Combinatorial properties of string polytopes of Type A

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What are string polytopes?

- $G = \mathsf{SL}_{n+1}(\mathbb{C})$.
- $\mathbf{i} = (i_1, \dots, i_N)$: a reduced decomposition of the longest element in $\mathfrak{S}_{n+1}$, i.e., $s_{i_1} s_{i_2} \cdots s_{i_N} = (n+1 \ n \ \cdots \ 2 \ 1)$.
- λ: dominant integral weight.

Using these data, Littelmann defined the string polytope $\Delta_{\mathbf{i}}(\lambda)$, which

1. is a rational polytope living in $\mathbb{R}^N$, where $N = \dim_{\mathbb{C}} G/B = \frac{n(n+1)}{2}$,
2. $\Delta_{\mathbf{i}}(\lambda) \cap \mathbb{Z}^N \leftrightarrow$ weights of $V(\lambda)$,
3. is a Newton–Okounkov body of $(G/B, \mathcal{L}_\lambda)$ (by [Kaveh, 15]).
4. For $\mathbf{i} = (1, 2, 1, 3, 2, 1, \dots, n, n-1, \dots, 1)$,

$$\Delta_{\mathbf{i}}(\lambda) \simeq \text{Gelfand–Cetlin polytope } \mathrm{GC}(\lambda).$$

Combinatorics of $\Delta_{\mathbf{i}}(\lambda)$ depends on $\mathbf{i}$.

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$$\downarrow$$

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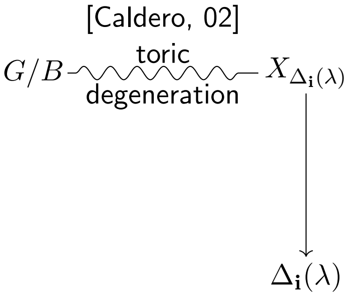

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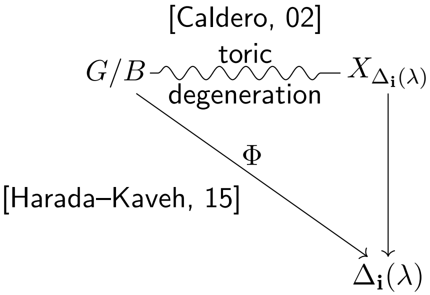

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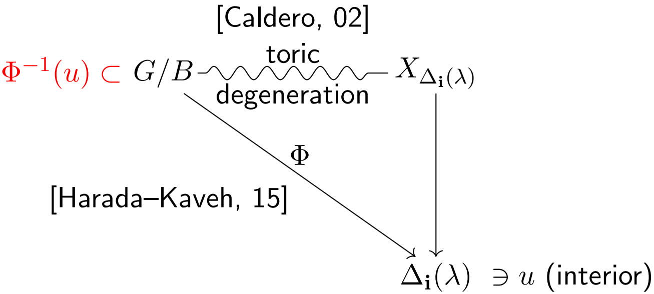

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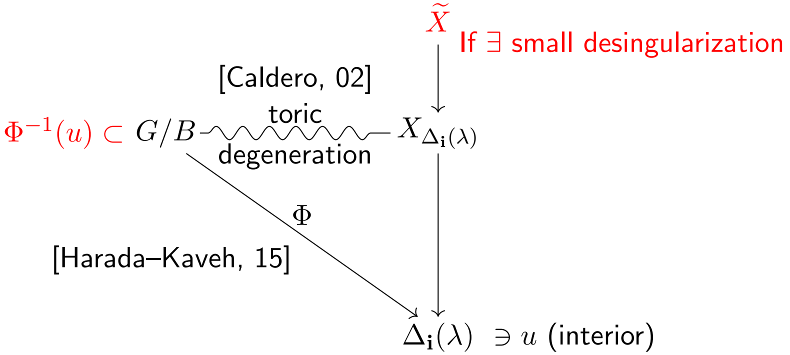

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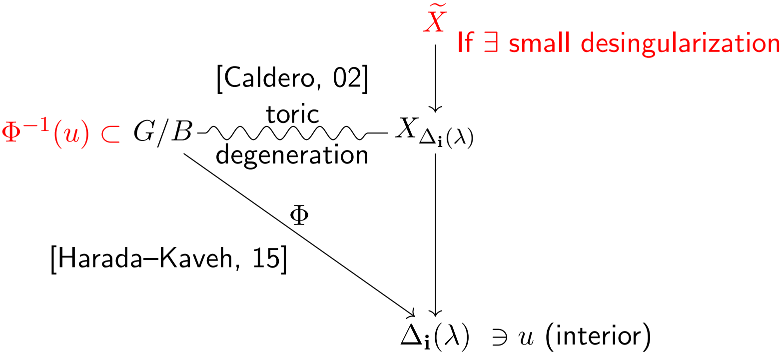

Theorem [Nishinou–Nohara–Ueda, 10]

One can get *symplectic information* (so called *disk potential*) of $\Phi^{-1}(u)$ using the combinatorics of $\Delta_{\mathbf{i}}(\lambda)$.

