On generalized Bott towers and twisted cubes

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Generalized Bott tower

$B_0 := \{\text{a point}\}$.
$\gamma_1^{(k)} \to B_0$: holomorphic line bundles over B_0 for $k = 1, \dots, n_1$.
$\to B_1 := \mathbb{P}(\underline{\mathbb{C}} \oplus \gamma_1^{(1)} \oplus \cdots \oplus \gamma_1^{(n_1)}) = \mathbb{C}P^{n_1}$.
$\gamma_2^{(k)} \to B_1$: holomorphic line bundles over B_1 for $k = 1, \dots, n_2$.
$\to B_2 := \mathbb{P}(\underline{\mathbb{C}} \oplus \gamma_2^{(1)} \oplus \cdots \oplus \gamma_2^{(n_2)})$: a bundle over B_1 with a fiber $\mathbb{C}P^{n_2}$.

Definition

Repeating this process m times, we obtain a sequence as follows:

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_3} B_2 \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where the fiber of $\pi_j : B_j \to B_{j-1}$ for $j = 1, \dots, m$ is $\mathbb{C}P^{n_j}$. This sequence is called a *generalized Bott tower* of height m, and B_j is called a *j-stage generalized Bott manifold*.

Generalized Bott tower

Remark

When $n_j = 1$ for all j, the above sequence is called the *Bott tower*.

We get a m-stage generalized Bott manifold B_m from any collection of integers $\{c_{i,j}^{(k)}\}$:

$$B_m = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times)/G,$$

where $G = (\mathbb{C}^\times)^m$, and its i-th factor acts on the right by

$$\begin{aligned}(z_{1,0}, \ldots, z_{m,n_m}) \cdot a_i \\ = (z_{1,0}, \ldots, z_{i-1,n_{i-1}}, z_{i,0}a_i, z_{i,1}a_i, \ldots, z_{i,n_i}a_i, \\ \ldots, z_{j,0}, a_i^{c_{i,j}^{(1)}} z_{j,1}, \ldots, a_i^{c_{i,j}^{(n_j)}} z_{j,n_j}, \ldots).\end{aligned}$$

Generalized Bott tower

We can construct a line bundle over B_m from the integers $\{l_j\}$ by

$$\mathbf{L} = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times) \times_G \mathbb{C},$$

where the i-th factor of $G = (\mathbb{C}^\times)^m$ acts by

$$((z_{1,0}, \ldots, z_{m,n_m}), v) \cdot a_i = ((z_{1,0}, \ldots, z_{m,n_m}) \cdot a_i, a_i^{l_i} v).$$

Twisted cube for generalized Bott tower

$$A_i(x) = A_i(x_{i+1,1}, \ldots, x_{m,n_m}) := \begin{cases} l_m & (i = m) \\ l_i + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k} & (1 \leq i \leq m-1). \end{cases}$$

$N := \sum_{j=1}^{m} n_j$.

$$C := \left\{ x = (x_{1,1}, \ldots, x_{m,n_m}) \in \mathbb{R}^N \;\middle|\; \begin{array}{l} \text{for all } 1 \leq i \leq m, \\ A_i(x) \leq \sum_{k=1}^{n_i} x_{i,k} \leq 0, x_{i,k} \leq 0 \\ \quad \text{or } 0 < \sum_{k=1}^{n_i} x_{i,k} < A_i(x), x_{i,k} > 0 \end{array} \right\}$$

C is called a *twisted cube* for generalized Bott tower.
When $n_j = 1$ for $j = 1, \ldots, m$, C is the twisted cube for the Bott tower which was defined by M. Grossberg and Y. Karshon (1994).

Twisted cube for generalized Bott tower

$$\mathsf{sgn}(x_{j,k}) := \begin{cases} -1 & (x_{j,k} \leq 0) \\ 1 & (x_{j,k} > 0). \end{cases}$$

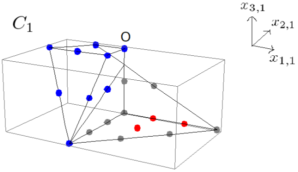
The function $\rho_0 : \mathbb{R}^N \to \{-1, 0, 1\}$ is defined by

$$\rho_0(x) = \begin{cases} (-1)^N \mathsf{sgn}(x_{1,1}) \cdots \mathsf{sgn}(x_{m,n_m}) & \text{if } x \in C \\ 0 & \text{elsewhere.} \end{cases}$$

This function is called the *density function* of the twisted cube.

