Classification of compact transformation groups on complex quadrics with codimension one orbits

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Abstract

Let G be a compact connected Lie group and M a rational cohomology complex quadric of real dimension divisible by 4 (where $\dim M \neq 4$). The aim of this paper is to classify pairs (G, M) such that G acts smoothly on M with codimension one principal orbits. There exist eight such pairs up to essential isomorphism. The underlying manifold M is diffeomorphic to the genuine complex quadric except one pair.

1 Introduction

One of the central problems in transformation groups is to classify compact Lie group actions on a fixed smooth manifold M such as a sphere and a complex projective space. Unfortunately the problem is beyond our reach in general, but it becomes within our reach if we put some assumption on the actions. For instance, when the actions are transitive, M is a homogeneous space and the problem reduces to finding a pair of a compact Lie group G and its closed subgroup H such that $G/H = M$. As is well known, there are a rich history and an abundant work in this case (e.g. [3], [11]). In particular, the transitive actions on a sphere are completely classified. The complete list can be found in [2] and [6].

The orbit of a transitive action is of codimension zero. So we are naturally led to study actions with codimension one principal orbits. In 1960 H. C. Wang ([19]) initiated the work in this direction. He investigated compact Lie group actions on spheres with codimension one principal orbits. In 1977 F. Uchida ([16]) classified compact connected Lie group actions on rational cohomology projective spaces with codimension one principal orbits. The same problem has been studied by K. Iwata on rational cohomology quaternion projective spaces ([7]), on rational cohomology Cayley projective planes ([8]) and by T. Asoh on $\mathbf{Z}_2$-cohomology spheres ([2]).

The purpose of this paper is to classify compact connected Lie group actions on a rational cohomology complex quadric with codimension one principal orbits. The complex quadric

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Q_r of complex dimension r is a degree two hypersurface $\sum_i z_i^2 = 0$ in the complex projective space $P_{r+1}(\mathbf{C})$ of complex dimension $r+1$. The linear action of $SO(r+2)$ on $P_{r+1}(\mathbf{C})$ leaves Q_r invariant and is transitive on Q_r. Hence Q_r is diffeomorphic to $SO(r+2)/(SO(r)\times SO(2))$. When r is odd, Q_r is a rational cohomology complex projective space and this case is already treated by Uchida ([16]) mentioned above. Therefore we assume that $r = 2n$, i.e., our rational cohomology complex quadric is of real dimension $4n$.

A pair (G, M) denotes a smooth G-action on M and we say that (G, M) is *essentially isomorphic* to (G', M') if their induced effective actions are isomorphic. Our main theorem is the following.

Theorem 1.1. *Let M be a rational cohomology complex quadric of real dimension $4n$ $(n \geq 2)$ and let G be a compact connected Lie group. If (G, M) has codimension one principal orbits, then (G, M) is essentially isomorphic to one of the pairs in the following list.*

n	G	M	*action*
$n \geq 2$	$SO(2n+1)$	Q_{2n}	$SO(2n+1) \to SO(2n+2)$
$n \geq 2$	$U(n+1)$	Q_{2n}	$U(n+1) \to SO(2n+2)$
$n \geq 2$	$SU(n+1)$	Q_{2n}	$SU(n+1) \to SO(2n+2)$
$n = 2m-1 \geq 3$	$Sp(1) \times Sp(m)$	Q_{4m-2}	$Sp(1) \times Sp(m) \to SO(4m)$
7	$Spin(9)$	Q_{14}	$Spin(9) \to SO(16)$
3	G_2	Q_6	$G_2 \to SO(7) \to SO(8)$
3	$G_2 \times T^1$	$G_2 \times_{SU(3)} P_3(\boldsymbol{C})$	G_2 *acts on* G_2 *canonically and* T^1 *acts on the first coordinate of* $P(\boldsymbol{C} \oplus \boldsymbol{C}^3) = P_3(\boldsymbol{C})$
2	$Sp(2)$	$S^7 \times_{Sp(1)} P_2(\boldsymbol{C})$	$Sp(2)$ *acts transitively on* S^7

Here $G_2 \times_{SU(3)} P_3(\boldsymbol{C})$ denotes the quotient of $G_2 \times P_3(\boldsymbol{C})$ by the diagonal $SU(3)$-action where $SU(3)$ acts on G_2 canonically and on $P_3(\boldsymbol{C})$ by $A([z_0 : z]) = [z_0 : Az]$ where $[z_0 : z] \in P(\boldsymbol{C} \oplus \boldsymbol{C}^3) = P_3(\boldsymbol{C})$ and $A \in SU(3)$. $S^7 \times_{Sp(1)} P_2(\boldsymbol{C})$ also denotes the quotient of $S^7 \times P_2(\boldsymbol{C})$ by the diagonal $Sp(1)$-action where $Sp(1)$ acts on S^7 canonically and on $P_2(\boldsymbol{C})$ through a double covering $Sp(1) \to SO(3)$.

Remark. The manifold $S^7 \times_{Sp(1)} P_2(\mathbf{C})$ is not diffeomorphic to Q_4 (see Proposition 6.2.1). On the other hand, the manifold $G_2 \times_{SU(3)} P_3(\mathbf{C})$ is diffeomorphic to Q_6 (see Section 7.2.2).

Closed connected subgroups of $SO(r+2)$ whose restricted actions on Q_r have codimension one principal orbits are classified by Kollross [13]. Comparing his result with our list above, the action of $G_2 \times T^1$ on $G_2 \times_{SU(3)} P_3(\mathbf{C}) \cong Q_6$ does not arise through a homomorphism to $SO(8)$. In this paper we use the notation $\cong$ as a diffeomorphism, $\simeq$ as an isomorphism and $\approx$ as a local isomorphism.

There are some works on compact connected Lie group actions with codimension two principal orbits, see [15] and [16], but the actions get complicated according as the codimension of principal orbit gets large. The classification of compact connected Lie group actions with codimension two principal orbits is studied by Uchida ([17]) on rational cohomology complex projective space. Nakanishi ([15]) completed the classification of homology spheres with an action of $SO(n)$, $SU(n)$ or $Sp(n)$.

The organization of this paper is as follows. In Section 2 we review a key theorem by F. Uchida on compact connected Lie group actions on M with codimension one principal orbits. It says that if $H^1(M;\mathbf{Z}_2)=0$, then there are exactly two singular orbits and M decomposes into a union of closed invariant tubular neighborhoods of the singular orbits. In Section 3 we compute the Poincaré polynomials of the singular orbits. To do this, we distinguish three cases according to orientability of singular orbits. In Section 4 we determine the possible transformation groups G from the Poincaré polynomials using a well known fact on Lie theory([14]). We also recall some facts used in later sections and state an outline of our steps to the classification. Sections 5 through 11 are devoted to classifying the pairs (G,M). By looking at the slice representations of the singular orbits, we completely determine the transformation groups G and the tubular neighborhood of singular orbits. Then we check whether the G-manifold obtained by gluing those two tubular neighborhoods along their boundary is a rational cohomology complex quadric. Finally we give all actions in Section 12.

2 Preliminary

In this section, we present some basic facts on a complex quadric and the key theorem to solve the classification problem on a rational cohomology complex quadric. Let us recall the definition of complex quadric.

DEFINITION(**complex quadric** Q_r).

$$\begin{aligned} Q_r &= \{\mathbf{z}\in P_{r+1}(\mathbf{C})|\ z_0^2+z_1^2+\cdots+z_{r+1}^2=0\} \\ &\cong SO(r+2)/SO(r)\times SO(2), \end{aligned}$$

where $\mathbf{z}=[z_0:z_1:\cdots:z_{r+1}]\in P_{r+1}(\mathbf{C})$. A simply connected closed manifold of dimension $2r$ is called a *rational cohomology complex quadric* if it has the same cohomology ring as Q_r with $\mathbf{Q}$ coefficient. It is well known that the rational cohomology ring of Q_{2n} is given by

$$H^*(Q_{2n};\mathbf{Q})=\mathbf{Q}[c,x]/(c^{n+1}-cx,x^2,c^{2n+1}),$$

where $\deg(x)=2n,\deg(c)=2$ for $n\geq 2$. Remark $Q_2=SO(4)/SO(2)\times SO(2)\cong Spin(4)/T^2\cong SU(2)/T^1\times SU(2)/T^1\cong S^2\times S^2$. Hence $H^*(Q_2;\mathbf{Q})$ is different from the above ring. In this paper we will classify the case $n\geq 2$.

