Algebraic properties of the equivariant cohomology rings of moment-angle complexes.

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Moment-angle complexes

$\mathcal{K}$ — simplicial complex on the set $[m] = \{1, \dots, m\}$.
$I = \{i_1, \dots, i_k\} \in \mathcal{K}$ — simplex.

For each simplex I define the set:

$$(D^2, S^1)^I = \{(x_1, \dots, x_m) \in (D^2)^m : x_i \in S^1 \text{ when } i \notin I\} \cong \prod_{i \in I} D^2 \times \prod_{i \notin I} S^1.$$

The moment-angle complex is the polyhedral product

$$\mathcal{Z}_{\mathcal{K}} := (D^2, S^1)^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (D^2, S^1)^I \subset (D^2)^m$$

Example

$\mathcal{K} = \bullet\ \bullet$ (2 points), then $\mathcal{Z}_\mathcal{K} = D^2 \times S^1 \cup S^1 \times D^2 \cong S^3$.

$\mathcal{K} = \partial\Delta^2$, then
$\mathcal{Z}_\mathcal{K} = (D^2 \times D^2 \times S^1) \cup (D^2 \times S^1 \times D^2) \cup (S^1 \times D^2 \times D^2) \cong S^5$.

The face ring of $\mathcal{K}$ (the Stanley–Reisner ring)

$$\mathbb{Z}[\mathcal{K}] := \mathbb{Z}[v_1, \ldots, v_m] \big/ \big(v_{i_1} \cdots v_{i_k} = 0 : \{i_1, \ldots, i_k\} \notin \mathcal{K}\big)$$

where $\deg v_i = 2$.

The cohomology ring of a moment-angle complex

Theorem (Buchstaber, Panov)

There are isomorphisms of graded commutative algebras

$$\begin{aligned} H^*(\mathcal{Z}_{\mathcal{K}}) &\cong \mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}],\mathbb{Z}) \\ &\cong H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[\mathcal{K}],d\big) \\ &\cong \bigoplus_{I\subset[m]} \widetilde{H}^*(\mathcal{K}_I) \qquad\qquad \mathcal{K}_I=\mathcal{K}|_I \end{aligned}$$

Here, the second row is the cohomology of the differential bigraded algebra with *deg* $u_i = 1$, *deg* $v_i = 2$, $du_i = v_i$, $dv_i = 0$. In the third row, $\widetilde{H}^*(\mathcal{K}_I)$ denotes the reduced simplicial cohomology of the full subcomplex $\mathcal{K}_I \subset \mathcal{K}$ (the restriction of $\mathcal{K}$ to $I \subset [m]$).

The equivariant cohomology ring of a moment-angle complex

There is an action of the $T^m = \{(t_1, \ldots, t_m) \in \mathbb{C}^m : |t_i| = 1,\, i = 1, \ldots, m\}$ on $\mathcal{Z}_\mathcal{K}$, obtained by the restriction of the coordinatewise action of T^m on $\mathbb{C}^m$.
We consider the action of the ith coordinate circle $S^1_i \subset T^m$ on $\mathcal{Z}_\mathcal{K}$ and the corresponding equivariant cohomomology ring $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$. We have a ring isomorphism:

Theorem (Masuda, Panov)

$$H^*_{S^1_i}(\mathcal{Z}_\mathcal{K}) \cong Tor_{\mathbb{Z}[\upsilon_1,\ldots,\upsilon_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}[\upsilon_i])$$
$$\cong H(\Lambda[u_1, \ldots, \hat{u}_i, \ldots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d),$$

где $du_j = \upsilon_j, d\upsilon_j = 0.$

We consider the equivariant cohomology ring $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$. The equivariant cohomology $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ may be not free as a module over $\mathbb{Z}[\upsilon_i]$. We have:

