The universal covers of hypertoric varieties and Bogomolov's decomposition

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Background and Motivation

Background - conical symplectic varieties

Definition

(Y, ω) is a **conical symplectic variety** if

- ω is a holomorphic symplectic form on Y_{reg} (+ conditions)
- Y is **affine**, and $\exists \mathbb{C}^* \curvearrowright Y$ with positive weight (like cone)

Example

- $(\mathbb{C}^{2n}, \omega_{\mathbb{C}})$ with scalar action
- ADE-type surface singularity (ex. $\mathbb{C}^2/\mathbb{Z}_\ell = \{xy - z^\ell = 0\}$)
- Affine hypertoric variety $Y_A(0) :=$ "$\mu_{\mathbb{C}}^{-1}(0)/\!/\mathbb{T}_{\mathbb{C}}^d$"

Study the "universal covering" of Y !

Motivation 1 - the "singular" universal cover

Conjecture

$$|\pi_1(Y_{\mathrm{reg}})| < \infty.$$

Proposition (The existence of the universal cover)

Assume $|\pi_1(Y_{\mathrm{reg}})| < \infty$.

Then, $\exists!$a conical symplectic variety $(\widetilde{Y}, \widetilde{\omega})$ satisfying

$$\begin{array}{ccc} (\widetilde{Y}, \widetilde{\omega}) & \supset & \varphi^{-1}(Y_{\mathrm{reg}}) \\ \downarrow \exists!\varphi \ : \ \text{finite} & & \downarrow \varphi| \ : \ \text{universal cover} \\ (Y, \omega) & \supset & Y_{\mathrm{reg}} \end{array}.$$

Problem 1

What are $\pi_1(Y_{\mathrm{reg}})$ and the universal cover $\widetilde{Y}$?

Motivation 2 - Bogomolov's decomposition

Problem 2 ([Namikawa] Analogue of Bogomolov's decomposition)

For any conical sympl. variety (Y, ω) with $|\pi_1(Y_{\mathrm{reg}})| < \infty$,

$$(\widetilde{Y}, \widetilde{\omega}) \cong \exists\textbf{?} \prod_{m=1}^{r} (Y_m, \omega_m),$$

where (Y_m, ω_m) is *irreducible*, i.e., ω_m is the unique sympl. form.

Goal

Answer these problems for hypertoric varieties.

Hypertoric varieties $Y_A(\alpha)$

$$\textbf{Combinatorics} \longrightarrow \textbf{Geometry}$$

$$\{\text{polytope } \mathcal{P}_B^\alpha\} \longrightarrow \{\text{toric variety } X_A(\alpha)\}$$

$$\{\text{hyperplane arrangement } \mathcal{H}_B^\alpha\} \longrightarrow \{\text{hypertoric variety } Y_A(\alpha)\}$$

What is hypertoric variety?

Example

Let $B^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{pmatrix}$ and $\tilde{\alpha} = (1, 1, 1)$.

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Set $\mathcal{H}_B^\alpha := \{H_i : \langle \boldsymbol{b_i}, - \rangle = -\tilde{\alpha}_i\}$.

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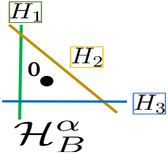

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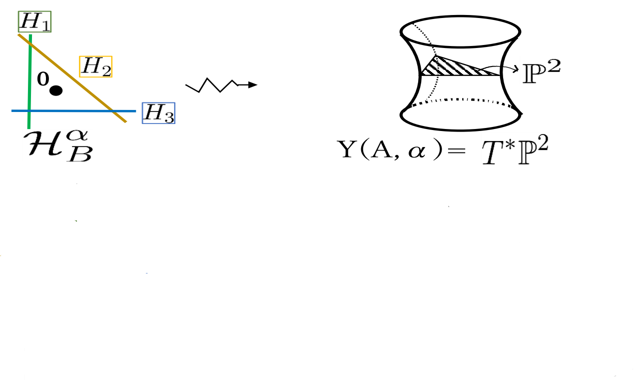

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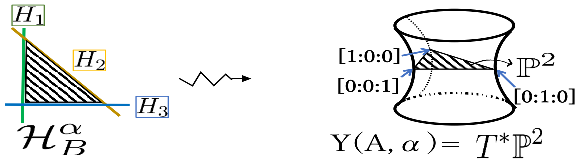