Let us recall the key theorem about the structure of (G,M).

Theorem 2.1 (Uchida[16] Lemma 1.2.1)**.** *Let G be a compact connected Lie group and M a compact connected manifold without boundary. Assume*

$$H^1(M; \boldsymbol{Z}_2) = 0,$$

and G acts smoothly on M with codimension one orbits $G(x)$. Then $G(x) \cong G/K$ is a principal orbit and (G, M) has just two singular orbits $G(x_1) \cong G/K_1$ and $G(x_2) \cong G/K_2$. Moreover there exists a closed invariant tubular neighborhood X_s of $G(x_s)$ such that

$$M = X_1 \cup X_2 \quad \textit{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Note that X_s is a k_s-dimensional disk bundle over G/K_s ($k_s \geq 2$).

3 Poincaré polynomial

Let M be a rational cohomology complex quadric of dimension $2r = 4n$ and G a compact connected Lie group which acts smoothly on M with codimension one principal orbits. Then the pair (G, M) satisfies the assumptions of Theorem 2.1. Therefore M is divided into X_1 and X_2 where X_i is the tubular neighborhood of the singular orbit G/K_i ($i = 1, 2$). Let us calculate the Poincaré polynomial of the singular orbits G/K_1 and G/K_2.

First we prepare some notations. Let $f_s^* : H^*(M; \mathbf{Q}) \to H^*(X_s; \mathbf{Q})$ be the homomorphism induced by the inclusion $f_s : X_s \to M$ and n_s a non-negative integer such that $f_s^*(c^{n_s}) \neq 0$ and $f_s^*(c^{n_s+1}) = 0$ where $c \in H^2(M; \mathbf{Q})$ is a generator. The following theorem is the goal of this section. The result in the case where the two singular orbits are orientable is due to an unpublished note by S. Kikuchi.

Theorem 3.1. *Two singular orbits G/K_1 and G/K_2 satisfy one of the following* (I)–(III).

(I) *If the two singular orbits are both orientable, then these singular orbits satisfy one of the following* (i)–(iii).

(i) $G/K_s \sim P_n(\boldsymbol{C})$, $k_1 = 2n = k_2$, $n_1 = n = n_2$.

(ii) $G/K_1 \sim P_{2n-1}(\boldsymbol{C})$, $G/K_2 \sim S^{2n}$, $k_1 = 2$, $k_2 = 2n$, $n_1 = 2n-1$, $n_2 = 0$.

(iii) $P(G/K_1; t) = (1 + t^{k_2 - 1})(1 + t^2 + \cdots + t^{2n})$ *and*
$P(G/K_2; t) = (1 + t^{k_1 - 1})(1 + t^2 + \cdots + t^{2n})$ (n_1, $n_2 \in \{n-1, n\}$) *or*
$P(G/K_2; t) = (1 + t^{2n+1})(1 + t^2 + \cdots + t^{2n_2})$ ($n_1 > n$),
k_2 *is odd*, k_1 *is even and* $k_1 + k_2 = 2n + 1$.

(II) *If G/K_1 is orientable and G/K_2 is non-orientable, then*

(iv) $G/K_1 \sim P_{2n-1}(\boldsymbol{C})$, $P(G/K_2; t) = 1 + t^{2n}$, $P(G/K_2^o; t) = (1 + t^n)(1 + t^{2n})$,
$G/K^o \sim S^{4n-1}$, $n_1 = 2n-1$, $n_2 = 0$, $k_1 = 2, k_2 = n$.

(III) *If the two singular orbits are both non-orientable, then*

$$\begin{aligned}
\text{(v)}\quad & P(G/K_s;t) = 1 + t^2 + t^4,\ P(G/K_s^o;t) = (1+t^2)(1+t^2+t^4), \\
& P(G/K;t) = P(G/K^o;t) = (1+t^3)(1+t^2+t^4) \textit{ or} \\
& P(G/K;t) = P(G/K^o;t) = (1+t^5)(1+t^2) \\
& n = k_1 = k_2 = 2 \textit{ and } n_1 = n_2 \in \{1,\ 2\} \textit{ or } n_1 = 2,\ n_2 = 1.
\end{aligned}$$

Here k_s is a codimension of G/K_s, $M \sim N$ means $P(M;t) = P(N;t)$, $P(X;t)$ is the Poincaré polynomial of X, K is a principal isotropy group, and K^o is the identity component of K.

To prove Theorem 3.1, we will consider three cases according to orientability of two singular orbits. Before we consider three cases, we shall show Proposition 3.0.1. Let us set

$$\begin{aligned}
P(\mathrm{Im} f_s^*;t) &= \sum t^q \dim(\mathrm{Im} f_s^q) \text{ and} \\
P(\mathrm{Ker} f_s^*;t) &= \sum t^q \dim(\mathrm{Ker} f_s^q)
\end{aligned}$$

where $\mathrm{Ker} f_s^q = \mathrm{Ker}(f_s^*) \cap H^q(M;\mathbf{Q})$ and $\mathrm{Im}(f_s^q) = \mathrm{Im}(f_s^*) \cap H^q(X_s;\mathbf{Q})$. First we prepare the following equations to prove Proposition 3.0.1.

Lemma 3.0.1. *Put $\epsilon_s = 1$ if $f_s^*(x) \neq \lambda f_s^*(c^n)$ for all $\lambda \in \boldsymbol{Q}$, $\epsilon_s = 0$ otherwise. Then we have*

$$\begin{aligned}
P(\mathrm{Im} f_s^*;t) &= 1 + t^2 + \cdots + t^{2n_s} + \epsilon_s t^{2n} \textit{ and} \\
P(\mathrm{Ker} f_s^*;t) &= t^{2n_s+2} + \cdots + t^{4n} + (1-\epsilon_s)t^{2n}.
\end{aligned}$$

We can easily check this lemma because of the isomorphism $H^*(M;\mathbf{Q}) \simeq H^*(Q_{2n};\mathbf{Q})$. Let us state a proposition.

Proposition 3.0.1.

1. *$n_1 + n_2 + \epsilon_1 + \epsilon_2 = 2n$.*
2. *$\epsilon_1 = \epsilon_2$ holds if and only if $n_1 = n_2$.*

We show the following two lemmas to prove Proposition 3.0.1.

Lemma 3.0.2. *We have the equation*

$$P(X_{3-s}, \partial X_{3-s};t) - tP(X_s;t) = P(\mathrm{Ker} f_s^*;t) - tP(\mathrm{Im} f_s^*;t).$$

Proof. We get $\dim(H^q(X_{3-s}, \partial X_{3-s})) = \dim(H^q(M, X_s))$ by the excision isomorphism. From this equality and the cohomology exact sequence of (M, X_s)

$$\longrightarrow H^{q-1}(X_s; \mathbf{Q}) \xrightarrow{\delta^{q-1}} H^q(M, X_s; \mathbf{Q}) \xrightarrow{j^q} H^q(M; \mathbf{Q}) \xrightarrow{f_s^*} H^q(X_s; \mathbf{Q}) \longrightarrow,$$

we get

$$\begin{aligned}
\dim(H^q(X_{3-s}, \partial X_{3-s})) &= \dim(\mathrm{Im}\delta^{q-1}) + \dim(\mathrm{Ker} f_s^q) \\
&= \dim(H^{q-1}(X_s)) - \dim(\mathrm{Im} f_s^{q-1}) + \dim(\mathrm{Ker} f_s^q). \blacksquare
\end{aligned}$$

From Lemma 3.0.2, we can show the following lemma.

Lemma 3.0.3. $P(\mathrm{Ker} f_1^*; t) - tP(\mathrm{Im} f_1^*; t) = t^{4n} P(\mathrm{Im} f_2^*; t^{-1}) - t^{4n+1} P(\mathrm{Ker} f_2^*; t^{-1})$.

