Multiplicities of points on Schubert varieties in the flag variety

Ryotaro Kawago

Overview

- Schubert varieties in the flag variety of classical types.
- We are interested in the singularity of Schubert varieties.
- The Hilbert-Samuel multiplicity is a positive integer. That measure how singular the points.

This research is a joint work with Dave Anderson, Takeshi Ikeda, and Minyoung Jeon.

1.1. Multiplicity

Definition (Hilbert-Samuel function)

$(R, \mathfrak{m})$:Noetherian local ring
We define the Hilbert-Samuel function

$$H_R(t) = \dim_k(\mathfrak{m}^t/\mathfrak{m}^{t+1})$$

$(k = R/\mathfrak{m})$

$H_R(t) = \frac{m}{n!}t^n+$(lower terms) $(t >> 1), n = \dim R$

Definition (Hilbert-Samuel multiplicity)

Let $\frac{m}{n!}t^n$ be the term of highest degree of Hilbert-Samuel function $H_R(t)$. $\mathrm{mult} R = m$

1.2. Results on multiplicity

Combinatorial formula of multiplicity of points on Schubert varieties:

- Schubert varieties in Grassmannian (Kodiyalam and Raghavan 2003)
- Schubert varieties in Lagrangian Grassmannian (Ghorpade and Raghavan 2006, Ikeda and Naruse 2009)
- Schubert varieties in Orthogonal Grassmannian (Ikeda and Naruse 2009, Raghavan and Upadhyay 2010)

These results can be interpreted using "Excited Young diagram". We want to extend these results.

1.3. Results on multiplicity

In the case of type A, the following result is known:

- Vexillary Schubert varieties in flag variety (Li and Yong 2012)

We consider “vexillary” Schubert varieties in flag variety of type B (C,D). We were able to get similar results for types B, C and D.

1.4. Flag variety of Orthogonal type

$\mathbb{C}^{2n+1}$: Basis $\mathbf{e}_i$ $(i \in I_n = \{\overline{n}, \cdots, \overline{1}, 0, 1, \cdots, n\})$

quadratic form

$\mathbf{a}, \mathbf{b} \in \mathbb{C}^{2n+1}$ $\langle \mathbf{a}, \mathbf{b} \rangle = {}^t\mathbf{a} J \mathbf{b} = a_{\overline{n}} b_n + \cdots + a_n b_{\overline{n}}$

$$J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

Isotropic subspaces

$V \subset \mathbb{C}^{2n+1}$ is an isotropic subspace $\Longleftrightarrow \langle V, V \rangle = \{0\}$

1.5. Flag variety of Orthogonal type

Definition (flag variety)

$Fl_n^B = \{V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{2n+1} | \dim V_i = i, V_n \text{ is isotropic}\}$

Definition (Borel subgroups)

$SO_{2n+1} = \{g \in GL_{2n+1} | \langle g\mathbf{a}, g\mathbf{b}\rangle = \langle \mathbf{a}, \mathbf{b}\rangle, \mathbf{a}, \mathbf{b} \in \mathbb{C}^{2n+1}\}$.
$B = \{g \in SO_{2n+1} | g \text{ is an upper triangular matrix}\}$,
$B_- = \{g \in SO_{2n+1} | g \text{ is a lower triangular matrix}\}$.
B, B_- are Borel subgroups.

We identify Fl_n^B with SO_{2n+1}/B.

1.6. Weyl group

Definition (Signed permutations)

Let S_{2n+1} be the symmetric group of I_n.
($I_n = \{\overline{n}, \cdots, 0, \cdots, n\}$).
$W_n = \{w \in S_{2n+1} | w(\overline{i}) = \overline{w(i)}, 0 \le i \le n\}$ $(\overline{0} = 0)$

$w \in W_n$ is written as $(w(1), \cdots, w(n))$.
Example:$(\overline{2}, 1, 0, \overline{1}, 2) \in W_2$ is written as $(\overline{1}, 2)$.

1.7. Torus fixed point

$v \in W_n$

Definition (Torus fixed point)

Let e_v be the isotropic flag that is determined by
$V_i = \langle \mathbf{e}_{v(\overline{n})}, \cdots, \mathbf{e}_{v(\overline{n-i+1})} \rangle (i \in \{1, \cdots, n\})$.

