Factorial Flagged Grothendieck Polynomials

Shogo Sugimto

November 22

Flagged partition and Set valued tableau

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flagged set valued tableau of (λ, f): i-th row's entries are not greater than f_i (Denote $FSVT(\lambda, f)$ the set of flgged set valued tableaux of (λ, f))

ex : $\lambda = (2, 1), f = (2, 2)$

$FSVT(\lambda, f) =$

1	1
2	

T_1

1	2
2	

T_2

1	1, 2
2	

T_3

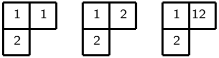
$\beta, x_i, b_i, i \in \mathbb{N}$: variables

$$[x|b]^T := \prod_{e \in T} x_e \oplus b_{e+c(e)-r(e)} \qquad u \oplus v := u + v + \beta uv.$$

$$[x|b]^{T_1} = (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_1)$$

$$[x|b]^{T_2} = (x_1 \oplus b_1)(x_2 \oplus b_3)(x_2 \oplus b_1)$$

$$[x|b]^{T_3} = (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_3)(x_2 \oplus b_1)$$



T_1 T_2 T_3

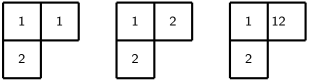
Definition factorial flagged Grothendieck polynomial $G_{\lambda,f}$

$$G_{\lambda,f}(x|b) = \sum_{T \in FSVT(\lambda,f)} \beta^{|T|-|\lambda|}[x|b]^T \tag{1}$$

$|T|$ is the total number of entries in T, $|\lambda|$ is the number of boxes of λ .

ex: $\lambda = (2,1), f = (2,2)$

$$\begin{aligned} G_{\lambda,f}(x|b) &= (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_1) + (x_1 \oplus b_1)(x_2 \oplus b_3)(x_2 \oplus b_1) \\ &\quad + \beta^1 (x_1 \oplus b_1)(x_1 \oplus b_2)(x_2 \oplus b_3)(x_2 \oplus b_1) \end{aligned}$$



T_1 T_2 T_3

Main Theorem

$$G_{\lambda,f}(x|b) = \det\left(\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s \mathcal{G}_{\lambda_i-i+j+s}^{[f_i,f_i+\lambda_i-i]}\right)_{1\le i,j\le r} \quad (=: \widetilde{G}_{\lambda,f}(x|b)) \qquad (2)$$

$\mathcal{G}_m^{[p,r]}$ is defined by the generating function

$$\sum_{m\in\mathbb{Z}} \mathcal{G}_m^{[p,r]} u^m = \frac{1}{1+\beta u^{-1}} \prod_{i=1}^{p} \frac{1+\beta x_i}{1-x_i u} \prod_{i=1}^{r} (1+(u+\beta)b_i).$$

generalized binomial coefficient

$$(1+u)^m = \sum_{i\ge 0}\binom{m}{i}u^i$$

Corollary

$m \ge 0,\ p > 0$ の時

$$\mathcal{G}_m^{[p,p+m-1]} = \sum_{T\in FSVT((m),(p))} [x|b]^T$$

Background in Geometry

A flag variety $Fl_n(\mathbb{C}) = \{U_\bullet = (U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset \mathbb{C}^n) | dim U_i = i\} = GL_n(\mathbb{C})/B$

$\Omega_w = \overline{B_- wB} \subset Fl_n(\mathbb{C})$ schubert variety $\quad (w \in S_n)$

Torus-equivariant K-theory $\quad K_T(Fl_n(\mathbb{C})) :=$ Grothendieck group of T-equiv. coherent sheaves

$\mathcal{O}_{\Omega_w}$: structure sheaf of Ω_w

$\Rightarrow [\mathcal{O}_{\Omega_w}] \in K_T(Fl_n(\mathbb{C}))$ equivariant Schubert class (a basis as $K_T(pt)$-module)

Schubert Calculus

Compute $c^u_{w,v} \in K_T(pt)$ where

$$[\mathcal{O}_{\Omega_w}] \cdot [\mathcal{O}_{\Omega_v}] = \sum_{u \in S_n} c^u_{w,v} [\mathcal{O}_{\Omega_u}]$$