Proof. By the Poincaré-Lefschetz duality and the universal coefficient theorem we get $H^q(X_s) \simeq H^{4n-q}(X_s, \partial X_s)$. Hence $P(X_s; t) = t^{4n} P(X_s, \partial X_s; t^{-1})$. From Lemma 3.0.2 we get

$$\begin{aligned}
P(\mathrm{Ker} f_1^*; t) - tP(\mathrm{Im} f_1^*; t) &= P(X_2, \partial X_2; t) - tP(X_1; t) \\
&= t^{4n} P(X_2; t^{-1}) - t^{4n+1} P(X_1, \partial X_1; t^{-1}) \\
&= -t^{4n+1}\{P(X_1, \partial X_1; t^{-1}) - t^{-1} P(X_2; t^{-1})\} \\
&= -t^{4n+1}\{P(\mathrm{Ker} f_2^*; t^{-1}) - t^{-1} P(\mathrm{Im} f_2^*; t^{-1})\}.
\end{aligned}$$

The last equal can be proved by using Lemma 3.0.2 with t replaced by t^{-1}. Therefore we get this statement. $\blacksquare$

Let us prove Proposition 3.0.1.

Proof of Proposition 3.0.1. From Lemma 3.0.1 and 3.0.3, we get the following equation

$$\begin{aligned}
&t^{2n_1+2}(1 + t^2 + \cdots + t^{4n-2n_1-2}) + (1 - \epsilon_1)t^{2n} - t(1 + t^2 + \cdots + t^{2n_1}) - \epsilon_1 t^{2n+1} \\
&= t^{4n}(1 + t^{-2} + \cdots + t^{-2n_2}) + \epsilon_2 t^{2n} - t(t^{4n-2n_2-2} + \cdots + t^2 + 1) - (1 - \epsilon_2)t^{2n+1}.
\end{aligned}$$

Put $t = 1$ then we get the first statement in Proposition 3.0.1.

When $\epsilon_1 = \epsilon_2 = 0$, compare the degree of this obtained equation by using the first statement then we get the equation $n_1 = n_2 = n$. When $\epsilon_1 = \epsilon_2 = 1$, similarly we get $n_1 = n_2 = n - 1$. Conversely if $n_1 = n_2$, then we have $\epsilon_1 + \epsilon_2 = 2(n - n_1)$ from the first statement. Since ϵ_1, $\epsilon_2 = 0$ or 1, we get $\epsilon_1 = \epsilon_2$. Hence the second statement holds. $\blacksquare$

From the next section we will consider three cases according to orientability of two singular orbits.

3.1 Both singular orbits are orientable

Suppose the two singular orbits G/K_1 and G/K_2 are orientable. The goal of this section is to prove Theorem 3.1 (i)–(iii). From now on we put $k_s = \mathrm{codim} G/K_s$ and $r = 3 - s$ for $s = 1, 2$. The following Poincaré duality will be used many times in this section.

Theorem 3.2 (Poincaré duality). *Let M^n be an n-dimensional closed orientable manifold. Then the following isomorphism holds*

$$\varphi : H^t(M^n; \boldsymbol{Q}) \simeq \mathrm{Hom}(H^{n-t}(M^n; \boldsymbol{Q}),\ \boldsymbol{Q})$$

by $(\varphi(x)(y))\mu = xy$ where $x \in H^t(M^n; \boldsymbol{Q})$, $y \in H^{n-t}(M^n; \boldsymbol{Q})$ and μ is a generator of $H^n(M^n; \boldsymbol{Q}) \simeq \boldsymbol{Q}$. Hence we have $H^t(M^n; \boldsymbol{Q}) \simeq H^{n-t}(M^n; \boldsymbol{Q})$.

First we prove the following equality.

Lemma 3.1.1. *The following equation holds.*

$$\begin{aligned} & (1 - t^{k_1+k_2-2})P(G/K_s; t) \\ = \ & (1 + t^{-1})\{P(\mathrm{Im} f_s^*; t) + t^{k_r-1}P(\mathrm{Im} f_r^*; t)\} - t^{-1}(1 + t^{k_r-1})P(M; t). \end{aligned}$$

Proof. By the Thom isomorphism, we get $t^{k_s}P(G/K_s; t) = P(X_s, \partial X_s; t)$. Since G/K_s is a deformation retract of X_s, $P(X_s; t) = P(G/K_s; t)$. Hence by Lemma 3.0.2, we get $t^{k_r}P(G/K_r; t) - tP(G/K_s; t) = P(\mathrm{Ker} f_s^*; t) - tP(\mathrm{Im} f_s^*; t)$ and we also get $P(G/K_r; t) = t^{k_s-1}P(G/K_s; t) - t^{-1}P(\mathrm{Ker} f_r^*; t) + P(\mathrm{Im} f_r^*; t)$. Using these equations and $P(\mathrm{Ker} f_s^*; t) = P(M; t) - P(\mathrm{Im} f_s^*; t)$, we can easily check the above equation. ■

Putting $t = -1$ in Lemma 3.1.1, we get $(1 - (-1)^{k_1+k_2})\chi(G/K_s) = (1 - (-1)^{k_r})\chi(M)$ where $\chi(X)$ is the Euler characteristic of X. From this equation, we see

Lemma 3.1.2. *If k_1+k_2 is even, then k_1 and k_2 are even. Hence the case $k_1 \equiv k_2 \equiv 1 (\mathrm{mod}\ 2)$ does not occur.*

Let us set $g_s(t) = (1-t^{k_1+k_2-2})P(G/K_s; t)$, which is the left side of the identity in Lemma 3.1.1. Next we consider two cases for ϵ_s ($s = 1,\ 2$) and prove (i)–(iii) in Theorem 3.1.

3.1.1 The cases $\epsilon_1 = \epsilon_2$.

Let us prove Theorem 3.1 (i) and (iii) occur in these cases.

If $\epsilon_1 = \epsilon_2 = 0$ then $n_1 = n_2 = n$ and if $\epsilon_1 = \epsilon_2 = 1$ then $n_1 = n_2 = n - 1$ by Proof of Proposition 3.0.1. In both of these cases we have

$$P(\mathrm{Im} f_s^*; t) = 1 + t^2 + \cdots + t^{2n}$$

by the definitions of ϵ_s and n_s $(s = 1,\ 2)$. If we put $a(n) = P(\mathrm{Im}f_s^*; t) = 1 + t^2 + \cdots + t^{2n}$, we have $P(M; t) = (1 + t^{2n})a(n)$. Then by Lemma 3.0.1 and 3.1.1, we have the following equation

$$g_s(t) = (1 + t^{k_r-1})(1 - t^{2n-1})a(n). \tag{1}$$

Let us consider three cases for k_s $(s = 1,\ 2)$.

Suppose $k_1 \equiv k_2 \equiv 0 (\mathrm{mod}\ 2)$. Dividing both sides of the equation (1) by $1+t$ and putting $t = -1$, we get $\chi(G/K_s) \neq 0$ for $s = 1, 2$. Now we have the following lemma.

Lemma 3.1.3. *If the Euler characters $\chi(G/K_s)$ are non-zero for $s = 1, 2$, then the Poincaré polynomials $P(G/K_s; t)$ are even functions for $s = 1, 2$, that is, $P(G/K_s; t) = P(G/K_s; -t)$.*

Proof. Because $\chi(G/K_s) \neq 0$, we have rank $K_s^o =$ rank G (see [14] Chapter III). Hence $H^{odd}(G/K_s^o; \mathbf{Q}) = 0$ from [14] Theorem 3.21 in Chapter VII. Since the induced map from the natural inclusion

$$H^*(G/K_s; \mathbf{Q}) \to H^*(G/K_s^o; \mathbf{Q})$$

is injective, the Poincaré polynomials $P(G/K_1; t)$ and $P(G/K_2; t)$ are even functions. ■

From this lemma, we see $(1+t^{k_r-1})(1-t^{2n-1}) = (1-t^{k_r-1})(1+t^{2n-1})$ by the equation (1). Consequently $k_1 = k_2 = 2n$. By the equation (1), the equation $P(G/K_s; t) = a(n)$ holds. Hence we have $G/K_s \sim P_n(\mathbf{C})$ because $P(P_n(\mathbf{C}); t) = a(n)$. This means Theorem 3.1 (i).