Gelfand–Cetlin polytopes

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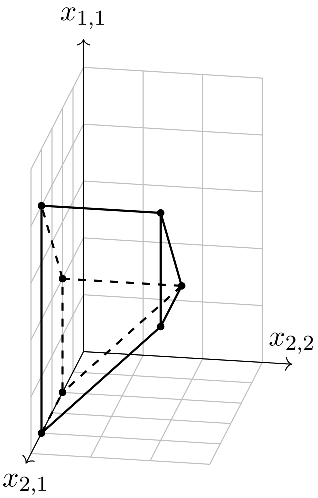

Gelfand–Cetlin toric varieties

$X_{\widetilde{\Sigma}}$ is a small desingularization of X_{Σ} if $\widetilde{\Sigma}$ is smooth and it is a refinement of Σ satisfying $\widetilde{\Sigma}(1) = \Sigma(1)$.

Theorem [Batyrev–Ciocan-Fontanine–Kim–van Straten, 00]

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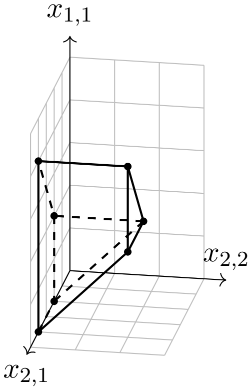
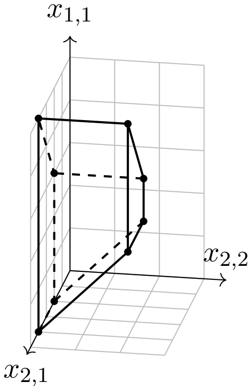
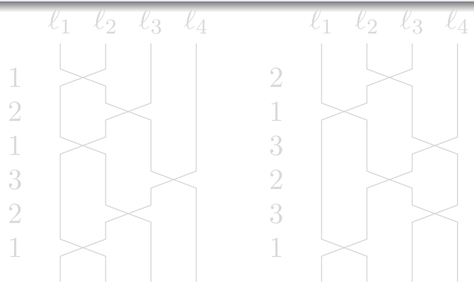




Wiring diagrams

Definition

For a reduced word $\mathbf{i}$, the wiring diagram $G(\mathbf{i})$ is a pseudoline arrangement consisting of a family of $(n+1)$-vertical piecewise straight lines such that the jth crossing of wires (from the top) is located in the i_j-column of the diagram for each $j = 1, \ldots, N$.



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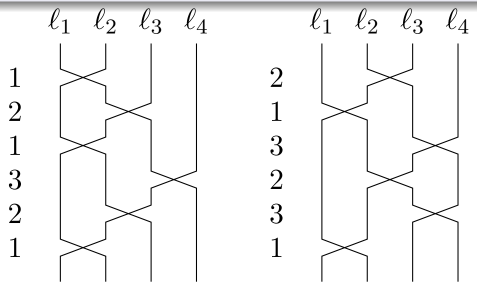

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Indices

$$\textrm{ind}_D(\mathbf{i}) = \#\text{ of crossings below } \ell_{n+1},$$
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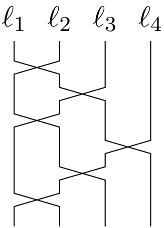	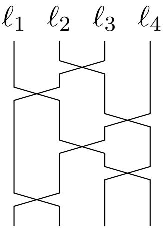
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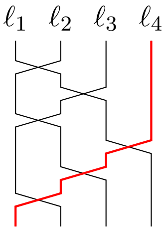	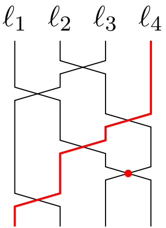
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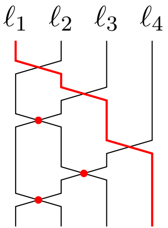	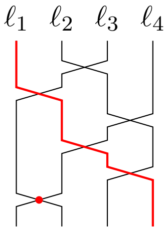
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Contractions

- $\color{red}{C_D(\mathbf{i})}$: erase ℓ_{n+1} and rearrange.
- $\color{red}{C_A(\mathbf{i})}$: erase ℓ_1 and rearrange.

Contraction maps a reduced word of the longest element in $\mathfrak{S}_{n+1}$ to a reduced word of the longest element in $\mathfrak{S}_n$.

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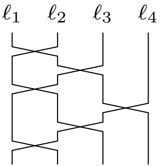		

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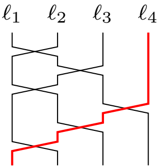	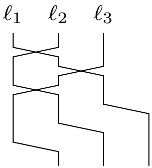	

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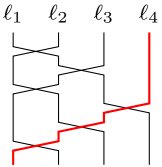	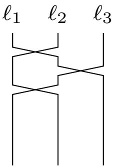	

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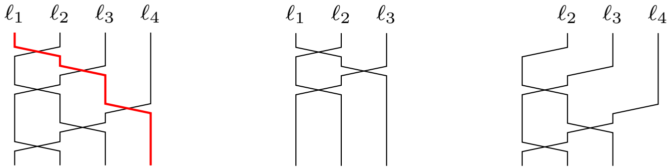

