On the topology of some torus invariant subsets in the Grassmannians

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1 Toric varieties in $G_k(\mathbb{C}^n)$

(I.M. Gel'fand, and R.D. MacPherson)
Geometry in Grassmannians and a Generalization of the Dilogarithm, 1982

(I.M. Gel'fand, R.M. Goresky, R.D. MacPherson, and V.V. Serganova)
Combinatorial geometries, convex polyhedra, and Schubert cells, 1987.

(Victor M. Buchstaber and Svjetlana Terzić)

Toric topology of the complex grassmann manifolds, 2018.

1 Toric varieties in $G_k(\mathbb{C}^n)$

$G_k(\mathbb{C}^n)$: set of all k-dimensional vector subspaces in $\mathbb{C}^n$

$(\mathbb{C}^*)^n$: algebraic torus, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

$(\mathbb{C}^*)^n \curvearrowright G_k(\mathbb{C}^n)$

For an orbit $\mathcal{O} \subset G_k(\mathbb{C}^n)$, the closure $\overline{\mathcal{O}}$ is a toric variety.

1 Toric varieties in $G_k(\mathbb{C}^n)$

$\exists \mu$ (moment map) $: G_k(\mathbb{C}^n) \to \mathbb{R}^n$ s.t.

- $\mu(\overline{\mathcal{O}})$ is a (convex) polytope in $\mathbb{R}^n$,
  $\{\text{fixed points of } \overline{\mathcal{O}}\} \longleftrightarrow \{\text{vertices of } \mu(\overline{\mathcal{O}})\}$;
- $\overline{\mathcal{O}}$ is smooth $\Leftrightarrow \mu(\overline{\mathcal{O}})$ is a simple polytope.

Question. Which $\overline{\mathcal{O}}$'s are smooth?

$\to$ Find simple $\mu(\overline{\mathcal{O}})'s$.

1 Toric varieties in $G_k(\mathbb{C}^n)$

Actually, $\mu(\overline{\mathcal{O}})$ is a *matroid polytope*.

(combinatorial)Theorem (Noji-O).

A simple matroid polytope is a product of simplices.

(geometric)Theorem (Noji-O).

If $\overline{\mathcal{O}}$ is smooth, then $\overline{\mathcal{O}}$ is a product of complex projective spaces.

$(\mu(\overline{\mathcal{O}}) = \Delta^{d_1} \times \cdots \times \Delta^{d_r} \;\Rightarrow\; \overline{\mathcal{O}} = \mathbb{C}P^{d_1} \times \cdots \times \mathbb{C}P^{d_r})$

1 Remark on our results

We found a paper which prove the following:

Theorem.

Every simple 0/1-polytope is the (cartesian) product of some 0/1-simplices. (Volker Kaibel and Martin Wolff, Simple 0/1-polytopes, Europ. J. Combinatorics (2000) 21, 139–144.)

Matroid polytopes are 0/1-polytopes and this easily implies our combinatorial Theorem in the previous slide.

However, at least for matroid polytopes, I and Noji proved a stronger statement in our paper to prove our combinatorial Theorem in the previous slide.

1 Toric varieties in $G_k(\mathbb{C}^n)$

$\exists \mu$ (moment map) : $G_k(\mathbb{C}^n) \to \mathbb{R}^n$ s.t.

- $\mu(\overline{\mathcal{O}})$ is a (convex) polytope in $\mathbb{R}^n$,
  $\{\text{fixed points of } \overline{\mathcal{O}}\} \longleftrightarrow \{\text{vertices of } \mu(\overline{\mathcal{O}})\}$;

- $\overline{\mathcal{O}}$ is smooth $\Leftrightarrow$ $\mu(\overline{\mathcal{O}})$ is a simple polytope.

One viewpoint.

The more edges emanate from a vertex of $\mu(\overline{\mathcal{O}})$, the "more singular " $\overline{\mathcal{O}}$ is, at the corresponding fixed point.

Question.

How many edges do emanate from a vertex of $\mu(\overline{\mathcal{O}})$?

We may assume that $\overline{\mathcal{O}}$ is non-degenerate i.e. $\dim\ \overline{\mathcal{O}} = n-1$,

because

$$\dim\ \overline{\mathcal{O}} = n - r$$
$$\Rightarrow$$
$$\overline{\mathcal{O}} = \overline{\mathcal{O}}_1 \times \cdots \times \overline{\mathcal{O}}_r$$

and each $\overline{\mathcal{O}}_i$ can be regarded as non-degenerate torus orbit closure in $G_{k_i}(\mathbb{C}^{n_i})$.