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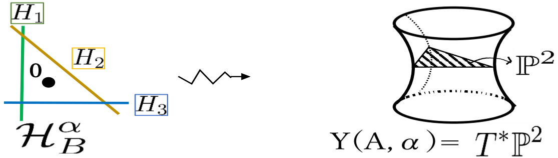

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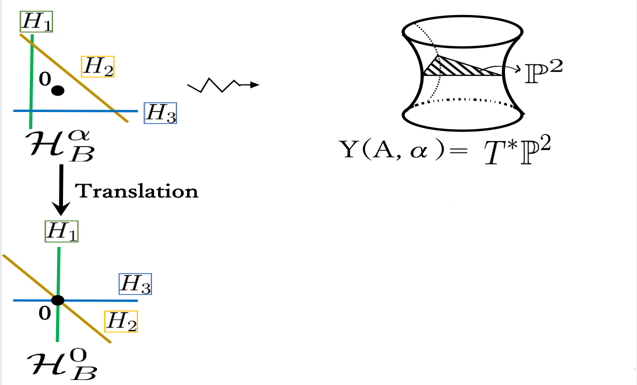

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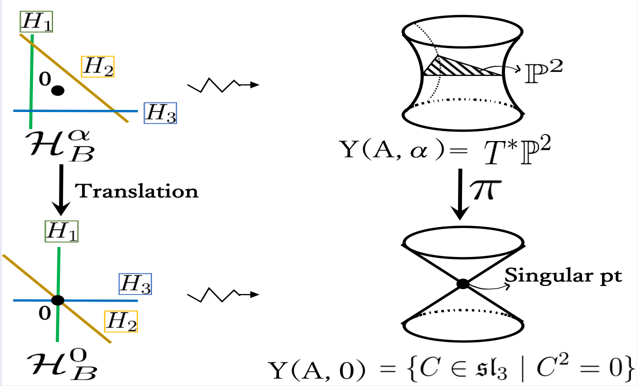


Definition of hypertoric variety as symplectic reduction

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} N_A := \mathbb{Z}^d \longrightarrow 0$$

$$T_A := \mathbb{T}^d_{\mathbb{C}} \overset{A^T}{\hookrightarrow} \mathbb{T}^n_{\mathbb{C}} \curvearrowright (\mathbb{C}^{2n} = \mathbb{C}^n \oplus (\mathbb{C}^n)^*, \omega_{\mathbb{C}})$$

$$\downarrow \mu_{\mathbb{C}} : \mathbb{T}^n_{\mathbb{C}}\text{-inv. moment map}$$

$$\mathbb{C}^d$$

Definition (Hypertoric variety)

$$\boldsymbol{Y_A(\alpha)} := \mu_{\mathbb{C}}^{-1}(0) - \{\alpha\text{-unstable pts}\} /\!/ T_A \; : \text{ hypertoric variety}$$

$(\mathcal{H}^{\alpha}_B := \{H_i : \langle \boldsymbol{b_i}, - \rangle = -\tilde{\alpha}_i\} \subset \mathbb{R}^{n-d} :$ the associated arrangement)

Fact

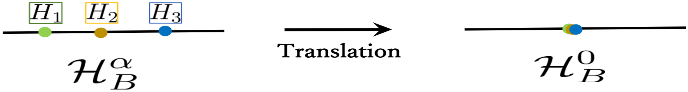
1. $\mathbb{T}^{n-d}_{\mathbb{C}} \curvearrowright (Y_A(0), \omega)$ is a $2(n-d)$ dim conical sympl. var.
2. $\exists \pi_\alpha : (Y_A(\alpha), \omega) \to (Y_A(0), \omega)$ is a resolution for **generic** α.

