Higher Whitehead products and L_∞ structures in toric topology

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Whitehead products

We consider the spaces $\mathcal{Z}_{\mathcal{K}}$ and $(\mathbb{C}P^{\infty})^{\mathcal{K}}$.
Canononical inclusions: $\mu_i : S^2 \longrightarrow \mathbb{C}P^{\infty} \longrightarrow (\mathbb{C}P^{\infty})^{\vee m} \longrightarrow (\mathbb{C}P^{\infty})^{\mathcal{K}}$, $1 \leqslant i \leqslant m$.

Definition of Whitehead products

Higher k-fold Whitehead product in $(\mathbb{C}P^{\infty})^{\mathcal{K}}$ is the element in $\pi_{2k-1}\left((\mathbb{C}P^{\infty})^{\mathcal{K}}\right)$, defined as:

$$[\mu_{i_1}, \ldots, \mu_{i_k}]_w : S^{2k-1} \xrightarrow{w} (S^2)^{\partial\Delta} \longrightarrow (\mathbb{C}P^{\infty})^{\partial\Delta} \longrightarrow (\mathbb{C}P^{\infty})^{\mathcal{K}},$$

Note

The Higher Whitehead product for an arbitrary topological space X is the set in $\pi_{2k-1}(X)$, not just an element.

The problem is how to describe the relations in $\pi_*\left(\Omega\left(\mathbb{C}P^\infty\right)^{\mathcal{K}}\right)$ and in Pontryagin algebra $H_*\left(\Omega\left(\mathbb{C}P^\infty\right)^{\mathcal{K}}\right)$.

It is well known that the (2-fold) Whitehead product satisfies the Jacobi identity on $\pi_*(\Omega(X))$.

Gererally, it is unknown what relations higher Whitehead products satisfy.

Algebraic interpretation of higher Whitehead products

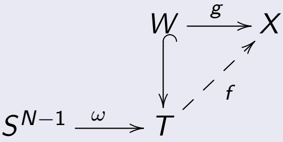
According to (Quillen, 1969), we can construct DGL L_X for any topological simply connected space X.
The reduced DGL L is a *model* of X if there is a sequence of DGL quasi-isomorphisms between L and L_X. For every model, we have

$$H(L) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$$

We consider the models of the form $L = (\mathbb{L}(V), \partial)$.

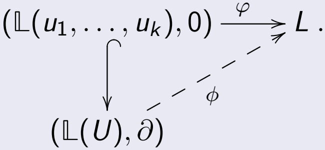
Topological higher Whitehead product

$$[x_1, \dots, x_k]_W = \{f \circ \omega | f : T \to X \text{ is an extension of } g\}.$$



Algebraic higher Whitehead product

$$[x_1, \dots, x_k]_W = \{\overline{\phi(\omega)} | \phi : (\mathbb{L}(U), \partial) \to L \text{ is an extension of } \varphi\}$$



The approaches are equivalent.

L_∞ algebra

Definition

An L_∞ algebra is a graded vector space L together with linear maps $\ell_k : L^{\otimes k} \to L$ of degree $k-2$, $k \geqslant 1$, satisfying the following two conditions:

(i) For every permutation σ

$$\ell_k\left(x_{\sigma(1)} \dots x_{\sigma(k)}\right) = sgn(\sigma)\varepsilon(\sigma)\ell_k\left(x_1 \dots x_k\right).$$

(ii) The generalized Jacobi identities hold:

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i,n-i)} \pm \ell_j\left(\ell_i\left(x_{\sigma(1)} \cdots x_{\sigma(i)}\right) x_{\sigma(i+1)} \dots x_{\sigma(n)}\right) = 0,$$

where $S(i, n-i)$ denotes the set of $(i, n-i)$ shuffles.

The notion of L_∞ algebra generalizes the notion of DGL.

Kadeishvili theorem

The notion of A_∞ algebra generalizes the notion of DGA.

Theorem (Kadeishvili, 1980)

Given a DGA A there is a A_∞ structure on its cohomology $H(A)$, for which $H(A)$ and A are quasi-isomorphic A_∞ algebras. The following holds: $m_1 = 0, m_2$ induced from the multiplication in A.

There is an L_∞ analog of this theorem:

Theorem (1)

Given a DGL L there is a L_∞ structure on its cohomology $H(L)$, for which $H(L)$ and L are quasi-isomorphic L_∞ algebras. The following holds: $\ell_1 = 0, \ell_2$ induced from Lie brackets in L.

Theorem (2)

Let X be a topological space, let L be its Quillen model. Let $x \in [x_1, \dots, x_k]_W$, $k \geqslant 3$ be an element from higher Whitehead product set.

Then, there exists an L_∞ structure on $H(L) = \pi_*(\Omega X) \otimes \mathbb{Q}$, for which

$$x = \varepsilon \ell_k (x_1 \otimes \dots \otimes x_k),$$

where $\varepsilon = (-1)^{1+|x_{k-1}|+|x_{k-3}|+\cdots}$.

The constructed L_∞ structure recovers the element x.

The case of $(\mathbb{C}P^\infty)^\mathcal{K}$

In this case, we have the canonical choice of the elements $[\mu_{i_1}, \dots, \mu_{i_k}]_W$. Let us consider the example. Let the elements $[\mu_1, \mu_2, \mu_3]_W$ and $[\mu_2, \mu_3, \mu_4]_W$ be defined. The diagram is commutative:

$$\begin{array}{ccc} {S^2 \times S^2}_{2,3} & \longrightarrow & T(S^2, S^2, S^2)_{1,2,3} \\ \downarrow & & \downarrow \rho \\ T(S^2, S^2, S^2)_{2,3,4} & \longrightarrow & (\mathbb{C}P^\infty)^\mathcal{K} \end{array}, \qquad (1)$$

Thus, the diagram consisting of DG Lie algebras is also commutative.

$$\begin{array}{ccc} (\mathbb{L}_{S^2 \times S^2},, \partial)_{2,3} & \xrightarrow[i_{2,3}]{} & (\mathbb{L}_{T(S^2,3)}, \partial)_{1,2,3} \\ \downarrow i_{1,2} & & \downarrow \phi_{1,2,3} \\ (\mathbb{L}_{T(S^2,3)}, \partial)_{2,3,4} & \xrightarrow[\phi_{2,3,4}]{} & L \end{array}, \qquad (2)$$

Theorem (3)

There exists an L_∞ *structure on* $H(L) = \pi_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}) \otimes \mathbb{Q}$ *such that for any well-defined canonical higher Whitehead product* $[\mu_{i_1}, \ldots, \mu_{i_k}]_W$, $k \geqslant 3$ *we have*

$$\ell_k(\mu_{i_1} \otimes \cdots \otimes \mu_{i_k}) = (-1)^{[\frac{k}{2}]+1} [\mu_{i_1}, \ldots, \mu_{i_k}]_W .$$

This particular L_∞ structure recovers all Whitehead products of this type.

Corollary

Let us consider the following homotopy classes:

$$\mu_{i_1}, \dots \mu_{i_k} \in \pi_1(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}) \otimes \mathbb{Q}, i_1 < \dots < i_k.$$

Let the higher Whitehead products be well-defined:

$$\left[\mu_{i_1}, \dots, \widehat{\mu_{i_j}}, \dots \mu_{i_k}\right]_W, j = 1, \dots k.$$

Then the identity holds:

$$\sum_{j=1}^{k} \left[\left[\mu_{i_1}, \dots, \widehat{\mu_{i_j}}, \dots \mu_{i_k}\right]_{k-1}, \mu_{i_j}\right]_2 = 0.$$