Topological contact toric manifolds

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(joint work with S. Sarkar)
arXiv:1909.00994 [math.AT]

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(Workshop for Young Researchers)
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Questions

Given two topological spaces X_1 and X_2...

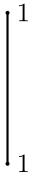
Cohomological rigidity

$$H^*(X_1) \cong H^*(X_2) \quad \overset{?}{\Longrightarrow} \quad {}^{\exists}f\colon X_1 \xrightarrow{\cong} X_2$$

If both X_1 and X_2 are equipped with G-actions...

Equivariant Cohomological Rigidity

$$H_G^*(X_1) \cong H_G^*(X_2) \quad \overset{?}{\Longrightarrow} \quad \begin{array}{ccc} G \times X_1 & \xrightarrow[\cong]{\theta \times f} & G \times X_2 \\ \downarrow & \circlearrowright & \downarrow \\ X_1 & \xrightarrow[\cong]{f} & X_2 \end{array}$$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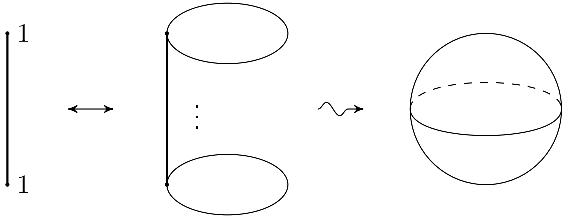

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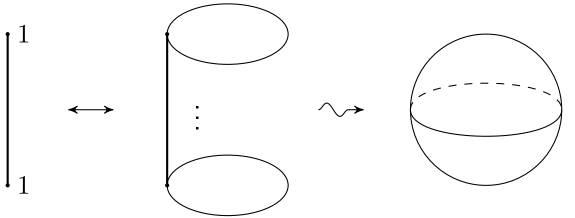

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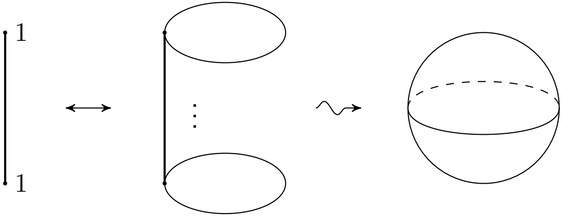

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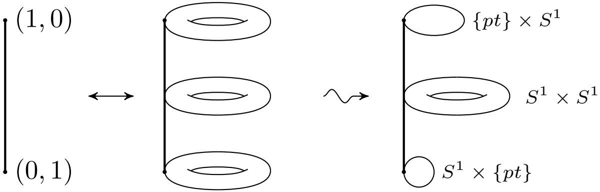
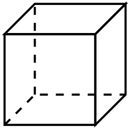


$$= I \times T^2/_{\sim}$$
$$= S^1 * S^1$$
$$= S^3$$

Objects (by examples)

$$T^3 \curvearrowright X^6$$

$$\pi \downarrow$$



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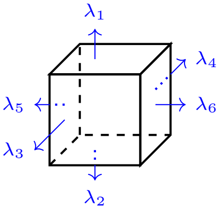

$$\lambda\colon \mathcal{F}(P^3) \to \mathbb{Z}^3$$
$$F_i \mapsto \lambda_i$$

such that

$\{\lambda_i, \lambda_j, \lambda_k\}$: $\mathbb{Z}$-basis of $\mathbb{Z}^3$,

whenever $F_i \cap F \cap F_k \neq \emptyset$.

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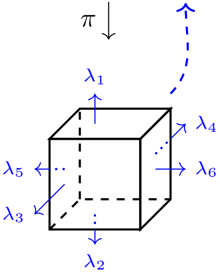

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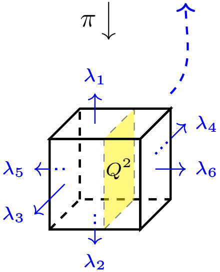

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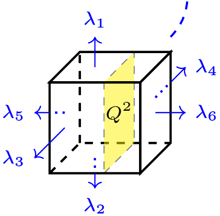

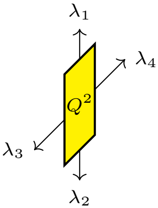

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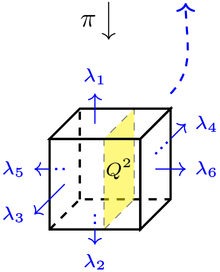

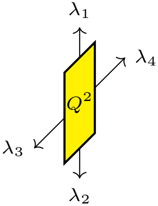

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$\rightsquigarrow$

$\widetilde{\lambda}\colon \mathcal{F}(Q^2) \to \mathbb{Z}^3$
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$\mathrm{span}_{\mathbb{Z}}\{\lambda_i, \lambda_j\}$: direct summand of $\mathbb{Z}^3$,
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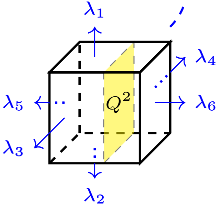
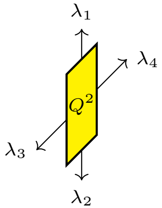
Objects (by examples)

$$T^3 \curvearrowright X^6 \cong (P^3 \times T^3)/_\sim \qquad \supset \qquad (Q^2 \times T^3)/\sim$$

$$\pi \downarrow \qquad\qquad\qquad\qquad \downarrow \pi|_{\pi^{-1}(Q)}$$





$$\lambda \colon \mathcal{F}(P^3) \to \mathbb{Z}^3 \qquad\qquad \widetilde{\lambda} \colon \mathcal{F}(Q^2) \to \mathbb{Z}^3$$

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span$_{\mathbb{Z}}\{\lambda_i, \lambda_j\}$: direct summand of $\mathbb{Z}^3$, whenever $F_i \cap F_j \neq \emptyset$.