Example

Example (A_2-type surface singularity $\mathbb{C}^2/\mathbb{Z}_3$)

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix},\ B^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{ccccl} Y_A(\alpha) & \xrightarrow{\sim} & \widetilde{\mathbb{C}^2/\mathbb{Z}_3} & & : \text{the minimal resolution of } S_{A_2} \\ \downarrow \pi_\alpha & & \downarrow & & \\ Y_A(0) & \xrightarrow{\sim} & \mathbb{C}^2/\mathbb{Z}_3 & \cong \{u^3 - xy = 0\} & : A_2\text{-type singularity} \end{array}$$



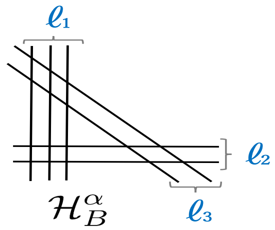
4-dimensional classification

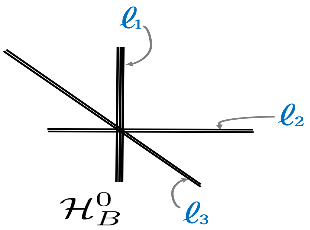
Fact (Classification of 4-dimensional $Y_A(0)$)

(i) $\mathbb{C}^2/\mathbb{Z}_{\ell_1} \times \mathbb{C}^2/\mathbb{Z}_{\ell_2}$.

(ii) $\overline{\mathcal{O}^{\min}}(\ell_1, \ell_2, \ell_3) :=$

$$\left\{ \begin{pmatrix} u_1 & x_1 & x_3 \\ y_1 & u_2 & x_2 \\ y_3 & y_2 & u_3 \end{pmatrix} \in \mathfrak{sl}_3 \;\middle|\; \text{All } 2\times 2\text{-minors of } \begin{pmatrix} u_1^{\ell_1} & x_1 & x_3 \\ y_1 & u_2^{\ell_2} & x_2 \\ y_3 & y_2 & u_3^{\ell_3} \end{pmatrix} = 0 \right\}.$$

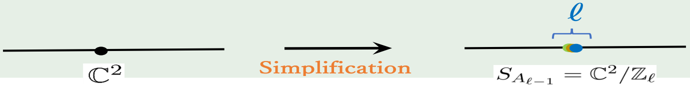




$\pi_1(Y_A(0)_{\mathrm{reg}})$ & universal coverings

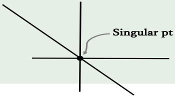
Example ($Y_A(0) := \mathbb{C}^2/\mathbb{Z}_\ell \cong \{xy - u^\ell = 0\}$)

$\mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}_\ell =: Y_A(0)$ is universal cover & $\pi_1(Y_A(0)_{\mathrm{reg}}) = \mathbb{Z}_\ell$.

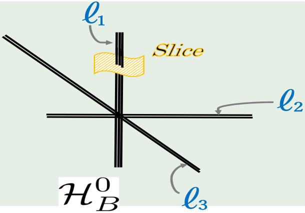


Example ($Y_A(0) := \overline{\mathcal{O}^{\min}_{A_2}}$)

4-dim isolated singularity & $\pi_1((\overline{\mathcal{O}^{\min}_{A_2}})_{\mathrm{reg}}) = 1$.



$\mathrm{Sing}(Y_A(0)) \leftrightarrow$ "multiplicated" or "non-general" locus of $\mathcal{H}^0_B$.



The slice $\cong \mathbb{C}^2/\mathbb{Z}_{\ell_1}$.

$\rightsquigarrow \pi_1(Y_A(0)_{\mathrm{reg}}) \neq 1$? (if $\ell_1 \geq 2$).

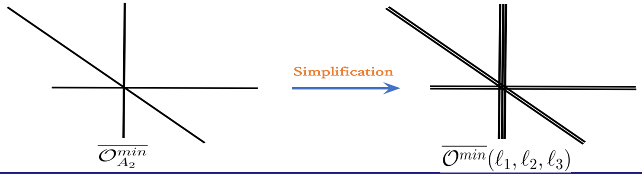
Simplification

We can assume $B^T = \big(\overbrace{\boldsymbol{b}^{(1)} \dots \boldsymbol{b}^{(1)}}^{\ell_1} \dots \overbrace{\boldsymbol{b}^{(s)} \dots \boldsymbol{b}^{(s)}}^{\ell_s}\big)$

Lemma

$\ell_1 = \cdots = \ell_s = 1$, i.e., $Y_A(0)$ is simple $\Rightarrow$ $\pi_1(Y_A(0)_{\mathrm{reg}}) = 1$.