Example

$e_{\overline{3}\ 4\ \overline{1}\ 2} = (\langle \mathbf{e}_{\overline{2}} \rangle \subset \langle \mathbf{e}_{\overline{2}}, \mathbf{e}_1 \rangle \subset \langle \mathbf{e}_{\overline{2}}, \mathbf{e}_1, \mathbf{e}_{\overline{4}} \rangle \subset \langle \mathbf{e}_{\overline{2}}, \mathbf{e}_1, \mathbf{e}_{\overline{4}}, \mathbf{e}_3 \rangle)$

1.8. Schubert varieties

Definition (Schubert variety)

The Schubert variety Ω_w is the closure of the B_--orbit $B_- e_w$

$$e_v \in \Omega_w \iff v \geq w$$

We use Hilbert-Samuel multiplicity to measure singularity of e_v on Ω_w. The singularity is more singular when multiplicity is larger.
We write multiplicity as $\mathrm{mult}_{e_v} \Omega_w$.

2.1. Rothe diagram

Rothe diagram

$w = (\overline{1}, \overline{3}, 2) \in W_3$.

	$\overline{3}$	$\overline{2}$	$\overline{1}$	0	1	2	3
$\overline{3}$							
$\overline{2}$							
$\overline{1}$							
0							
1							
2							
3							

2.1. Rothe diagram

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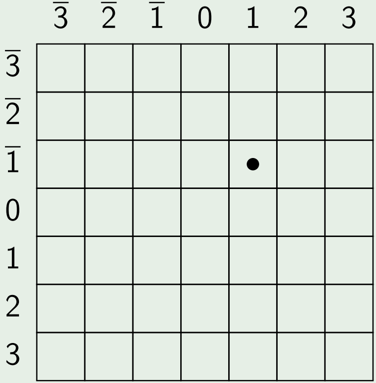

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$\overline{2}$	•						
$\overline{1}$					•		
0				•			
1			•				
2							•
3		•					

We remove the box to the right and bottom of the point.

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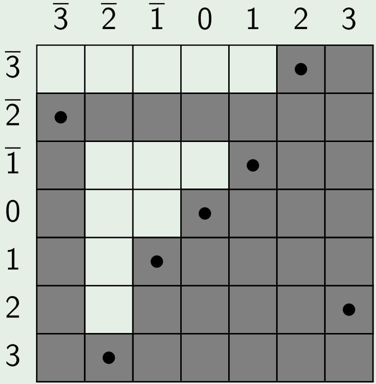

Let $D(w)$ be the set of remaining boxes.

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Rothe diagram

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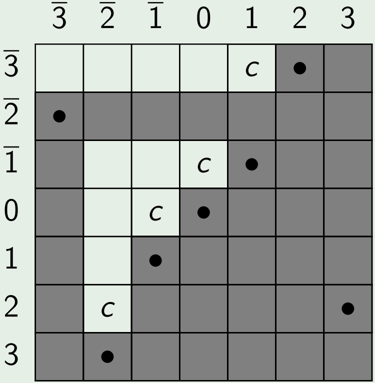
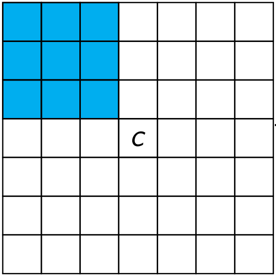


Let c be the lower right corner of $D(w)$.

2.2. Vexillary permutation

Definition

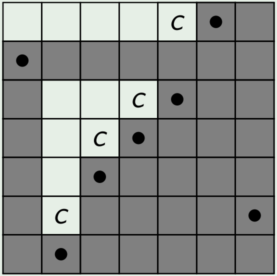
$w \in W_n$ is vexillary signed permutation. $\iff$ About for all corner, $(p', q'), p' < p, q' < q$ is not corner.

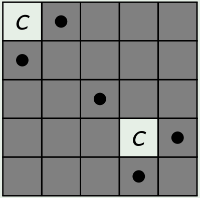


There is no corner in the blue area.

2.3. Vexillary signed permutation

Example

$w = (\overline{1}, \overline{3}, 2)$.  w is vexillary

$w = (2, 1)$.  w is non-vexillary

2.4. Shifted Young diagram

Definition (Shifted Young diagram)

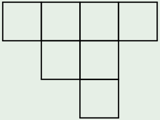
$\lambda_1 > \cdots > \lambda_m > 0 (\lambda_i \in \mathbb{N})$

Shifted Young diagram is the set of square boxes with coordinate

$(i, j) \in \{(i, j) \in \mathbb{N} \times \mathbb{N} | 1 \leq i \leq m, i \leq j \leq \lambda_i + i - 1\}$.

