# Cohomological Rigidity Problem of Toric Fano Manifolds

Kazuki Kurimoto (Kyoto Sangyo University)

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J.W.W. Akihiro Higashitani and Mikiya Masuda

## Variety isomorphism classes of toric Fano manifolds

| dimension | variety types | proved by |
|---|---|---|
| 2 | 5 | |
| 3 | 18 | Watanabe-Watanabe('82), Batyrev ('91) |
| 4 | 124 | Batyrev('99), Sato('00) |
| 5 | 866 | Øbro ('07) |
| ⋮ | ⋮ | ⋮ |

## Toric Fano 2-folds

| $\rho$ | varieties |
|---|---|
| 1 | $\mathbb{P}^2$ |
| 2 | $\mathbb{F}_0$ $(= \mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathbb{F}_1$ |
| 3 | 2 points blow up of $\mathbb{P}^2$ |
| 4 | 3 points blow up of $\mathbb{P}^2$ |

## Theorem 1 (Higashitani-K.-Masuda)

*Toric Fano 3-folds are cohomologically rigid.*

**Invariants**

$$\text{s.v.e.} : \{f \in H^*(M) \mid f^2 = 0\}$$

$$\text{c.v.e.} : \{f \in H^*(M) \mid f^3 = 0\}$$

$$\vdots$$

**3-stage Bott towers**

| ID. | ideals | s.v.e. over $\mathbb{Z}$ | s.v.e. over $\mathbb{Z}/2\mathbb{Z}$ |
|---|---|---|---|
| 11. | $(x^2, y^2, z(z-x-y))$ | $x, y$ (2) | $x, y, x+y$ (3) |
| 12. | $(x^2, y(y-x), z(z-y))$ | $x, x-2y$ (2) | $x$ (1) |
| 17. | $(x^2, y^2, z(z-x))$ | $x, y, x-2z$ (3) | $x, y, x+y$ (3) |
| 18. | $(x^2, y^2, z(z-x+y))$ | $x, y$ (2) | $x, y, x+y$ (3) |
| 21. | $(x^2, y^2, z^2)$ | $x, y, z$ (3) | all (7) |

from the database by Øbro (http://www.grdb.co.uk/forms/toricsmooth)

**11.** & **18.**

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & {\color{red}1} \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & {\color{red}-1} \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

**10.** & **13.**

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & {\color{red}1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & {\color{red}-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

from the database by Øbro (http://www.grdb.co.uk/forms/toricsmooth)

## Lemma

$M_1, M_2$ : toric manifolds, $A$ : fan of $M_1$, $B$ : fan of $M_2$.
If two $M_2$-bundles over $M_1$ have fans as follows, they are diffeomorphic.

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix} \approx \begin{pmatrix} A & -C \\ O & B \end{pmatrix}$$

It is shown to use moment angle manifolds.

**Isomorphism classes of toric Fano 3-folds**

| variety types | $H^*(V;\mathbb{Z})$ | diffeomorphism types |
|---|---|---|
| 18 | 16 | 16 |

**11.**

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathcal{V}_{11} : \mathbb{T}^6 \to \mathbb{T}^3$$

$$\mathcal{V}_{11}(g_1, \dots, g_6) = (g_1 g_2^{-1}, g_3 g_4^{-1}, g_2 g_4 g_5 g_6^{-1})$$

**18.**

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathcal{V}_{18} : \mathbb{T}^6 \to \mathbb{T}^3$$

$$\mathcal{V}_{18}(g_1, \dots, g_6) = (g_1 g_2^{-1}, g_3 g_4^{-1}, g_2^{-1} g_4 g_5 g_6^{-1})$$

$$\begin{aligned} ker\mathcal{V}_{11} &= (g_1, g_1, g_3, g_3, g_5, g_1 g_3 g_5) \\ ker\mathcal{V}_{18} &= (g_1, g_1, g_3, g_3, g_5, g_1^{-1} g_3 g_5) \\ &= (g_1^{-1}, g_1^{-1}, g_3, g_3, g_5, g_1 g_3 g_5) \end{aligned}$$

$$
\begin{array}{rcl}
P & : & \text{octahedron} \\
\mathcal{Z}_P & : & \text{moment angle manifold corresponding to } P
\end{array}
$$

We get a diffeomorphism as follows.

$$
\mathcal{Z}_P/ker\mathcal{V}_{11} \overset{\approx}{\longrightarrow} \mathcal{Z}_P/ker\mathcal{V}_{18}
$$

$$
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} \mapsto \begin{pmatrix} z_1^{-1} \\ z_2^{-1} \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix}
$$