Equivariant Schubert classes and double Grothendieck polynomials

(K-theoretic) divided difference operator π_i

$$\pi_i(f) = \frac{(1+\beta x_{i+1})f - (1+\beta x_i)s_i(f)}{x_i - x_{i+1}}$$

where s_i permutes x_i and x_{i+1}.

double Grothendieck polynomials (Lascoux–Schützenberger)

$w_0 \in S_n$:longest element

$\Rightarrow \mathcal{G}_{w_0}(x|b) = \prod_{i+j\leq n}(x_i \oplus b_j)$

$w \neq w_0$ $l(w) = l(ws_i) - 1$

$\Rightarrow \mathcal{G}_w(x|b) = \pi_i(\mathcal{G}_{ws_i})$

fact

$$\mathcal{G}_w(x|b) \text{``="} [\mathcal{O}_{\Omega_w}]$$

Background

purpose

Want to find an explicit formula of $\mathcal{G}_w$.

Known facts:

If $w \in S_n$ is vexillary, then

(1)$\mathcal{G}_w = G_{\lambda(w),f(w)}$ (Knutson–Miller–Yong) (tableau formula)

(2)$\mathcal{G}_w = \widetilde{G}_{\lambda(w),f(w)}$ (Hudson–Matsumura, Anderson)(determinant formula)

Goal of this talk:

Define $G_{\lambda,f}$ in general by tableau formula and prove its determinant formula.

Note:

When $b = 0$ (non-factorial), it was shown by Matsumura. We generalize his proof to the factorial case.

Proof of the main theorem 1

Main Theorem

$$G_{\lambda,f}(x|b) = \det\left(\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s \mathcal{G}_{\lambda_i-i+j+s}^{[f_i,f_i+\lambda_i-i]}\right)_{1\le i,j\le r} \qquad (=: \widetilde{G}_{\lambda,f}(x|b))$$

proof) We prove the theorem by induction on $|f| = f_1 + f_2 + \cdots + f_r$.

(1) $|f| = 1$ (base case) :

then $\lambda = (\lambda_1), f = (1)$

so it follows from Lemma 1.

Lemma 1

For any $d \in \mathbb{N}$

$$\left(G_{(d),(1)} =\right) \prod_{i=1}^{d}(x_1 \oplus b_i) = \mathcal{G}_d^{[1,d]}.$$

Proof of the main theorem2

(2) When $f_1 = 1$: it follows from Lemma 2 by the induction hypothesis .

Lemma 2

If $f_1 = 1$

(1) $\widetilde{G}_{\lambda,f}(x|b) = \mathcal{G}_{\lambda_1}^{[1,\lambda_1]} \cdot \left(\widetilde{G}_{\lambda',f'}(x|b)\right)^{\star}$

(2) $G_{\lambda,f}(x|b) = \mathcal{G}_{\lambda_1}^{[1,\lambda_1]} \cdot (G_{\lambda',f'}(x|b))^{\star}$

Where ,

- $\lambda' = (\lambda_2, \lambda_3 \cdots, \lambda_r)$, $f' = (f_2 - 1, f_3 - 1, \cdots, f_r - 1)$
- $\star$ replaces each x_i by x_{i+1}.

Proof of the main theorem3

(3) If $f_1 > 1$ the claim follows from Lemma 3 by induction hypothesis .

Lemma3

If $f_1 > 1$

(1) $\widetilde{G}_{\lambda,f}(x|b) = \pi_{f_1-1}\widetilde{G}_{\lambda',f'}(x|b)$

(2) $G_{\lambda,f}(x|b) = \pi_{f_1-1}G_{\lambda',f'}(x|b)$

ただし $\lambda' = (\lambda_1+1, \lambda_2, \cdots, \lambda_r), f' = (f_1-1, f_2, \cdots, f_r)$ とする.