Objects

Constructive definition

- Q^n: n-dimensional simple polytope,
- $\lambda\colon \mathcal{F}(Q^n) \to \mathbb{Z}^{n+1}$ such that
  $\mathrm{span}_{\mathbb{Z}}\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\}$: direct summand of $\mathbb{Z}^{n+1}$, whenever $F_{i_1} \cap \cdots \cap F_{i_\ell} \neq \emptyset$.

$$X^{2n+1} = Q^n \times T^{n+1}/_{\sim}$$

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Axiomatic definition

A smooth manifold X^{n+k} with effective T^k-action s.t.

- locally isomorphic to $T^n \times T^{k-n} \curvearrowright \mathbb{C}^n \times T^{k-n}$,
- X^{n+k}/T^k is homeomorphic to a simple polytope P^n as manifold with corners.

Examples

Two extreme cases:

$\boldsymbol{k = n}$**,** {quasitoric manifolds} ($\supset$ {symplectic toric manifolds})

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$$\begin{cases} \lambda(F_i) = e_i & i = 1, \dots, n \\ \lambda(F_{n+1}) = (-q_1, \dots, -q_n, p) & \gcd(q_1, p) = \cdots = \gcd(q_n, p) = 1, \end{cases}$$

$$X^{2n+1} = L(p; q_1, \dots, q_n).$$

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$k = n + 1$, {topological contact toric manfiolds}

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$\boldsymbol{k = n+2}$**,** hyperplane cut of t.c.t.m.

⋮

Contact toric manifolds

Definition (Conact toric manifold)

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Theorem [Lerman, 2002] The classification of c.c.c.t.

	$T^{n+1} \curvearrowright M$ freely	$T^{n+1} \curvearrowright M$ not freely
$\dim M = 3$	T^3	Lens space
$\dim M > 3$	Principal T^{n+1}-bundle over S^n	Classified by moment cone.

Moment cone

- (M, α): contact G-manifold,
- $(M \times \mathbb{R}, d(e^t \alpha))$: symplectization of (M, α),
- $G \curvearrowright M \times \mathbb{R}$, (trivially on $\mathbb{R}$.)
- $\Phi \colon M \times \mathbb{R} \to \mathfrak{g}^*$ moment map.

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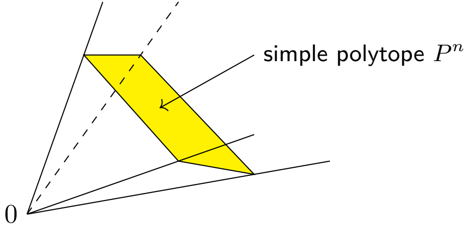
- When $G = T^{n+1}$, $\Phi(M \times \mathbb{R}) \cup \{0\} = C(P^n)$ such that...



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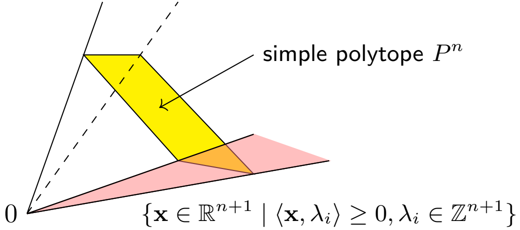

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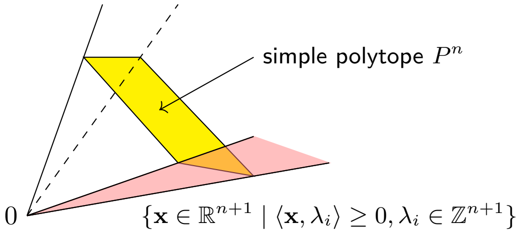

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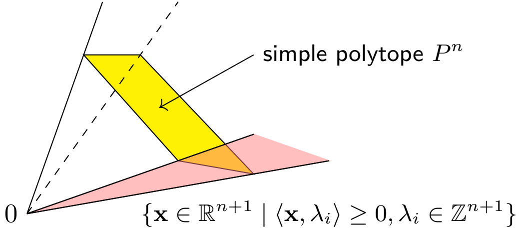

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- $M \cong P^n \times T^{n+1}/_\sim$, (T^{n+1}-equivariantly homeomorphic.)

Answer for the equivariant cohomological rigidity

For $T^{n+k} \curvearrowright X_1^{n+k}$ and $T^{n+k} \curvearrowright X_2^{n+k}$,

(1) When $k = n$:

Theorem (Masuda, 2008)

X_1, X_2: quasitoric manifolds.
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(2) When $k = m$ and $\lambda\colon \mathcal{F}(Q) \to \mathbb{Z}^m$ is given by $F_i \mapsto e_i$:

Theorem (Davis–Januszkiewicz, 1991)

$H^*_{T^m}(\mathcal{Z}_Q) \cong \mathrm{SR}[Q]$ ring isom.

(3) When $k = n+1$,

Theorem (Sarkar-S, 2019, arXiv)

Let X_1 and X_2 be two topological contact toric manifolds. Then,

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«**Thank you for your attention.**»