SMOOTH ACTIONS ON COMPLEX PROJECTIVE SPACES

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An *almost complex* structure on a smooth manifold M of dimension $2n$ (for an integer $n \geq 1$) is a fiberwise complex structure on the tangent bundle $T(M)$, i.e., an automorphism $J\colon T(M) \to T(M)$ of real vector bundles, such that $J^2 = -\mathrm{id}$.

A smooth manifold M of dimension $2n$ is called *homotopically symplectic* if M is *almost complex* (i.e., M admits an almost complex structure) and there exists an element $x \in H^2(M;\mathbb{R})$ such that $x^n \neq 0$.

A smooth manifold M of dimension $2n$ is called *symplectic* if M is equipped with a *symplectic form*, i.e., a differential 2-form ω on M which is both closed ($d\omega = 0$) and nondegenerate (ω^n is a volume form on M, $\omega^n \neq 0$).

A *stable almost complex* structure on a smooth manifold M of dimension m (for an integer $m \geq 1$) is a fiberwise complex structure on the Whitney sum

$$T(M) \oplus \underset{\cdots}{\mathbb{R}^k} \quad \text{for an integer } k \geq 0 \text{ with } m + k = 2n,$$

where $\underset{\cdots}{\mathbb{R}^k}$ denotes the product vector bundle $M \times \mathbb{R}^k$ over M.

We have defined four classes of manifolds:

$$\text{symplectic} \quad \subset \quad \text{homotopically symplectic} \quad \subset$$
$$\subset \quad \text{almost complex} \quad \subset \quad \text{stable almost complex.}$$

The goal of the lecture is to present constructions of smooth actions of compact Lie groups G on complex projective spaces, such that the manifold M consisting of points fixed under the action of G in question has a specific geometric structure while missing another one. Precisely speaking, we show how to construct a smooth action of G on a complex projective space $\mathbb{C}\mathrm{P}^n$ so that the G-fixed point set

(1) M is stable almost complex and M is not almost complex, or
(2) M is almost complex and M is not homotopically symplectic, or
(3) M is homotopically symplectic and M is not symplectic.

Smooth actions that we construct show that, in general, geometric structures (almost complex, homotopically symplectic, and symplectic) present on $\mathbb{C}\mathrm{P}^n$ do not appear on the manifold M occurring as the G-fixed point set in $\mathbb{C}\mathrm{P}^n$. Hence, smooth actions, almost complex actions, homotopically symplectic actions, and symplectic actions of compact Lie groups G on $\mathbb{C}\mathrm{P}^n$ form different classes of groups of transformations on $\mathbb{C}\mathrm{P}^n$.

References

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