant j\leqslant k}]}{\left\langle \prod_{j=1}^{k} \tilde{p}^{(j)} = \prod_{l=1}^{n}(1+\alpha_l^2); (\tilde{e}^{(j)})^2 = \tilde{p}_{n_j}^{(j)} \mid_{1\leqslant j\leqslant k}; \prod_{j=1}^{k} \tilde{e}^{(j)} = \prod_{l=1}^{n} \alpha_l \right\rangle}$$

T-equivariant cohomology of pure odd oriented flag manifolds

$$T^n \curvearrowright \frac{SO(2n+k)}{SO(2n_1+1)\times\cdots\times SO(2n_k+1)}, \qquad n = n_1+\cdots+n_k$$

has the exterior factor of rational cohomology as

$$\frac{SO(2n+k)}{SO(2n+1)}$$

and has the rational equivariant cohomology ring:

$$\frac{\mathbb{Q}[\alpha_l \mid_{1\leqslant l\leqslant n}][\tilde{p}_i^{(j)} \mid_{1\leqslant j\leqslant k, 1\leqslant i\leqslant n_j}]}{\langle \prod_{j=1}^k \tilde{p}^{(j)} = \prod_{l=1}^n (1+\alpha_l^2)\rangle} \otimes_{\mathbb{Q}} H^*\Big(\frac{SO(2n+k)}{SO(2n+1)}, \mathbb{Q}\Big).$$

Complex vs real flag manifolds

Let $n = n_1 + \cdots n_k$, there is the degree-halving map:

$$H^*_{T^n}(Fl(n_1, \dots, n_k, \mathbb{C}^n), \mathbb{F}_2) \longrightarrow H^*_{\mathbb{Z}_2^n}(Fl(n_1, \dots, n_k, \mathbb{R}^n), \mathbb{F}_2)$$

$$\frac{\mathbb{F}_2[\alpha_l \mid_{1 \leqslant l \leqslant n}][\tilde{c}_i^{(j)} \mid_{1 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j}]}{\langle \prod_{j=1}^k \tilde{c}^{(j)} = \prod_{l=1}^n (1+\alpha_l) \rangle} \longrightarrow \frac{\mathbb{F}_2[\beta_l \mid_{1 \leqslant l \leqslant n}][\tilde{w}_i^{(j)} \mid_{1 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j}]}{\langle \prod_{j=1}^k \tilde{w}^{(j)} = \prod_{l=1}^n (1+\beta_l) \rangle}$$

$$\tilde{c}_i^{(j)} \longmapsto \tilde{w}_i^{(j)}$$
$$\alpha_l \longmapsto \beta_l.$$

Complex vs real flag manifolds

Let $n = n_1 + \cdots n_k$, there is the degree-doubling map:

$$H^*_{T^n}(Fl(n_1, \dots, n_k, \mathbb{C}^n), \mathbb{Q}) \longrightarrow H^*_{T^n}(Fl(2n_1, \dots, 2n_k, \mathbb{R}^{2n}), \mathbb{Q})$$

$$\frac{\mathbb{Q}[\alpha_l \mid_{1 \leqslant l \leqslant n}][\tilde{c}_i^{(j)} \mid_{1 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j}]}{\langle \prod_{j=1}^k \tilde{c}^{(j)} = \prod_{l=1}^n (1 + \alpha_l) \rangle} \longrightarrow \frac{\mathbb{Q}[\alpha_l \mid_{1 \leqslant l \leqslant n}][\tilde{p}_i^{(j)} \mid_{1 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j}]}{\langle \prod_{j=1}^k \tilde{p}^{(j)} = \prod_{l=1}^n (1 + \alpha_l^2) \rangle}$$

$$\tilde{c}_i^{(j)} \longmapsto \tilde{p}_i^{(j)}$$

$$\alpha_l \longmapsto \alpha_l^2.$$

Thank you!