1 Toric varieties in $G_k(\mathbb{C}^n)$

When $k = 0$ or 1,
$G_k(\mathbb{C}^n) = \{\text{a point}\}$.

When $k = 1$ or $n - 1$,
$G_k(\mathbb{C}^n) =$ complex projective space.

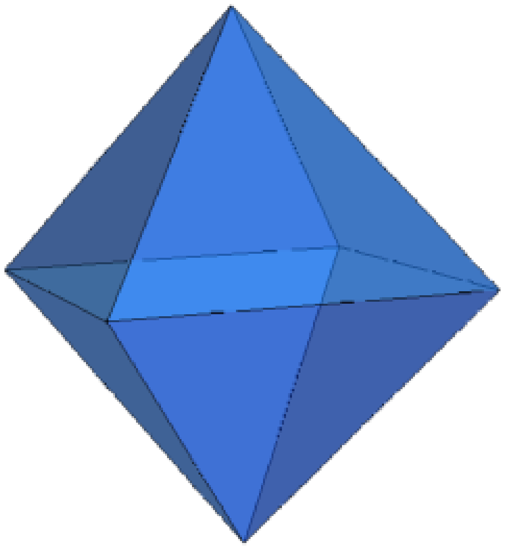
And then (non-degenerate $\overline{\mathcal{O}}$)$= G_k(\mathbb{C}^n)$.
($\because \mathcal{O}$ is unique open dense orbit in $G_k(\mathbb{C}^n)$,
since $G_k(\mathbb{C}^n)$ is the complex projective space)

Thus, we consider the cases when $2 \leq k \leq n - 2$.

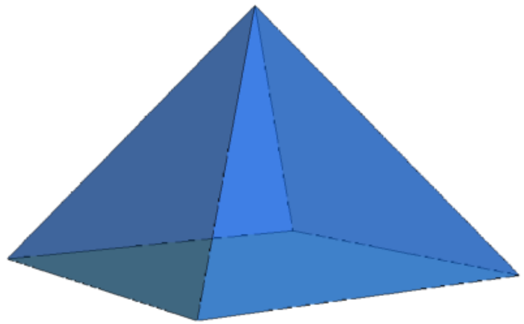
2 $\mu(\overline{\mathcal{O}})$ for non-degenerate $\overline{\mathcal{O}} \subset G_2(\mathbb{C}^4)$

$\overline{\mathcal{O}} \subset G_2(\mathbb{C}^4)$

$\mu(\exists\overline{\mathcal{O}}) =$ Octahedron



$\mu(\exists\overline{\mathcal{O}}) =$ Pyramid



3 A iniquality for the number of edges from a vertex

For a polytope P, let

$$D(P) = \max_{v \in V(P)} d_v - \dim P.$$

$V(P)$: set of all vertices of polytope and

d_v: the number of edges which emanate from v

Then $$P \text{ is simple} \Leftrightarrow D(P) = 0.$$

“Theorem” (O).

When $2 \leq k \leq n-2$, for non-degenerate $\overline{\mathcal{O}}$

$$\sqrt[3]{\frac{3}{4}(n-3)} - 1 \leq D(\mu(\overline{\mathcal{O}})).$$

3 A iniquality for the number of edges from a vertex

When $5 \leq n$,

$$0 < \sqrt[3]{\frac{3}{4}(5-3)} - 1 \leq \sqrt[3]{\frac{3}{4}(n-3)} - 1 \leq D(\mu(\overline{\mathcal{O}})).$$

This means that $\overline{\mathcal{O}}$ is not smooth when $5 \leq n$,

which reprove the previous theorem.

Remark.

More generally, the above assertion holds for matroid polytopes

of connected matroids on $[n]$ of rank k.

4 Remark on our viewpoint

Other viewpoint can be taken. Indeed, we can think that the more facets intersect at a vertex of $\mu(\overline{\mathcal{O}})$, the "more singular" $\overline{\mathcal{O}}$ is, at corresponding fixed point.

For generic $\overline{\mathcal{O}} \subset G_k(\mathbb{C}^n)$,
$\mu(\overline{\mathcal{O}}) = \Delta_{n,k}$ is called a *hypersimlex*.

The $\Delta_{n,k}$ has n facets at each vertex
while it has $k(n-k)$ edges at each vertices.

Since for a general $(n-1)$-dimensional polytope, the minimum number of its edgs and facets at a vertex of it are both $n-1$, the two viewpoints give different impressions of the singularity of $\overline{\mathcal{O}}$.