Example of the twisted cube (1)

$m = 3, n_j = 1$ for $j = 1, 2, 3$
$l_1 = 1, l_2 = 1, l_3 = 2,$
$c^{(1)}_{1,2} = -1, c^{(1)}_{1,3} = 2, c^{(1)}_{2,3} = -1$

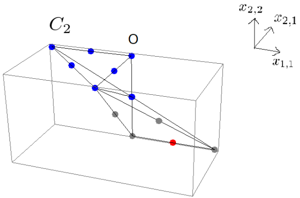


$$\begin{cases} -2 \leq x_{3,1} \leq 0 \\ -1 + x_{3,1} \leq x_{2,1} \leq 0 \\ -1 + x_{2,1} - 2x_{3,1} \leq x_{1,1} \leq 0 \text{ or } 0 < x_{1,1} < -1 + x_{2,1} - 2x_{3,1} \end{cases}$$

●$\cdots$ the lattice point of $\rho_0 = 1$
●$\cdots$ the lattice point of $\rho_0 = -1$
●$\cdots$ not contained in C_1

Example of the twisted cube (2)

$m = 2, n_1 = 1, n_2 = 2$
$l_1 = 2, l_2 = 2$
$c_{1,2}^{(1)} = 1, c_{1,2}^{(2)} = 2$



$$\begin{cases} -2 \leq x_{2,1} + x_{2,2} \leq 0, x_{2,1}, x_{2,2} \leq 0 \\ -2 - x_{2,1} - 2x_{2,2} \leq x_{1,1} \leq 0 \text{ or } 0 < x_{1,1} < -2 - x_{2,1} - 2x_{2,2} \end{cases}$$

●$\cdots$ the lattice point of $\rho_0 = 1$
●$\cdots$ the lattice point of $\rho_0 = -1$
●$\cdots$ not contained in C_2

Multiplicity function of virtual character

Now, consider the torus action as follows:
$T = T^{N+1} = S^1 \times \cdots \times S^1 \curvearrowright \mathbf{L}$:
$(t_{1,1}, \ldots, t_{m,n_m}, t_{m+1}) \cdot [z_{1,0}, \ldots, z_{m,n_m}, v] =$
$[z_{1,0}, t_{1,1}z_{1,1}, \ldots, t_{1,n_1}z_{1,n_1}, \ldots, z_{m,0}, t_{m,1}z_{m,1}, \ldots, t_{m,n_m}z_{m,n_m}, t_{m+1}v]$.
Let $\mathcal{O}_{\mathbf{L}}$ be the sheaf of holomorphic sections.
The *virtual character* is the function $\chi : T \to \mathbb{C}$ which is given by

$$\chi = \sum (-1)^i \chi^i$$

where $\chi^i(a) = \mathrm{trace}\{a : H^i(B_m, \mathcal{O}_{\mathbf{L}}) \to H^i(B_m, \mathcal{O}_{\mathbf{L}})\}$ for $a \in T$.
Every μ in the integral weight lattice $l^* \subset i\mathfrak{t}^*$ defines a homomorphism $\lambda^\mu : T \to S^1$. We can write

$$\chi = \sum_{\mu \in l^*} m_\mu \lambda^\mu.$$

The coefficients are given by a function mult $: l^* \to \mathbb{Z}$, sending $\mu \mapsto m_\mu$, called the *multiplicity function*.

Multiplicity function of virtual character

Theorem

Fix integers $\{c_{i,j}^{(k)}\}$ and $\{l_j\}$. Let $\mathbf{L} \to B_m$ be the corresponding line bundle over a generalized Bott manifold. Let $\rho_0 : \mathbb{R}^N \to \{-1, 0, 1\}$ be the density function of the twisted cube which is determined by these integers. Consider the above torus action of T. Then the multiplicity function for $l^*(\cong \mathbb{Z}^N) \times \mathbb{Z}$ is given by

$$\mathrm{mult}(\alpha, k) = \begin{cases} \rho_0(\alpha) & (k = 1) \\ 0 & (k \neq 1). \end{cases}$$

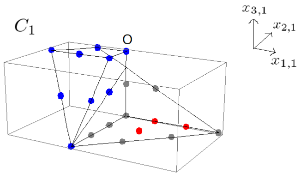
Remark

When B_m is a Bott manifold, the above theorem was proved by M. Grossberg and Y. Karshon (1994).

Example

Now, consider the twisted cube C_1.

The blue dots and the red dots correspond to the integral weight lattice μ. Then the virtual character is



$$\chi = \lambda^{e_4} + \lambda^{e_4 - e_{1,1}} + \lambda^{e_4 - e_{2,1}} + \lambda^{e_4 - e_{2,1} - e_{1,1}} + \lambda^{e_4 - e_{2,1} - 2e_{1,1}} + \lambda^{e_4 - e_{3,1}} + \lambda^{e_4 - e_{3,1} - e_{2,1}} + \lambda^{e_4 - e_{3,1} - e_{2,1} - e_{1,1}} - \lambda^{e_4 - 2e_{3,1} + e_{1,1}} - \lambda^{e_4 - 2e_{3,1} + 2e_{1,1}} - \lambda^{e_4 - e_{3,1} - e_{2,1} + e_{1,1}} + \lambda^{e_4 - 2e_{3,1} - 3e_{2,1}},$$

where $\{e_{1,1}, e_{2,1}, e_{3,1}, e_4\}$ is the standard basis of $\mathbb{R}^4$.

Example

$L(C)$: the number of the lattice points with sign of twisted cube C
$=$ the number of the terms λ^{μ} with multiplicity in the virtual character χ.
When $C = C_1$,

$$L(C_1) = 9 + (-3) = 6.$$

We hope to compute the number of lattice points of a twisted cube for a (generalized) Bott tower in general case.

References

[GK] M. Grossberg, and Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke math. J., 76(1) (1994), 23–58.