Example

Let K be an m–cycle (the boundary of an m-gon), with vertices numbered counter-clockwise.
When $m = 3$ or $m = 4$, $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ is free over $\mathbb{Z}[\upsilon_i]$ for all i.
For $m \geqslant 5$: the cohomology class in $H^3_{S^1_m}(\mathcal{Z}_\mathcal{K})$ represented by the cocycle $u_1\upsilon_3 \in \Lambda[u_1, \dots, u_{m-1}] \otimes \mathbb{Z}[\mathcal{K}]$ is a $\mathbb{Z}[\upsilon_m]$-torsion element.
Indeed, $\upsilon_m \cdot u_1\upsilon_3 = u_1(\upsilon_3\upsilon_m) = 0$, as $\upsilon_3\upsilon_m = 0$ in $\mathbb{Z}[\mathcal{K}]$ for $m \geqslant 5$. Hence, $H^*_{S^1_m}(\mathcal{Z}_\mathcal{K})$ is not free as a $\mathbb{Z}[\upsilon_m]$-module.

So, for what type of $\mathcal{K}$ the equivariant cohomology ring $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ is free as a module over $\mathbb{Z}[\upsilon_i]$?

Lemma

For simplicial complex $\mathcal{K}$ such as

$$\partial\Delta^{k_1} * \cdots * \partial\Delta^{k_p} * \Delta^l, l \geq -1, k_i \geq 0$$

the equivariant cohomology ring $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ is a free module over $\mathbb{Z}[\upsilon_i]$ for all i.

Criterion for flag complexes

Lemma

Let $\mathcal{K}$ be such simplicial complex for which in the set of missing faces $MF(\mathcal{K})$ there are such faces I_1, I_2, so that $I_1 \backslash I_2 = \{i\}$. Then $H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$ is not free $\mathbb{Z}[\upsilon_i]$–module.

Proof

Let I_1, I_2 be such faces in $MF(\mathcal{K})$ so that $I_1 \backslash I_2 = \{i\}$. Let us consider cohomology class $[u_s \upsilon_{I_2 \backslash s}] \in H^*_{S^1_i}(\mathcal{Z}_{\mathcal{K}})$, where $s \neq i$. We have

$$\upsilon_i \cdot u_s \upsilon_{I_2 \backslash s} = u_s \upsilon_i \upsilon_{I_2 \backslash s} = u_s \upsilon_i \upsilon_{I_1 \cap I_2} \upsilon_{I_2 \backslash (s, I_1 \cap I_2)} = 0,$$

as $\upsilon_i \upsilon_{I_1 \cap I_2} = \upsilon_{I_1} = 0$ in $\mathbb{Z}[\mathcal{K}]$.

Criterion for flag complexes

Flag complex is a simplicial complex in which any set of vertices pairwise connected by edges forms a simplex.

Theorem

Let $\mathcal{K}$ be a flag complex. Then the next conditions are equivalent:
a) $\mathcal{K} = \partial\Delta^{k_1} * \dots * \partial\Delta^{k_p} * \Delta^l, l \geq -1, k_j = 1\ \forall j$
b) $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ *is a free module over* $\mathbb{Z}[\upsilon_i]$ *for all* i

Proof

Implication a) $\Rightarrow$ b) is a lemma 1. Implication b) $\Rightarrow$ a) follows from lemma 2.
Indeed, if $H^*_{S^1_i}(\mathcal{Z}_\mathcal{K})$ is a free module, then in $MF(\mathcal{K})$ there are no such I_k, I_l, that $I_k \backslash I_l = \{i\}$. For the flag complex $\mathcal{K}$ all $I_k, I_l \in MF(\mathcal{K})$ consist of two vertices. It means that $I_k \cap I_l = \varnothing\, \forall\, k, l$.
This is equivalent to $\mathcal{K} = \partial\Delta^{k_1} * \dots * \partial\Delta^{k_p} * \Delta^l$.

References

- Victor Buchstaber and Taras Panov. *Toric Topology*. Mathematical Surveys and Monographs, 204, Amer. Math. Soc., Providence, RI, 2015.
- Mikiya Masuda, Taras Panov. *Cohomological rigidity of moment-angle manifolds*, preprint.