Suppose k_1 is even and k_2 is odd. Then we have $\chi(G/K_1) \neq 0$, dividing both sides of the equation (1) by $1 - t$ and putting $t = -1$. So $P(G/K_1; t)$ is an even function by Lemma 3.1.3. When $s = 1$ $(r = 2)$ in the equation (1), compare even degree terms and odd degree terms. Then we have $k_1 + k_2 = 2n + 1$ and $P(G/K_1; t) = (1 + t^{k_2-1})a(n)$. When $s = 2$ in the equation (1), we also have $P(G/K_2; t) = (1 + t^{k_1-1})a(n)$ by $k_1 + k_2 = 2n + 1$. This means Theorem 3.1 (iii). If k_1 is odd and k_2 is even, then we get a similar result.

By Lemma 3.1.2, there does not exist the case that k_1 and k_2 are odd. Therefore in the case $\epsilon_1 = \epsilon_2$, Theorem 3.1 (i) and (iii) occur. Let us consider the case $\epsilon_1 \neq \epsilon_2$.

3.1.2 The case $\epsilon_1 \neq \epsilon_2$.

The goal of this section is to prove Theorem 3.1 (ii) and (iii) occur in the case $\epsilon_1 \neq \epsilon_2$.

If we put $\epsilon_1 = 0$ and $\epsilon_2 = 1$, we have $n_1 + n_2 = 2n - 1$ by Proposition 3.0.1 and we also have $P(\mathrm{Im}f_1^*; t) = a(n_1)$ and $P(\mathrm{Im}f_2^*; t) = a(n_2) + t^{2n}$ by definitions of ϵ_s and n_s $(s = 1,\ 2)$. Hence we easily get

$$g_1(t) = (1 - t^{2n_2+k_2})a(n_1) + (t^{k_2-1} - t^{2n_1+1})a(n_2) - t^{2n-1}(1 - t^{k_2}), \tag{2}$$
$$g_2(t) = (1 - t^{2n_1+k_1})a(n_2) + (t^{k_1-1} - t^{2n_2+1})a(n_1) + t^{2n}(1 - t^{k_1-2}) \tag{3}$$

by Lemma 3.0.1 and Lemma 3.1.1. Let us consider four cases for k_s ($s = 1,\ 2$).

Suppose $k_1 \equiv k_2 \equiv 0 (\mathrm{mod}\ 2)$. Dividing both sides of (2), (3) by $1 + t$ and putting $t = -1$, we see $P(G/K_1; t)$ and $P(G/K_2; t)$ are even functions by $k_s \geq 2$ and Lemma 3.1.3. So $k_1 = 2n_2 + 2$ by comparing the odd degree terms in (3).

Consider the odd degree terms in (2). Then we see $(t^{k_2-1} - t^{2n_1+1})a(n_2) - t^{2n-1}(1-t^{k_2}) = 0$. So we have

$$t^{k_2-1}a(n_2) + t^{2n+k_2-1} = t^{2n_1+1}a(n_2) + t^{2n-1}.$$

The minimum degree of the left side is $k_2 - 1$, while that of right side is $2n_1 + 1$ or $2n - 1$. If $k_2 - 1 = 2n_1 + 1$, then we get $t^{2n+k_2-1} = t^{2n-1}$ by this equation. This contradicts $k_2 \geq 2$. Hence we have $k_2 - 1 = 2n - 1$, and we also have $n_1 = n$ (if $n_2 \neq 0$) and $n_1 = 2n - 1$ (if $n_2 = 0$) by comparing the second lower degree in this equation. When $n_1 = n$, we see $n_2 = n - 1$ by Proposition 3.0.1 and $\dim G/K_2 = 2n$ by $k_2 = 2n$. In particular we have $G/K_2 \sim P_n(\mathbf{C})$ by the equation (3). However $f_2^*(c)f_2^*(c^{n_2}) = f_2^*(c^{n_2+1}) = 0 \in H^{2n}(G/K_2; \mathbf{Q})$ by the definition of n_2. This contradicts the Poincaré duality (Theorem 3.2).

Hence $n_1 = 2n - 1$ and $n_2 = 0$. So we see $k_1 = 2n_2 + 2 = 2$. Hence we have $G/K_1 \sim P_{2n-1}(\mathbf{C})$ from the equation (2), and we also have $G/K_2 \sim S^{2n}$ from the equation (3) and $k_2 = 2n$. This result is Theorem 3.1 (ii).

Suppose k_1 is even and k_2 is odd. Put $t = -1$ in (2). Then we see $P(G/K_1; t)$ is an even function by Lemma 3.1.3. So we get from (2)

$$P(G/K_1; t) \quad = \quad a(n_1) + t^{k_2-1}a(n_2) + t^{2n-1+k_2}. \tag{4}$$

Since G/K_1 is orientable, we have $\dim G/K_1 = \max\{2n_1,\ k_2 - 1 + 2n_2,\ 2n - 1 + k_2\}$.

If $\dim G/K_1 = 2n_1$ then $k_2 - 1 = 2n_1 - (k_2 - 1 + 2n_2)$ or $2n_1 - (2n - 1 + k_2)$ from the Poincaré duality about G/K_1, the inequality $n \geq 2$ ($k_2 - 1 < 2n - 1 + k_2$) and the equation (4). Hence $k_2 - 1 = n_1 - n_2$ or $n_1 - n$. Since $n_1 + n_2 = 2n - 1$, $n_1 - n_2$ is an odd number. Now k_2 is an odd number. So $k_2 - 1 = n_1 - n$. Therefore $k_2 - 1 = n_1 - n = n - n_2 - 1$ by Proposition 3.0.1. In this case $2n - 1 + k_2 = (k_2 - 1 + 2n_2) + 2$ from the Poincaré duality about G/K_1 and the equation (4). So $n_2 = n - 1$. However we have $k_2 - 1 = n - n_2 - 1 = n - (n - 1) - 1 = 0$. This contradicts $k_2 \geq 2$. Hence $\dim G/K_1 \neq 2n_1$.

If $\dim G/K_1 = k_2 - 1 + 2n_2$, then $2(n_2 - n) = k_2 - 1$ or $n_2 = n_1$ from the Poincaré duality about G/K_1, the inequality $k_2 - 1 < 2n - 1 + k_2$ and the equation (4). Now $n_1 + n_2 = 2n - 1$ that is $n_1 \neq n_2$. So $2(n_2 - n) = k_2 - 1$ and we also have $n_1 + 1 = n_2$ by the Poincaré duality about G/K_1 and the equation (4). Since $n_1 + n_2 = 2n - 1$, we have $n_2 = n$. This contradicts $k_2 \geq 2$.

Hence $\dim G/K_1 = 2n - 1 + k_2$. In this case $2n - 1 + k_2 - 2 = 2n_1$ or $k_2 - 1 + 2n_2$ from the Poincaré duality and the equation (4). If $2n - 1 + k_2 - 2 = 2n_1$, then $\dim G/K_1 = 2n_1 + 2$. However $f_1^*(c)f_1^*(c^{n_1}) = f_1^*(c^{n_1+1}) = 0 \in H^{2n_1+2}(G/K_1; \mathbf{Q})$ by the definition of n_1. This contradicts the Poincaré duality. Therefore we have $2n-1+k_2-2=k_2-1+2n_2$. So $n_1=n$ and $n_2=n-1$. Hence we have $P(G/K_1;t)=(1+t^{k_2-1})a(n)$ from the equation (4). Moreover we have $P(G/K_2;t)=(1+t^{k_1-1})a(n)$ by the equation (3) and $k_1+k_2=2n+1$. This result is Theorem 3.1 (iii).

Suppose k_1 is odd and k_2 is even. In this case we get $P(G/K_2;t)=a(n_2)+t^{k_1-1}a(n_1)+t^{2n}$ because $P(G/K_2;t)$ is an even function and the equation (3) holds. Hence we have $\dim G/K_2=4n-k_2=\max\{2n_2,\ k_1-1+2n_1,\ 2n\}$.