$\rightsquigarrow$ Consider the **simplification** $\overline{\boldsymbol{B}}^{\boldsymbol{T}} := (\boldsymbol{b}^{(1)} \dots \boldsymbol{b}^{(s)})$



Natural expectation

$\exists$? finite quotient $\varphi : Y_{\overline{\boldsymbol{A}}}(0) \to Y_A(0)$: universal cover,

where $0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{\overline{B}} \mathbb{Z}^s \xrightarrow{\overline{\boldsymbol{A}}} N_{\overline{A}} := \mathbb{Z}^{d-(n-s)} \longrightarrow 0$: exact.

The fundamental group $\pi_1(Y_A(0)_{\mathrm{reg}})$

Theorem ([N.])

$\exists$ finite $\varphi : Y_{\overline{A}}(0) \twoheadrightarrow Y_A(0)$: universal cover, and

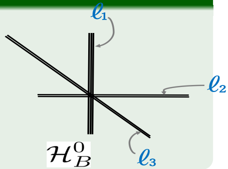
$$\pi_1(Y_A(0)_{\mathrm{reg}}) \cong \Gamma/\Gamma \cap T_{\overline{A}},$$

where $\Gamma := \prod_{k=1}^{s} \mathbb{Z}/\ell_k\mathbb{Z} \longrightarrow \mathbb{T}^s_{\mathbb{C}} \xleftarrow{\overline{A}^T} T_{\overline{A}} := \mathbb{T}^{d-(n-s)}_{\mathbb{C}}$.

Example ($\overline{\mathcal{O}^{\min}}(\ell_1, \ell_2, \ell_3)$)

$$\pi_1(\overline{\mathcal{O}^{\min}}(\ell_1, \ell_2, \ell_3)_{\mathrm{reg}}) \cong \frac{\mathbb{Z}_{\ell_1} \times \mathbb{Z}_{\ell_2} \times \mathbb{Z}_{\ell_3}}{\mathbb{Z}\langle \left(\frac{\ell_1}{g}, \frac{\ell_2}{g}, \frac{\ell_3}{g}\right) \rangle},$$

where $g := \gcd(\ell_1, \ell_2, \ell_3)$.



Bogomolov's decomposition

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Problem ([Namikawa] Analogue of Bogomolov's decomposition)

For any conical sympl. var. (Y, ω) with $|\pi_1(Y_{\mathrm{reg}})| < \infty$,

$$(\widetilde{Y}, \widetilde{\omega}) \cong \exists ? \prod_{m=1}^{r} (Y_m, \omega_m), \text{ where } (Y_m, \omega_m) \text{ is } \textit{irreducible}.$$

A trivial example

$(\mathbb{C}^{2n}, \omega_{\mathbb{C}}) \cong \prod_{m=1}^{n} (\mathbb{C}^2, dz_m \wedge dw_m)$.

If A admits a block decomposition $A = \bigoplus_{m=1}^{r} A_m$, then clearly

$$Y_A(0) \cong \prod_{m=1}^{r} Y_{A_m}(0).$$

Naive Question

A is block indecomposable

$\overset{?}{\Leftrightarrow}$ $Y_A(0)$ is an irreducible conical symplectic variety.

Bogomolov's decomposition

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Theorem ([N.])

A is block indecomposable
$\Leftrightarrow$ $Y_A(0)$ is an *irreducible* conical symplectic variety.

Byproduct - Combinatorial classification

Theorem (Berchtold)

$\forall$ toric varieties X and X' of dim $n-d$,

$$X \cong X' \Leftrightarrow X \cong X' : \mathbb{T}_{\mathbb{C}}^{n-d}\text{-equivariant}.$$

Corollary ([N.])

$\forall$ two smooth (or affine) hypertoric $Y_A(\alpha)$ and $Y_{A'}(\alpha')$, TFAE:

(i) $Y_A(\alpha) \cong Y_{A'}(\alpha')$: $\mathbb{C}^*$-equivariant.

(ii) $(Y_A(\alpha), \omega) \cong (Y_{A'}(\alpha'), \omega')$: $\mathbb{C}^* \times \mathbb{T}_{\mathbb{C}}^{n-d}$-equivariant.

(iii) $\mathcal{H}_B^{\alpha} \cong \mathcal{H}_{B'}^{\alpha'}$, i.e., induces the same "fan".

Thank you for listening !