Proof of Lemma 1

Lemma 1

For any $d \in \mathbb{N}$

$$\left(G_{(d),(1)} =\right) \prod_{i=1}^{d} (x_1 \oplus b_i) = \mathcal{G}_d^{[1,d]}.$$

proof) We prove Lemma 1 by induction on d. When $d = 1$. From the generating function $\mathcal{G}_m^{[1,1]}$

$$\sum_{m\in\mathbb{Z}} \mathcal{G}_m^{[1,1]} u^m := \frac{1}{1+\beta u^{-1}} \frac{1+\beta x_1}{1-x_1 u} (1 + (u+\beta) b_1) = \left(\sum_{m\in\mathbb{Z}} \mathcal{G}_m^{[1,0]} u^m \right) (1 + \beta b_1 + b_1 u)$$

See u's coefficient ($\mathcal{G}_m^{[1,0]} = x_1^m, m \geq 0$)

$$(1+\beta b_1) x_1 + b_1 = x_1 + b_1 + \beta x_1 b_1 = x_1 \oplus b_1$$

Proof of Lemma 1

Suppose $d > 0$. By definition of $\mathcal{G}_m^{[1,d+1]}$

$$\sum_{m\in\mathbb{Z}} \mathcal{G}_m^{[1,d+1]} u^m = \left(\sum_{m\in\mathbb{Z}} \mathcal{G}_m^{[1,d]} u^m \right) (1 + \beta b_{d+1} + b_{d+1} u).$$

See the coefficients of u^{d+1} , $(\mathcal{G}_{d+1}^{[1,d]} = x_1 \mathcal{G}_d^{[1,d]})$,

$$\mathcal{G}_{d+1}^{[1,d+1]} = \mathcal{G}_{d+1}^{[1,d]}(1 + \beta b_{d+1}) + \mathcal{G}_d^{[1,d]} b_{d+1} = \mathcal{G}_d^{[1,d]}(x_1 \oplus b_{d+1}) = \prod_{i=1}^{d+1}(x_1 \oplus b_i).$$

(By induction hypothesis $\mathcal{G}_d^{[1,d]} = \prod_{i=1}^{d}(x_1 \oplus b_d)$.)

Proof of Lemma 2

Lemma 2

If $f_1 = 1$

(1) $\widetilde{G}_{\lambda,f}(x|b) = \mathcal{G}_{\lambda_1}^{[1,\lambda_1]} \cdot \left(\widetilde{G}_{\lambda',f'}(x|b)\right)^{\star}$

$\lambda' = (\lambda_2, \lambda_3 \cdots, \lambda_r)$, $f' = (f_2 - 1, f_3 - 1, \cdots, f_r - 1)$, $\star$ replaces each x_i by x_{i+1}.

proof)

- $\mathcal{G}_{\lambda_1}^{[1,\lambda_1]}$ is the (1,1) entry of the determinant $\widetilde{G}_{\lambda,f}(x|b)$
- the column operation (the j-th column) $-$[(the (j-1)-st column) $\times\ x_1(1 + x_1\beta)^{-1}$]

  $(j = 2, 3, \cdots, r)$

  $\Rightarrow$ The first row becomes $(\mathcal{G}_{\lambda_1}^{[1,\lambda_1]}, 0, \cdots, 0)$.
- cofactor expansion along the first row

Proof of Lemma 2

Lemma 2

If $f_1 = 1$

(2) $G_{\lambda,f}(x|b) = \mathcal{G}_{\lambda_1}^{[1,\lambda_1]} \cdot (G_{\lambda',f'}(x|b))^{\star}$

$\lambda' = (\lambda_2, \lambda_3 \cdots, \lambda_r)$, $f' = (f_2 - 1, f_3 - 1, \cdots, f_r - 1)$, $\star$ replaces each x_i by x_{i+1}.

proof)

- the entries on the first row of $T \in FSVT(\lambda, f)$ are all 1
- $\prod_{i=1}^{\lambda_1}(x_1 \oplus b_i) = \mathcal{G}_d^{[1,\lambda_1]}$ is the common factor

Lemma 3

If $f_1 > 1$

(1) $\widetilde{G}_{\lambda,f}(x|b) = \pi_{f_1-1}\widetilde{G}_{\lambda',f'}(x|b)$

Where $\lambda' = (\lambda_1 + 1, \lambda_2, \cdots, \lambda_r), f' = (f_1 - 1, f_2, \cdots, f_r)$.

proof)

- The entries of the determinant are symmetric in x_{f_1-1}, x_{f_1} expect the ones on the first row
- Consider the cofactor expansion of $\widetilde{G}_{\lambda,f}(x|b)$ with respect to the first row
- Apply π_{f_1-1} to this expansion

proof of Lemma 3

Special case Lemma 3 (2)

(2)' $\pi_1 G_{(d),(1)} = G_{(d-1),(2)}$.

Prove the claim by induction on d .