If $\dim G/K_2=2n$, then we have $k_2=2n$. Because of the odd degree terms in the equation (3), we have $P(G/K_2;t)=t^{2n_1-2n+2}a(n_2)+t^{2n_2+3-k_1-2n}a(n_1)+1$. So $2n_1-2n+2\geq 2$ and $2n_2+3-k_1-2n\geq 2$. From $2n=n_1+n_2+1$, we have $k_1\leq n_2-n_1\leq -1$. This contradicts $k_1>2$.

If $\dim G/K_2=k_1-1+2n_1$, we have the following cases by making use of the Poincaré duality for the even function $P(G/K_2;t)=a(n_2)+t^{k_1-1}a(n_1)+t^{2n}$;

- $\dim G/K_2-(k_1-1)=2n_2$,
- $2n=(k_1-1)-2$ and $\dim G/K_2-2n=2n_2$,
- $2n=2n_2+2$ and $\dim G/K_2-2n=k_1-1$.

When $\dim G/K_2-(k_1-1)=2n_2$, we have $n_1=n_2$. However this does not occur because $n_1+n_2+1=2n$. When $2n=(k_1-1)-2$ and $\dim G/K_2-2n=2n_2$, we have $n_1=n-1$, $n_2=n$ because $2n=n_1+n_2+1$. So we have

$$\begin{aligned}\dim G/K_2 &= 4n-k_2\\ &= (k_1-1)+2n_1\\ &= (2n+2)+2n-2=4n.\end{aligned}$$

Hence $k_2=0$. This is a contradiction. Hence we have $2n=2n_2+2$. Then we can show $n_1=n,\ n_2=n-1,\ k_1+k_2=2n+1$ and $P(G/K_s;t)=(1+t^{k_r-1})a(n)$ $(s+r=3)$ from the equations (2) and (3). This result is Theorem 3.1 (iii).

If $\dim G/K_2=4n-k_2=2n_2$, then we have and $2n_2-2n=k_1-1$ from the Poincaré duality and the above equation of $P(G/K_2;t)$. Hence $k_1=n_2-n_1$ and we see $k_1+k_2=2n+1=n_1+n_2+2$. So we have

$$\begin{aligned}P(G/K_2;t) &= a(n_2)+t^{k_1-1}a(n_1)+t^{2n}\\ &= a(n_2)+t^{k_1-1}a(n_1+1)\\ &= \{a(n)+(t^{2n+2}+\cdots+t^{2n+k_1-1})\}+t^{k_1-1}(1+t^2+\cdots+t^{2n+1-k_1})\\ &= a(n)+t^{k_1-1}(1+t^2+\cdots+t^{2n+1-k_1})+(t^{2n+2}+\cdots+t^{2n+k_1-1})\\ &= a(n)+t^{k_1-1}+t^{k_1+1}+\cdots+t^{2n}+t^{2n+2}+\cdots+t^{2n+k_1-1}\\ &= a(n)+t^{k_1-1}a(n)\\ &= (1+t^{k_1-1})a(n).\end{aligned}$$

Moreover we have $P(G/K_1;t) = (1+t^{2n+1})a(n_1)$ by the equation (2). This result becomes the second case in Theorem 3.1 (iii).

By Lemma 3.1.2, there does not exist the case that k_1 and k_2 are odd.

We can get a similar result in the case $\epsilon_1 = 1$ and $\epsilon_2 = 0$. Therefore in the case $\epsilon_1 \neq \epsilon_2$, Theorem 3.1 (ii) and (iii) occur.

Consequently Theorem 3.1 (i)–(iii) occur in the case both G/K_1 and G/K_2 are orientable.

3.2 Preparation for non-orientable cases

In order to prove two non-orientable cases in Theorem 3.1 (iv)–(v), it is necessary to show the following proposition.

Proposition 3.2.1. *If G/K_2 is non-orientable, then we have*

$$\begin{aligned} P(G/K_2^o;t) &= (1+t^{k_2})P(G/K_2;t), \\ P(G/K^o;t) &= (1+t^{2k_2-1})P(G/K_2;t) - P(n_1,n_2;t) - \epsilon_2(1-\epsilon_1)(1+t^{-1})t^{2n}, \end{aligned}$$

where

$$P(n_1,n_2;t) = \begin{cases} t^{2n_1+1} + t^{2n_1+2} + \cdots + t^{2n_2} & (n_1 < n_2) \\ 0 & (n_1 \geq n_2). \end{cases}$$

The goal of Section 3.2 is to prove Proposition 3.2.1. Our proof is essentially due to Uchida ([16] 2.4, 2.5 and 2.6).

First we show the following lemma.

Lemma 3.2.1. *If $k_1 > 2$, then G/K_2 is simply connected, hence K_2 is connected.*

Proof. We see $\pi_1(M) = \pi_1(G/K_2)$ from the transversality theorem ([5] (14.7)), Theorem 2.1 and $k_1 > 2$. Hence G/K_2 is simply connected. So $K_2 = K_2^o$ because a canonical map $G/K_2^o \to G/K_2$ is a finite covering. ■

Next we prepare the following two lemmas (Lemma 3.2.2 and 3.2.3) which just come from the condition $k_1 = 2$.

Lemma 3.2.2 ([16] Lemma 2.4.1)**.** *If $k_1 = 2$, then $R_k^* = id : H^*(G/K^o;\boldsymbol{Q}) \to H^*(G/K^o;\boldsymbol{Q})$ for all $k \in K$, where $R_k : [g] \to [gk]$ and R_k^* is the homomorphism induced from R_k.*

From Lemma 3.2.2, we can show the following lemma.

Lemma 3.2.3. *If $k_1 = 2$, then $H^*(G/K_s^o;\boldsymbol{Q}) = \mathrm{Im}(q_s^*) + \mathrm{Ker}(p_s^{o*})$ (possibly non direct sum), where the homomorphisms q_s^* and p_s^{o*} are induced from $q_s : G/K_s^o \to G/K_s$ and $p_s^o : G/K^o \to G/K_s^o$.*

Proof. The natural map $K_s^o/K^o \to K_s/K$ is a surjection because K_s/K is a (k_s-1)-sphere. So we see $K_s = K_s^o K$. In particular for each $a \in K_s$ there exists $k \in K$ such that R_a and R_k are homotopic by the connectedness of K_s^o. Hence $R_a^* = R_k^* : H^*(G/K_s^o; \mathbf{Q}) \to H^*(G/K_s^o; \mathbf{Q})$. By Lemma 3.2.2 the right R_k^* is an identity map in the following commutative diagram for all $a \in K_s$,

$$\begin{array}{ccc} H^*(G/K_s^o;\mathbf{Q}) & \xrightarrow{p_s^{o*}} & H^*(G/K^o;\mathbf{Q}) \\ R_a^* = R_k^* \downarrow & & R_k^* = id \downarrow \\ H^*(G/K_s^o;\mathbf{Q}) & \xrightarrow{p_s^{o*}} & H^*(G/K^o;\mathbf{Q}). \end{array}$$

So we have $p_s^{o*}(u) = p_s^{o*}(R_a^*(u))$ for $u \in H^*(G/K_s^o; \mathbf{Q})$ and $a \in K_s$. K_s/K_s^o acts on $H^*(G/K_s^o; \mathbf{Q})$ by R_l^* for $l \in K_s/K_s^o$. Then we easily see $\mathrm{Im}(q_s^*) = H^*(G/K_s^o; \mathbf{Q})^{K_s/K_s^o}$. Hence $R_l^*(v) = v$ for all $l \in K_s/K_s^o$ and $v \in \mathrm{Im}(q_s^*)$. Moreover if we put $K_s/K_s^o = \{l_1, \cdots, l_i\}$ then $R_{l_1}^*(u) + \cdots + R_{l_i}^*(u) \in \mathrm{Im}(q_s^*)$ for all $u \in H^*(G/K_s^o; \mathbf{Q})$. Therefore there is $w \in H^*(G/K_s; \mathbf{Q})$ such that $p_s^{o*} \circ q_s^*(w) = ip_s^{o*}(u)$. So we see $\mathrm{Im}(p_s^{o*}) = \mathrm{Im}(p_s^{o*} \circ q_s^*)$. Consequently we get the equation $H^*(G/K_s^o; \mathbf{Q}) = \mathrm{Im}(q_s^*) + \mathrm{Ker}(p_s^{o*})$. ■

Put $J_k = q_2^* H^k(G/K_2; \mathbf{Q})$ and $J = \oplus_k J_k$. Next we show properties about this J in the following two lemmas (Lemma 3.2.4 and 3.2.5) by using Lemma 3.2.3.

