Lower bound of the number of seed PL-spheres of picard number 4

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Real Toric Spaces

Let K be a simplicial complex of dim. $n-1$ on m vertices and λ be a characteristic map of K.

- $Pic(K) := m - n$ is called the **picard number** of K.

Theorem(Fundamental Theorem for Toric Geometry)

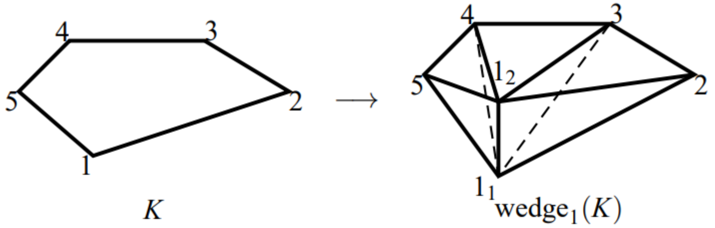
Each D-J class of real toric spaces can be indexed by (K, λ).

Wedge Operation

The **wedge** of K at v is a simplicial complex defined as

$$Wed_v(K) = (I * Lk_K(v)) \cup (\partial I * (K \setminus v)),$$

where I is a 1-simplex and $K \setminus v$ is the subcomplex without v.



Wedge Operation

- K is a **simplicial PL-sphere** if $|K| \overset{PL}{\cong} \partial\Delta^n$.
- K is **polytopal** if it is the boundary complex of a simplicial polytope.

Proposition(Choi-Park, 2016)

- K is a simplicial PL-sphere if and only if so is $Wed_v(K)$.
- K is polytopal if and only if so is $Wed_v(K)$.

Wedge Operation

Theorem(Choi-Park, 2016∼2017)

All characteristic maps of $Wed_v(K)$ can be constructed by characteristic maps of K.

Now, we want to know simplicial complexes that is not a result of wedge.

- K is called a **seed** if it can not be obtained from wedge operations of a simplicial complex.
- A **good seed** is a seed that supports a characteristic map.

Dual Characteristic maps

A **dual characteristic map** λ^* is a map

$$\lambda^* : \{1, 2, \ldots, m\} \longrightarrow \mathbb{Z}_2^{m-n}$$

whose rows span $\ker \lambda$.

Proposition

For a n-subset $J \subset \{1, 2, \ldots, m\}$, $\{\lambda(i) \mid i \in J\}$ is a basis if and only if $\{\lambda^*(i) \mid i \notin J\}$ is a basis.

Therefore, λ^* has still non-singularity information of λ.

Dual Characteristic maps

λ^* has same columns $\lambda^*(i) = \lambda^*(j)$
$\longrightarrow$ every facets have i or j as a vertex.
$\longrightarrow$ K is a suspension or a wedge.

Theorem(Choi-Park, 2017)

If K is a good seed, then $m < 2^{Pic(K)}$.

Therefore, there are finitely many good seeds with a fixed picard number.

Dual Characteristic maps(a modification of Garrison and Scott's algorithm, 2003)

- **Input:** CF = the union of the power sets of all cofacets of K.
- **Output:** Γ = the list of $\mathbb{Z}_2$ vectors $(\lambda_1, \cdots, \lambda_m)$ such that the last $m-n$ vectors form the standard basis $\mathbb{Z}_2^{m-n}$.
- **Initialization:**

  $\lambda_{n+1} \leftarrow (1,0,\cdots,0), \lambda_{n+2} \leftarrow (0,1,\cdots,0), \cdots, \lambda_m \leftarrow (0,0,\cdots,1)$.
  $\Gamma \leftarrow \emptyset$
  $S \leftarrow$ the list of nonzero elements of $\mathbb{Z}_2^{m-n}$
  $i \leftarrow n$
- **Procedure:**
  1. Set $S_i \leftarrow S$.
  2. For all $I \in CF$ of the form $I = \{i\} \cup \{i_1, \cdots, i_k\}$ with $1 \leq i \leq i_1 \leq \cdots \leq i_k$, remove the vector $\lambda_{i_1} + \cdots + \lambda_{i_k}$ from the list S_i.
  3. If $i = n$, then STOP.
  4. If $S_i = \emptyset$, then $i \leftarrow i+1$ and go to 3.
  5. Set $\lambda_i \leftarrow$ the 1st element of S_i and remove it from S_i
  6. If $i = 1$, then add $(\lambda_1, \cdots, \lambda_m)$ to Γ and go to 4.
  7. Set $i \leftarrow i-1$ and go to 1.