Example

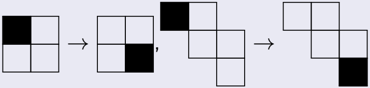
$\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1$



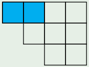
2.5. Excited Young diagrams

Definition (Exited Young diagrams)

Let $\mathcal{E}_\mu(\lambda)$ be the set of diagrams that is obtained by the following operation.

- Let $\lambda, \mu (\lambda \subset \mu)$ be shifted Young diagrams. We stack μ and λ in the upper left corner. A figure is obtained by μ, λ.
- We perform operation  on that diagram. And repeat this operation.

Example



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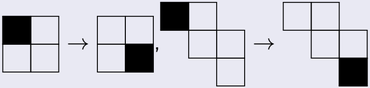
- We perform operation  on that diagram. And repeat this operation.

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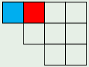

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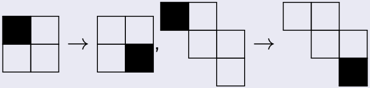
- We perform operation  on that diagram. And repeat this operation.

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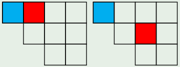

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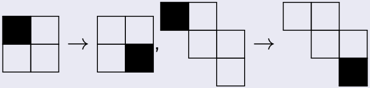
- We perform operation  on that diagram. And repeat this operation.

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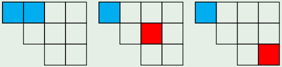

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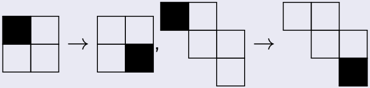
- We perform operation  on that diagram. And repeat this operation.

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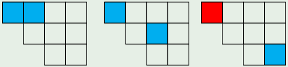

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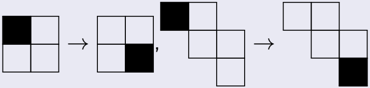
- We perform operation  on that diagram. And repeat this operation.

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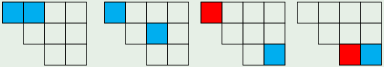

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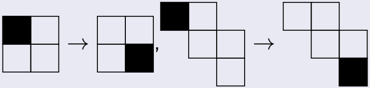
- We perform operation  on that diagram. And repeat this operation.

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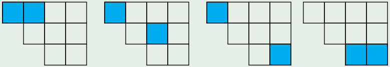

2.6. Main theorem

Theorem (Anderson-Ikeda-Jeon-K)

Let w be a vexillary signed permutation and $v \geq w$. We can determine Shifted Young diagrams λ, μ such that

$$\mathrm{mult}_{e_v} \Omega_w = \# \mathcal{E}_\mu(\lambda)$$

3.1. Rank function

Definition (Rank function)

$p, q \in I_n$. $r_w(p,q) := \#\{i \in I_n | w(i) \leq p, i \leq q\}$.

Example

$w = (1, \overline{2})$.

	$\overline{2}$	$\overline{1}$	0	1	2
$\overline{2}$					•
$\overline{1}$		•			
0			•		
1				•	
2	•				

$r_w(0,1) = 2$

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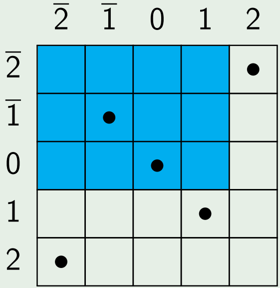


$r_w(0,1) = 2$

3.2. Essential boxes

$D^{-}(w)$:Left half of rothe diagram $D(w)$.

Definition (Essential box, Anderson and Fulton)

Essential boxes are lower right corner of $D^{-}(w)$. The following two cases are excluded.

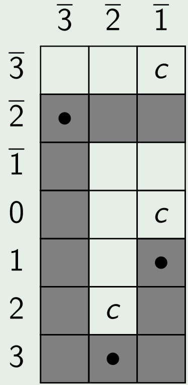
- $(p, \overline{1}), \overline{n} \leq p \leq \overline{1}$
- $(p-1, \overline{q})$:$p > 0, q > 1$,$(\overline{p}, \overline{q})$ is essential box. And $k = r_w(p, \overline{q}) = r_w(\overline{p}+1, \overline{q}) - p + 1$.

Let $Ess(w)$ be the set of essential boxes that determined by w.