**Theorem 2 (Higashitani-K.-Masuda)**

*Toric Fano 4-folds are cohomologically rigid except for one pair.*

**Isomorphism classes of toric Fano manifolds**

| dimension | variety types | $H^*(V;\mathbb{Z})$ | diffeo. types |
|---|---|---|---|
| 2 | 5 | 5 | 5 |
| 3 | 18 | 16 | 16 |
| 4 | 124 | 102 | 102 or 103 |
| 5 | 866 | ?? | ?? |
| ⋮ | ⋮ | ⋮ | ⋮ |

## Excepted one pair of dimesion 4

**50.** **57.**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & {\color{red}1} & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & {\color{red}0} & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathcal{I}_{50} = ((x - y - z + w)w, (x - y)y, (x - z)z,$$
$$x^2(x + w), x^2y, x^2z, (x - y)(x - z)w)$$

$$\mathcal{I}_{57} = ((x - y + w)w, \quad (x - y)y, (x - z)z,$$
$$x^2(x + w), x^2y, x^2z, (x - y)(x - z)w)$$

$$F : H^*(V_{50}) \xrightarrow{\cong} H^*(V_{57}) \qquad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

**Theorem 3 (Higashitani-K.-Masuda)**

*Toric Fano manifolds of dimension d whose Picard number is greater than or equal to 2d-2 are cohomologically rigid.*

They are some direct product or bundles of $\mathbb{P}^1$, $\mathbb{F}_1$, $BL_2(\mathbb{P}^2)$, $BL_3(\mathbb{P}^2)$ and $DP_4$ where

$$\begin{aligned} BL_2(\mathbb{P}^2) &: \text{2 points blow up of } \mathbb{P}^2 \\ BL_3(\mathbb{P}^2) &: \text{3 points blow up of } \mathbb{P}^2 \\ DP_4 &: \text{del Pezzo 4-fold} \end{aligned}$$

| $\rho$ | $d$ | varieties | proved by |
| --- | --- | --- | --- |
| $2d$ | even | $\prod^{\frac{d}{2}} BL_3(\mathbb{P}^2)$ | Casagrande('06) |
| $2d$ | odd | no such variety | |
| $2d-1$ | even | $BL_2(\mathbb{P}^2) \times \prod^{\frac{d-2}{2}} BL_3(\mathbb{P}^2)$ | Øbro('08) |
| $2d-1$ | odd | $\left(\prod^{\frac{d-1}{2}} BL_3(\mathbb{P}^2)\right)$-bundle over $\mathbb{P}^1$ | |
| $2d-2$ | even | $\left(\prod^{\frac{d-2}{2}} BL_3(\mathbb{P}^2)\right)$-bundle over $\mathbb{F}_0$ or $\mathbb{F}_1$<br>$BL_2(\mathbb{P}^2) \times BL_2(\mathbb{P}^2) \times \prod^{\frac{d-4}{2}} BL_3(\mathbb{P}^2)$<br>$DP_4 \times \prod^{\frac{d-4}{2}} BL_3(\mathbb{P}^2)$ | Assarf-Joswig-Paffenholz ('14) |
| $2d-2$ | odd | $\left(BL_2(\mathbb{P}^2) \times \prod^{\frac{d-3}{2}} BL_3(\mathbb{P}^2)\right)$-bundle over $\mathbb{P}^1$ | |

## A part of known results for Bott towers

**Theorem 4 (Masuda-Panov('08))**

*$M$ : n-stage Bott tower.* $H^*(M;\mathbb{Z}) \cong H^*((\mathbb{P}^1)^n;\mathbb{Z}) \Rightarrow M \approx (\mathbb{P}^1)^n$

**Theorem 5 (Choi-Masuda-Suh('10))**

2*-stage generalized Bott towers are cohomologically rigid.*

**Theorem 6 (Choi-Masuda-Suh('10))**

3*-stage Bott towers are cohomologically rigid.*

**Theorem 7 (Choi('15))**

4*-stage Bott towers are cohomologically rigid.*

**Theorem 8 (Higashitani-K.)**

*Fano Bott towers are cohomologically rigid.*

$$\begin{array}{ccc} \text{Fano Bott tower} & \longrightarrow & \text{rooted tree} \\ \downarrow & & \downarrow \\ \text{s.v.e. in } H^*(M) & \sim & \text{leaves of rooted tree} \end{array}$$

Thank you for your attention.