Lemma 3.2.4. *Let χ be the rational Euler class of the oriented (k_2-1)-sphere bundle $p_2^o : G/K^o \to G/K_2^o$. If $k_1 = 2$, then $\chi^2 \in J$ and* $\mathrm{Ker}(p_2^{o*}) = J \cdot \chi + J \cdot \chi^2$.

Proof. From the Thom-Gysin exact sequence of $p_2^o : G/K^o \to G/K_2^o$ that is,

$$\xrightarrow{p_2^{o*}} H^{q-1}(G/K_2^o) \xrightarrow{\delta^*} H^{q-k_2}(G/K_2^o) \xrightarrow{\cdot\chi} H^q(G/K_2^o) \xrightarrow{p_2^{o*}} H^q(G/K^o) \xrightarrow{\delta^*},$$

we see $\mathrm{Ker}(p_2^{oq}) = H^{q-k_2}(G/K_2^o; \mathbf{Q}) \cdot \chi$. By Lemma 3.2.3 $H^{q-k_2}(G/K_2^o; \mathbf{Q}) = J_{q-k_2} + \mathrm{Ker}(p_2^{oq-k_2})$. So we have $\mathrm{Ker}(p_2^{oq}) = J_{q-k_2} \cdot \chi + J_{q-2k_2} \cdot \chi^2 + \cdots + J_{q-Nk_2} \cdot \chi^N$ for some integer N. Because of the following bundle mapping

$$\begin{array}{ccc} G/K^o & \xrightarrow{R_k} & G/K^o \\ \downarrow p_2^o & & \downarrow p_2^o \\ G/K_2^o & \xrightarrow{R_k} & G/K_2^o, \end{array}$$

we see $R_k^*(\chi) = \chi$ or $-\chi$ for $k \in K$. Hence $R_k^*(\chi^2) = \chi^2$. Since the equation $J = \mathrm{Im}(q_2^*) = H^*(G/K_2^o; \mathbf{Q})^{K_2} = H^*(G/K_2^o; \mathbf{Q})^K$ holds (because of $K_s = K_s^o K$), we have $\chi^2 \in J$. So we get the equation $\mathrm{Ker}(p_2^{o*}) = J \cdot \chi + J \cdot \chi^2$. ■

We remark that non-orientability of G/K_2 is not assumed in Lemma 3.2.1 through 3.2.4 unlike Proposition 3.2.1. From now on we assume G/K_2 is non-orientable. Then $k_1 = 2$ from Lemma 3.2.1.

Lemma 3.2.5. *The following two properties hold.*

(1) $\dim(\mathrm{Ker}(p_2^{o*})) = \dim\, J + \dim(J \cap \mathrm{Ker}(p_2^{o*}))$.

(2) $J \cdot \chi \cap J \cdot \chi^2 = 0$, $J \cdot \chi^2 = J \cap \mathrm{Ker}(p_2^{o*})$ *and the homomorphism* $E : J \to \mathrm{Ker}(p_2^{o*})$ *is injective, where* E *is defined by* $E(y) = y \cdot \chi$.

Proof. First we show the property (1) by proving two inequalities. From Lemma 3.2.3 we get

$$\dim\, H^*(G/K_2^o; \mathbf{Q}) = \dim\, J + \dim(\mathrm{Ker}(p_2^{o*})) - \dim(J \cap \mathrm{Ker}(p_2^{o*})).$$

Since $q_2^* : H^*(G/K_2; \mathbf{Q}) \to H^*(G/K_2^o; \mathbf{Q})$ is an injective map, we have $\dim\, J = \dim\, H^*(G/K_2; \mathbf{Q})$. Since G/K_2 is non-orientable, there is $k \in K_2$ such that $R_k : G/K_2^o \to G/K_2^o$ reverses an orientation and an element in Im q_2^* is fixed by R_k^*. Because of the Poincaré duality theorem (Theorem 3.2) about G/K_2^o, for all $u \in \mathrm{Im}\ q_2^* \cap H^d(G/K_2^o)$ there exists some $v \in H^{2n-k_2-d}(G/K_2^o)$ such that $(\varphi(u)(v))\mu = uv$, where $\mu \in H^{2n-k_2}(G/K_2^o; \mathbf{Q})$ is the generator and $\varphi : H^d(G/K_2^o; \mathbf{Q}) \simeq \mathrm{Hom}(H^{2n-k_2-d}(G/K_2^o; \mathbf{Q}), \mathbf{Q})$. Now we have

$$-(\varphi(u)(v))\mu = R_k^*((\varphi(u)(v))\mu) = R_k^*(uv) = R_k^*(u)R_k^*(v) = -uv$$

and $R_k^*(u) = u$ because $u \in \mathrm{Im}\ q_2^*$. Hence we have $v \notin \mathrm{Im}\ q_2^*$. Consequently there is an element $v \in H^*(G/K_2^o; \mathbf{Q}) \backslash \mathrm{Im}\ q_2^*$ for $u \in \mathrm{Im}\ q_2^*$. So we see

$$2\ \mathrm{Im}\ q_2^* = 2\ \dim\, H^*(G/K_2; \mathbf{Q}) \leq \dim\, H^*(G/K_2^o; \mathbf{Q}).$$

Therefore we get

$$\dim\, H^*(G/K_2; \mathbf{Q}) = \dim J \leq \dim(\mathrm{Ker}(p_2^{o*})) - \dim(J \cap \mathrm{Ker}(p_2^{o*})).$$

From Lemma 3.2.4 we get $\chi^2 \in J$ and $J\chi^2 \subset \mathrm{Ker}(p_2^{o*})$. So $J \cdot \chi^2 \subset J \cap \mathrm{Ker}(p_2^{o*})$. Moreover we easily see $\dim(J \cdot \chi) \leq \dim\, J$. Hence we get

$$\dim(\mathrm{Ker}(p_2^{o*})) \leq \dim(J \cdot \chi) + \dim(J \cdot \chi^2) \leq \dim\, J + \dim(J \cap \mathrm{Ker}(p_2^{o*})).$$

So we have the property (1) from the two inequalities above .

Next we show the property (2). From the proof of the equation (1), we have $\dim(J \cdot \chi) = \dim\, J$ (so we get the injectivity of E) and $\dim(J \cdot \chi^2) = \dim(J \cap \mathrm{Ker}(p_2^{o*}))$ (so we get $J \cdot \chi^2 = J \cap \mathrm{Ker}(p_2^{o*})$). From Lemma 3.2.4 $\mathrm{Ker}(p_2^{o*}) = J \cdot \chi + J \cdot \chi^2$ and $\dim\, \mathrm{Ker}(p_2^{o*}) = \dim(J \cdot \chi) + \dim(J \cdot \chi^2)$, we have $J \cap J \cdot \chi = \{0\}$. Hence we get the property (2). ■

From Lemma 3.2.4 and 3.2.5, we can prove the following equation.

Proposition 3.2.2. $P(G/K_2^o; t) = (1 + t^{k_2})P(G/K_2; t)$.

Proof. From Lemma 3.2.5, we see $\dim J = \dim(\mathrm{Ker}(p_2^{o*})) - \dim(J \cap \mathrm{Ker}(p_2^{o*}))$. Moreover from Lemma 3.2.4 and 3.2.5 we have the equation

$$\mathrm{Ker}(p_2^{o*}) = J \cdot \chi \oplus (J \cap \mathrm{Ker}(p_2^{o*})).$$

Since $\chi \in H^{k_2}(G/K_2^o; \mathbf{Q})$ and $\dim H^*(G/K_2; \mathbf{Q}) = \dim J$, by the equation above we get

$$P(\mathrm{Ker}(p_2^{o*}); t) = t^{k_2} P(G/K_2; t) + P(J \cap \mathrm{Ker}(p_2^{o*}); t). \tag{5}$$

Comparing the equation (5) with

$$\begin{aligned} P(G/K_2^o; t) &= P(\mathrm{Im}(q_2^*); t) + P(\mathrm{Ker}(p_2^{o*}); t) - P(J \cap \mathrm{Ker}(p_2^{o*}); t) \\ &= P(G/K_2; t) + P(\mathrm{Ker}(p_2^{o*}); t) - P(J \cap \mathrm{Ker}(p_2^{o*}); t) \end{aligned}$$

(by Lemma 3.2.3) we get $P(G/K_2^o; t) = (1 + t^{k_2}) P(G/K_2; t)$ from the injectivity of q_2^*. ■

This result is a part of Proposition 3.2.1.