3.3. Essential boxes

Example

$w = (\overline{1}, \overline{3}, 2)$:



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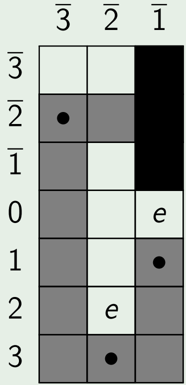
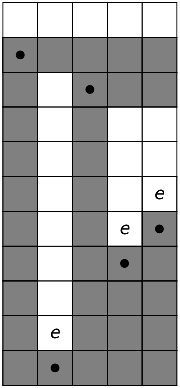


$Ess(w) = \{(2, \overline{2}), (0, \overline{1})\}$

3.4. How to make λ

λ is determined by vexillary w. $w = (\overline{1}, \overline{2}, 3, \overline{5}, 4)$.

$Ess(w)$:



Rank functions are $r_w(4, \overline{4}) = 1$, $r_w(1, \overline{2}) = 2$, $r_w(0, \overline{1}) = 2$.
e is shifted to the upper left by $r_w(e)$.

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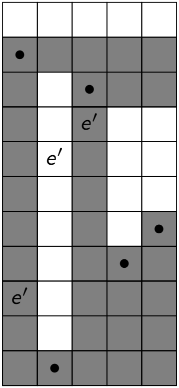

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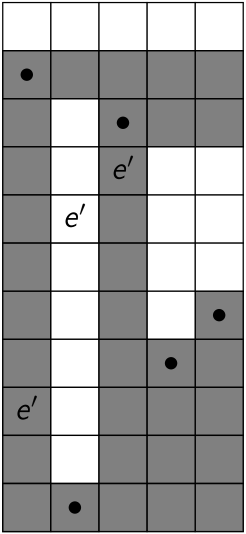

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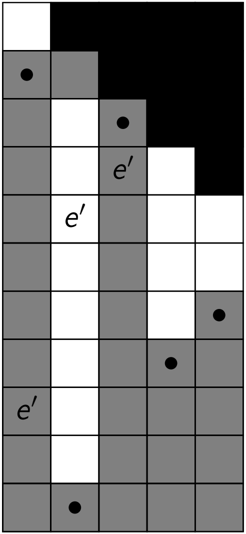

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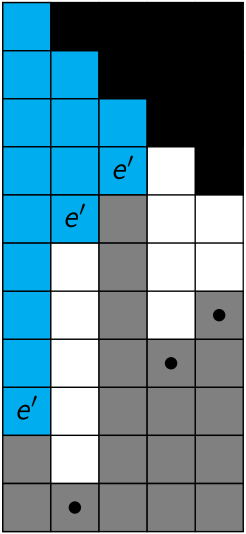

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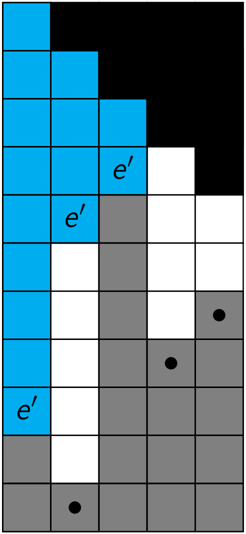
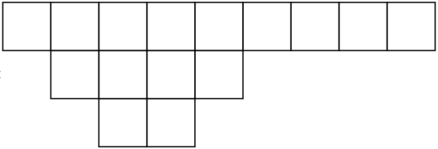


$\lambda^+ =$ 

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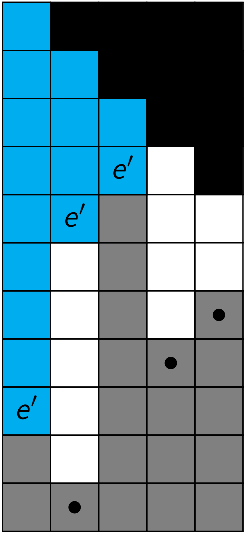

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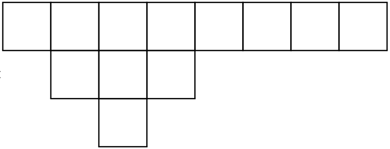

3.6. How to make μ

μ determined by w, v.
We find the $Ess(w)$ that determined by w.
$e \in Ess(w)$ is shifted to the upper left by $r_v(e)$.
The remaining operations are same operations as λ.

4.1. Conjecture

Theorem (Anderson-Ikeda-Jeon-K)

Let w be a vexillary signed permutation and $v \geq w$. We can determine Shifted Young diagrams λ, μ such that

$$\mathrm{mult}_{e_v}\Omega_w = \#\mathcal{E}_\mu(\lambda)$$

Theorem (Ikeda-Naruse, Raghavan-Upadhyay)

If w, v are Orthogonal Grassmannian Elements, Conjecture is right.

Idea of proof
Reduce to the case of maximal isotropic Grassmannians.