- Left side equals to $\pi_1 \prod_{i=1}^{d} (x_1 \oplus b_i)$
- Liebniz rule "$\pi_i(fg) = \pi_i(f)g + s_i(f)\pi_i(g) + \beta s_i(f)g$"
- $f = (x_1 \oplus b_d)$

$$
\begin{aligned}
\pi_1 \prod_{i=1}^{d}(x_1 \oplus b_i) &= \sum_{v=0}^{d-1}\left(\prod_{i=1}^{v}(x_1 \oplus b_{s+i}) \prod_{i=v+1}^{d-1}(x_2 \oplus b_{s+1+i})\right) \\
&+ \beta \sum_{v=1}^{d-1}\left(\prod_{i=1}^{v}(x_1 \oplus b_{s+i}) \prod_{i=v}^{d-1}(x_2 \oplus b_{s+1+i})\right) \\
&= G_{(d-1),(2)}
\end{aligned}
$$

Theorem

Let (λ, f) be flagged partition of length r . If $f_i > i-1$ for $i = 1, \dots, r$. Then we have ,

$$G_{\lambda,f} = \pi_w \prod_{i=1}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j).$$

Where

$$a_i = \lambda_i + f_i - i, \qquad w = (s_r s_{r+1} \cdots s_{f_r - 1})(s_{r-1} s_r \cdots s_{f_{r-1}-1}) \cdots (s_1 s_2 \cdots s_{f_1 - 1})$$

$(\pi_w = \pi_{f_1 - 1} \cdots \pi_2 \pi_1 \cdots \pi_{r-1})$

proof)Prove the induction on $|f| = f_1 + \cdots + f_r$

(1)If $|f|$=1 (base case)

$\Rightarrow f = (1), a_1 = \lambda_1, w = id$

$\Rightarrow G_{\lambda,f} = \prod_{j=1}^{\lambda_1}(x_1 \oplus b_j) = \pi_{id} \prod_{j=1}^{\lambda_1}(x_1 \oplus b_j)$

(2)If $f_1 = 1$：

$a_1 = \lambda_1$,so

$$G_{\lambda,f} = \left(\prod_{j=1}^{a_1}(x_1 \oplus b_j)\right)(G_{\lambda',f'})^{\star}.$$

Where,

- $\lambda' = (\lambda_2, \lambda_3 \cdots, \lambda_r)$, $f' = (f_2 - 1, f_3 - 1, \cdots, f_r - 1)$
- $\star$ replaces each x_i by x_{i+1}.

By the induction hypothesis ,

$$G_{\lambda',f'} = \pi_{w'} \prod_{i=1}^{r-1} \prod_{j=1}^{a_i'} (x_i \oplus b_j)$$

である．ここで $a_i' = a_{i+1}$ で $w' = (s_{r-1} s_r \cdots s_{f_r-2})(s_{r-2} s_{r-1} \cdots s_{f_r-2})(s_1 s_2 \cdots s_{f_2-2})$.

一方,

$$\left(\pi_{w'} \prod_{i=1}^{r-1} \prod_{j=1}^{a_i'} (x_i \oplus b_j)\right)^{\star} = \pi_w \prod_{i=2}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j)$$

w does not contain s_1,

$$G_{\lambda,f} = \left(\prod_{j=1}^{a_1} (x_1 \oplus b_j)\right) \pi_w \prod_{i=2}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j) = \pi_w \prod_{i=1}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j)$$

を得る．

3)If $f_1 > 1$: It follows Lemma 3 .

Lemma 3

If $f_1 > 1$

$$(2) \quad G_{\lambda,f}(x|b) = \pi_{f_1 - 1} G_{\lambda', f'}(x|b)$$

Where $\lambda' = (\lambda_1 + 1, \lambda_2, \cdots, \lambda_r), f' = (f_1 - 1, f_2, \cdots, f_r)$.

By induction hypothesis

$$G_{\lambda', f'} = \pi_{ws_{f_1 - 1}} \prod_{i=1}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j).$$

Apply Lemma 3 and definition π_w,

$$G_{\lambda,f} = \pi_{f_1 - 1} G_{\lambda', f'} = \pi_{f_1 - 1} \pi_{ws_{f_1 - 1}} \left(\prod_{i=1}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j) \right) = \pi_w \left(\prod_{i=1}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j) \right)$$