Next we show the following equation.

Proposition 3.2.3. $P(G/K^o; t) = (1 + t^{2k_2 - 1}) P(G/K_2; t) - (1 + t^{-1}) P(J \cap \mathrm{Ker}(p_2^{o*}); t)$.

Proof. From the Thom-Gysin exact sequence of $p_2^o : G/K^o \to G/K_2^o$ that is

$$\xrightarrow{p_2^{o*}} H^{q+k_2-1}(G/K^o) \xrightarrow{\delta^*} H^q(G/K_2^o) \xrightarrow{\cdot\chi} H^{q+k_2}(G/K_2^o) \xrightarrow{p_2^{o*}} H^{q+k_2}(G/K^o) \xrightarrow{\delta^*},$$

we easily get

$$\begin{aligned} P(\mathrm{Im}(\delta^*); t) &= P(G/K_2^o; t) - t^{-k_2} P(\mathrm{Ker}(p_2^{o*}); t), & (6) \\ P(G/K^o; t) &= t^{k_2 - 1} P(\mathrm{Im}(\delta^*); t) + P(\mathrm{Im}(p_2^{o*}); t). & (7) \end{aligned}$$

From the equation (5) and Proposition 3.2.2, we have

$$\begin{aligned} P(\mathrm{Im}(p_2^{o*}); t) &= P(G/K_2^o; t) - P(\mathrm{Ker}(p_2^{o*}); t) \\ &= (1 + t^{k_2}) P(G/K_2; t) - (t^{k_2} P(G/K_2; t) + P(J \cap \mathrm{Ker}(p_2^{o*}); t)) \\ &= P(G/K_2; t) - P(J \cap \mathrm{Ker}(p_2^{o*}); t). \end{aligned} \tag{8}$$

Substituting (7) for (6) and (8), we obtain the equation

$$\begin{aligned} P(G/K^o; t) &= t^{k_2 - 1} P(G/K_2^o; t) - t^{-1} P(\mathrm{Ker}(p_2^{o*}); t) \\ &+ P(G/K_2; t) - P(J \cap \mathrm{Ker}(p_2^{o*}); t). \end{aligned}$$

Moreover substituting the equation above for (5) and $P(G/K_2^o; t) = (1 + t^{k_2}) P(G/K_2; t)$, the identity of the proposition follows. ■

Let us concentrate on the term $(1+t^{-1})P(J \cap \mathrm{Ker}(p_2^{o*}); t)$. Consider the following commutative diagram

$$\begin{array}{ccc} H^*(G/K_2; \mathbf{Q}) & \xrightarrow{p_2^*} & H^*(G/K; \mathbf{Q}) \\ q_2^* \downarrow & & q^* \downarrow \\ H^*(G/K_2^o; \mathbf{Q}) & \xrightarrow{p_2^{o*}} & H^*(G/K^o; \mathbf{Q}), \end{array}$$

where q^* is the induced homomorphism from the natural covering map $q : G/K^o \to G/K$. Now q_2^* is an injection and moreover we show

Lemma 3.2.6. *$q^* : H^*(G/K; \boldsymbol{Q}) \to H^*(G/K^o; \boldsymbol{Q})$ is an isomorphism.*

Proof. Let $q^! : H^*(G/K^o; \mathbf{Q}) \to H^*(G/K; \mathbf{Q})$ be the transfer of the covering map $q : G/K^o \to G/K$. From Lemma 3.2.2 $R_k^* = id : H^*(G/K^o; \mathbf{Q}) \to H^*(G/K^o; \mathbf{Q})$, so $q^* \circ q^! : H^*(G/K^o; \mathbf{Q}) \to H^*(G/K^o; \mathbf{Q})$ is r times map where r is the covering degree of q. Hence q^* is surjective. The injectivity of q^* is well known. So q^* is an isomorphism. ■

Hence we have $\mathrm{Ker}(p_2^*) = \mathrm{Ker}(p_2^{o*} \circ q_2^*) \simeq \mathrm{Im}(q_2^*) \cap \mathrm{Ker}(p_2^{o*}) = J \cap \mathrm{Ker}(p_2^{o*})$. So we see $P(J \cap \mathrm{Ker}(p_2^{o*}); t) = P(\mathrm{Ker}(p_2^*); t)$. The inclusion $i_s : X_1 \cap X_2 \to X_s$ is homotopy equivalent to $p_s : G/K \to G/K_s$, hence $i_s^* = p_s^*$. Considering the following commutative diagram from the cohomology exact sequences of (M, X_1) and $(X_2, X_1 \cap X_2)$ and the excision isomorphism

$$\begin{array}{ccccc} H^*(M, X_1) & \longrightarrow & H^*(M) & \xrightarrow{f_1^*} & H^*(X_1) \\ \simeq\downarrow & & f_2^* \downarrow & & i_1^* \downarrow \\ H^*(X_2, X_1 \cap X_2) & \longrightarrow & H^*(X_2) & \xrightarrow{i_2^*} & H^*(X_1 \cap X_2), \end{array}$$

we get $f_2^*(\mathrm{Ker}(f_1^*)) = \mathrm{Ker}(i_2^*)$ by this diagram. Hence we obtain the following equations from the definition of n_1 and n_2, that is $f_s^*(c^{n_s}) \neq 0$ and $f_s(c^{n_s+1}) = 0$,

$$P(\mathrm{Ker}(i_2^*); t) = t^{2n_1+2} + \cdots + t^{2n_2} + \epsilon_2(1-\epsilon_1)t^{2n} \quad (n_1 < n_2)$$

and for $n_1 \geq n_2$

$$P(\mathrm{Ker}(i_2^*); t) = \epsilon_2(1-\epsilon_1)t^{2n}.$$

Because we have the two equations above, Proposition 3.2.3 and $P(J \cap \mathrm{Ker}(p_2^{o*}); t) = P(\mathrm{Ker}(i_2^*); t)$, we complete the proof of Proposition 3.2.1.

3.3 G/K_1 is orientable, G/K_2 is non-orientable

Let us prove Theorem 3.1 (iv). Assume G/K_1 is orientable and G/K_2 is non-orientable. From Proposition 3.2.1, we get the following equation.

Lemma 3.3.1. $t^{4n}P(G/K_2;t^{-1}) = t^{2k_2}P(G/K_2;t)$.

Proof. By Proposition 3.2.1, $P(G/K_2^o;t) = (1+t^{k_2})P(G/K_2;t)$. From the Poincaré duality of G/K_2^o, we see $P(G/K_2^o;t^{-1}) = t^{k_2-4n}P(G/K_2^o;t)$. ■

Since G/K_2 is non-orientable, we see $k_1 = 2$ by Lemma 3.2.1. Hence we can show the following equation.

Lemma 3.3.2. $P(G/K_2;t) = tP(G/K_1;t)+a(n_2)-t^{2n_2+1}a(2n-n_2-1)+t^{2n-1}(\epsilon_2+t\epsilon_2-1)$.

Proof. Since $k_1 = 2$, we see dim $G/K_1 = 4n-2$. By the Poincaré-Lefschetz duality and X_1 is a deformation retract to G/K_1,

$$H^q(X_1,\partial X_1;\mathbf{Q}) \simeq H_{4n-q}(X_1;\mathbf{Q}) \simeq H_{4n-q}(G/K_1;\mathbf{Q}) \simeq H^{q-2}(G/K_1;\mathbf{Q}).$$

So we get the equality $P(X_1,\partial X_1;t) = t^2P(G/K_1;t)$.