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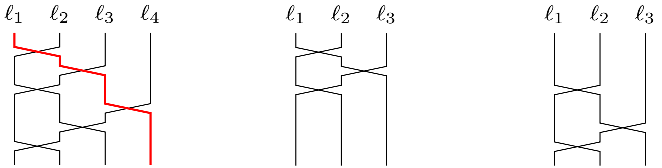

Theorem 1 ([Cho–Kim–L–Park])

Let $\mathbf{i}$ *be a reduced word of the longest element in* $\mathfrak{S}_{n+1}$*. Let* λ *be a regular dominant integral weight. Then the following are equivalent.*

1. $\Delta_{\mathbf{i}}(\lambda) \cong \mathrm{GC}(\lambda)$.
2. $\#$ *of facets of* $\Delta_{\mathbf{i}}(\lambda) = n(n+1)$.
3. *There exists a sequence* $(\sigma_1, \dots, \sigma_n) \in \{A, D\}^n$ *such that*
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- $\mathbf{i} = (2,1,2,3,2,1)$. $\operatorname{ind}_D(\mathbf{i}) = 0$, $C_D(\mathbf{i}) = (2,1,2)$. $\operatorname{ind}_A(2,1,2) = 0$, $C_D(2,1,2) = (1)$, $\operatorname{ind}_D(1) = 0$. Hence $\Delta_{(2,1,2,3,2,1)}(\lambda) \simeq \mathsf{GC}(\lambda)$.
- $\mathbf{i} = (2,1,3,2,3,1)$. $\operatorname{ind}_A(\mathbf{i}) = 1 \neq 0$ and $\operatorname{ind}_D(\mathbf{i}) = 1 \neq 0$. Hence $\Delta_{(2,1,2,3,2,1)}(\lambda) \not\simeq \mathsf{GC}(\lambda)$. Indeed, $\#$ of facets of $\Delta_{\mathbf{i}}(\lambda) = 13 \neq 12$.

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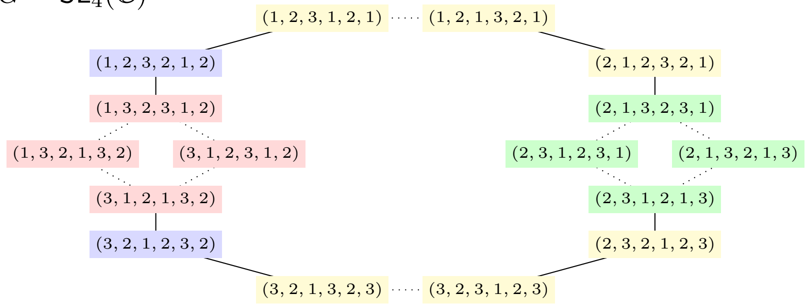

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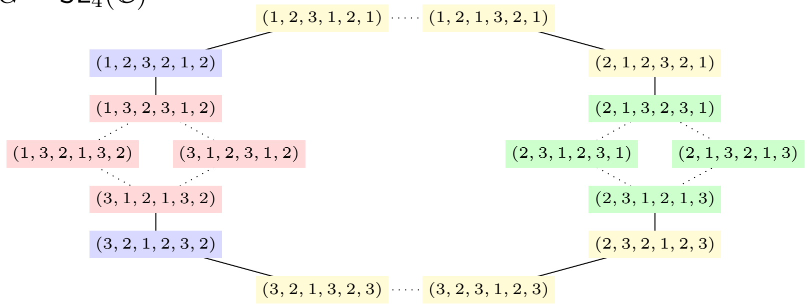

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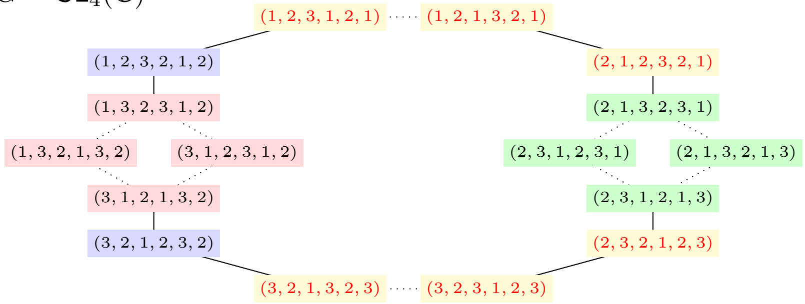

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then $X_{\Delta_{\mathbf{i}}(\lambda)}$ *admits a small desingularization* $\widetilde{X}$. *Moreover,* $\widetilde{X}$ *is obtained by a blow-up of a Bott manifold.*

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For $G = \mathrm{SL}_4(\mathbb{C})$, there are four string polytopes in $\mathbb{R}^6$ up to unimodular equivalence. (There are 16 reduced decompositions.) For $G = \mathrm{SL}_5(\mathbb{C})$, there are 28 string polytopes in $\mathbb{R}^{10}$ up to unimodular equivalence. (There are 768 reduced decompositions.)

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Thank you for your attention!!