Injective Dual Characteristic maps(a modification of Garrison and Scott's algorithm, 2003)

- **Input:** CF = the union of the power sets of all cofacets of K.
- **Output:** Γ = the list of $\mathbb{Z}_2$ vectors $(\lambda_1, \cdots, \lambda_m)$ such that the last $m-n$ vectors form the standard basis $\mathbb{Z}_2^{m-n}$.
- **Initialization:**

  $\lambda_{n+1} \leftarrow (1, 0, \cdots, 0), \lambda_{n+2} \leftarrow (0, 1, \cdots, 0), \cdots, \lambda_m \leftarrow (0, 0, \cdots, 1).$
  $\Gamma \leftarrow \emptyset$
  $S_{n+1} \leftarrow$ the list of nonzero elements of $\mathbb{Z}_2^{m-n}$
  $i \leftarrow n$
- **Procedure:**
  1. Set $S_i \leftarrow S_{i+1}$.
  2. For all $I \in CF$ of the form $I = \{i\} \cup \{i_1, \cdots, i_k\}$ with $1 \leq i \leq i_1 \leq \cdots \leq i_k$, remove the vector $\lambda_{i_1} + \cdots + \lambda_{i_k}$ from the list S_i.
  3. If $i = n$, then STOP.
  4. If $S_i = \emptyset$, then $i \leftarrow i + 1$ and go to 3.
  5. Set $\lambda_i \leftarrow$ the 1st element of S_i and remove it from S_i
  6. If $i = 1$, then add $(\lambda_1, \cdots, \lambda_m)$ to Γ and go to 4.
  7. Set $i \leftarrow i - 1$ and go to 1.

Known Results

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   $\longrightarrow$ there are 21 good seed PL-spheres and 2 among them are not polytopal.

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3. When m=9, n=5, there are 322 polytopal simplicial complexes (Fukuda-Miyata-Moriyama, 2013).
   $\longrightarrow$ there are 132 good seed polytopes.

Bistellar Moves

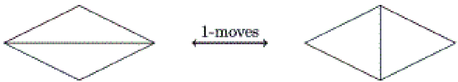
Definition

Let K be a simplicial complex. Assume that $F \in K$ has a link that is a boundary complex of a k-simplex $G \notin K$. The **bistellar k-move** $bm_F(K)$ of K at F is defined as follows:

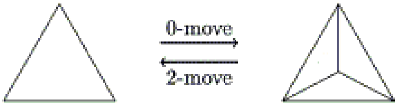
$$K \setminus (\bar{F} * \partial\bar{G}) \cup (\partial\bar{F} * \bar{G}).$$

Note that $Lk_{bm_F(K)}(G) = \partial\bar{F}$, so the inverse of a bistellar move is a bistellar move. In particular, $K = bm_G(bm_F(K))$.

Bistellar Moves



If $\dim(G) = 0$, then $bm_F(K)$ has one more vertex.



Bistellar Moves

Theorem(Pachner)

Let K, L be PL-manifolds. $|K| \overset{PL}{\cong} |L|$ if and only if they can be connected by a sequence of bistellar moves.
Moreover, polytopes are connected by a sequence of bistellar moves preserving vertices.

There is a simplicial sphere each face of which has the link that is not a simplex.
$\longrightarrow$ It can not be obtained from vertex preserving bistellar moves.

Bistellar Moves

- **Input:** P = PL-spheres of dim. $n-1$ on $\{1, 2, \ldots, m\}$.
- **Output:** BP = PL-spheres of dim. $n-1$ on $\{1, 2, \ldots, m+1\}$ that can be obtained by a sequence of bistellar moves without using vertex reduction.
- **Initialization:**

$$BP \leftarrow \{bm_F(K) \mid |F| = n, F \in K, K \in P\}/_{\cong}.$$
$$l \leftarrow |BP|$$
$$E_k \leftarrow \emptyset \text{ for each } k \leq l.$$
$$V \leftarrow \{1, 2, \ldots, m+1\}.$$
$$i \leftarrow 1.$$

- **Procedure:**
  1. If $i > l$, then STOP.
  2. $K \leftarrow BP_i$.
  3. $j \leftarrow 1$.
  4. $FP \leftarrow \{F \in K \mid Lk_K(F) = \partial\Delta^g$ for some $0 < g < n-1\}$.
  5. If $j > |FP|$, then $i \leftarrow i+1$ and go to 1.
  6. If for some $e \in E_i$ such that $e_2 = \dim(Lk(F_j)) + 2$, $BP_{e_1} \cong bm_{F_j}(K)$, then $j \leftarrow j+1$ and go to 5.
  7. For $i \leq p \leq l$, if $BP_p \cong bm_{F_j}(K)$, then add $(i, |F_j|)$ to E_p, $j \leftarrow j+1$, go to 5.
  8. Add $bm_{F_j}(K)$ to BP, $l \leftarrow l+1$, $E_l \leftarrow \{(i, |F_j|)\}$, $j \leftarrow j+1$.

Our Results

	PL-spheres/polytopes	seeds	good seeds
4	39/37	23/21	21/19
5	337≤/322	194≤/184	142≤/132
6	6257≤/6257≥	4237≤/4237≥	733≤/733≥

The bistellar program took 14 hours for m=9, n=5 case, 21 days for m=10, n=6 case.

The goodness checking program with the charateristic version and the injective dual version took 44 and 5 seconds, respectively for m=10, n=6 case.