From Lemma 3.0.1 and 3.0.2, we have the equation

$$\begin{aligned} & P(X_1,\partial X_1;t) - tP(X_2;t) \\ = \ & t^{2n_2+2}+\cdots+t^{4n}+(1-\epsilon_2)t^{2n} - t(1+t^2+\cdots+t^{2n_2}+\epsilon_2 t^{2n}) \\ = \ & t^{2n_2+2}a(2n-n_2-1) - ta(n_2) + (1-\epsilon_2-t\epsilon_2)t^{2n}. \end{aligned}$$

Putting $P(X_1,\partial X_1;t) = t^2P(G/K_1;t)$ and $P(X_2;t) = P(G/K_2;t)$ in this equation, we get this lemma. ■

From Lemma 3.3.1 and 3.3.2, we can get the following proposition.

Proposition 3.3.1. *$P(G/K_1;t)$ is an even function.*

Proof. Multiplying both sides of the identity in Lemma 3.3.2 by t^{2k_2-1}, we get

$$\begin{aligned} & t^{2k_2-1}P(G/K_2;t) \\ = \ & t^{2k_2}P(G/K_1;t) + t^{2k_2-1}a(n_2) - t^{2k_2+2n_2}a(2n-n_2-1) + t^{2k_2+2n-2}(\epsilon_2+t\epsilon_2-1). \end{aligned}$$

Moreover multiplying both sides of the equation which substitute t^{-1} for t in Lemma 3.3.2 by t^{4n-1}, we get

$$\begin{aligned} & t^{4n-1}P(G/K_2;t^{-1}) \\ = \ & t^{4n-2}P(G/K_1;t^{-1}) + t^{4n-2n_2-1}a(n_2) - a(2n-n_2-1) + t^{2n}(\epsilon_2+t^{-1}\epsilon_2-1). \end{aligned}$$

From Lemma 3.3.1, the above two equations are same, that is

$$\begin{aligned} & t^{2k_2}P(G/K_1;t) + t^{2k_2-1}a(n_2) - t^{2k_2+2n_2}a(2n-n_2-1) + t^{2k_2+2n-2}(\epsilon_2+t\epsilon_2-1) \\ = \ & t^{4n-2}P(G/K_1;t^{-1}) + t^{4n-2n_2-1}a(n_2) - a(2n-n_2-1) + t^{2n}(\epsilon_2+t^{-1}\epsilon_2-1). \end{aligned}$$

By the Poincaré duality of G/K_1, $P(G/K_1;t) = t^{4n-2}P(G/K_1;t^{-1})$. Hence we get

$$\begin{aligned}
& (1-t^{2k_2})P(G/K_1;t) \\
= \;& (1-\epsilon_2)t^{2n}(1-t^{2k_2-2}) - \epsilon_2 t^{2n-1}(1-t^{2k_2}) \\
+ \;& (t^{2k_2-1}-t^{4n-2n_2-1})a(n_2) + (1-t^{2n_2+2k_2})a(2n-n_2-1). \qquad (9)
\end{aligned}$$

So we easily see $\chi(G/K_1) \neq 0$. Hence $P(G/K_1;t)$ is an even function. ■

Since $P(G/K_1;t)$ is an even function, it follows from (9) that

$$(t^{2k_2-1}-t^{4n-2n_2-1})a(n_2) - \epsilon_2 t^{2n-1}(1-t^{2k_2}) = 0, \qquad (10)$$

$$(1-t^{2k_2})P(G/K_1;t) = (1-\epsilon_2)t^{2n}(1-t^{2k_2-2}) + (1-t^{2n_2+2k_2})a(2n-n_2-1). \qquad (11)$$

Comparing the minimal degree terms in (10), we get $k_2 = \min\{2n-n_2,\ n\}$. If $k_2 = 2n-n_2$, then we see $\epsilon_2 = 0$ from (10) and $k_2 \geq 2$. However we see easily $\chi(G/K_1) \notin \mathbf{Z}$ from (11) and $k_2 \geq 2$. So this case does not occur.

Hence $k_2 = n$. So we see $\epsilon_2 = 1$ from (10).

If $n_2 \neq 0$, then we see $n_2 = n-1$ from (10). In this case we can also prove $\chi(G/K_1) \equiv -(1/n) \pmod{\mathbf{Z}}$ from (11). Hence $\chi(G/K_1) \notin \mathbf{Z}$. This is a contradiction.

Hence $k_2 = n$, $\epsilon_2 = 1$, $n_2 = 0$. If $\epsilon_1 = \epsilon_2 = 1$, then $n_1 = n_2 = 0$ and $n = 1$ because of Proposition 3.0.1. Since we assume $n \geq 2$, we have $\epsilon_1 = 0$. Therefore we have $n_1 = 2n-1$ by Proposition 3.0.1. Consequently we see $P(G/K_1;t) = P(\mathrm{Im} f_1^*;t) = a(n_1) = a(2n-1)$, and $G/K_1 \sim P_{2n-1}(\mathbf{C})$ from (11). So we get $P(G/K_2;t) = 1+t^{2n}$ from Lemma 3.3.2. By Proposition 3.2.1, $P(G/K_2^o;t) = (1+t^n)(1+t^{2n})$ and $G/K^o \sim S^{4n-1}$. This is the case that G/K_1 is orientable and G/K_2 is non-orientable in Theorem 3.1 (iv).

3.4 Both singular orbits are non-orientable

Let us prove Theorem 3.1 (v). Suppose G/K_1 and G/K_2 are non-orientable. By Lemma 3.2.1 and Proposition 3.2.1, we have $k_1 = k_2 = 2$, and

$$P(G/K_s^o;t) = (1+t^2)P(G/K_s;t), \qquad (12)$$

$$P(G/K^o;t) = (1+t^3)P(G/K_s;t) - P(n_r,n_s;t) - \epsilon_s(1-\epsilon_r)(1+t^{-1})t^{2n} \qquad (13)$$

where

$$P(p,q;t) = \begin{cases} t^{2p+1}+t^{2p+2}+\cdots+t^{2q} & (p<q) \\ 0 & (p \geq q). \end{cases}$$

From the Mayer-Vietoris exact sequence of $M = X_1 \cup X_2$, we have the following lemma.

Lemma 3.4.1. *The following equation holds.*

$$\begin{aligned}
& P(G/K_1;t) + P(G/K_2;t) \\
= \;& P(G/K;t) - t^{-1}(1+t^{2n})(1+t^2+\cdots+t^{2n}) + P(\mathrm{Im} f_1^* \oplus f_2^*;t)(1+t^{-1})
\end{aligned}$$

Proof. By the Mayer-Vietoris exact sequence

$$\cdots \longrightarrow H^q(M) \overset{f_1^*\oplus f_2^*}{\longrightarrow} H^q(X_1)\oplus H^q(X_2) \longrightarrow H^q(X_1\cap X_2) \longrightarrow H^{q+1}(M) \longrightarrow \cdots$$

where M is a rational cohomology complex quadric, we see

$$\begin{aligned} & P(X_1;t)+P(X_2;t) \\ = \ & P(X_1\cap X_2;t) - t^{-1}(1+t^{2n})(1+t^2+\cdots+t^{2n}) + P(\mathrm{Im} f_1^*\oplus f_2^*;t)(1+t^{-1}). \end{aligned}$$

Since X_s is a tubular neighborhood of G/K_s, $H^*(X_s)=H^*(G/K_s)$ and $X_1\cap X_2 = G/K$. So we get this lemma. ■

3.4.1 The case $\epsilon_1=\epsilon_2$.

We will prove this case is one of Theorem 3.1 (v). In this case we see $n_1=n_2$ from Proposition 3.0.1. So we get the following two equations from (13),

$$\begin{aligned} P(G/K_1;t) &= P(G/K_2;t), \\ P(G/K^o;t) &= (1+t^3)P(G/K_s;t). \end{aligned}$$

Now we have

$$P(\mathrm{Im} f_s^*;t) = 1+t^2+\cdots+t^{2n}$$

from Lemma 3.0.1 and Proposition 3.0.1. We can get the following lemma because of Lemma 3.4.1 and $\epsilon_1=\epsilon_2$.

Lemma 3.4.2. *The following equation holds.*

$$P(G/K_1;t)+P(G/K_2;t) = (1-t^{2n-1})(1+t^2+\cdots+t^{2n}) + P(G/K;t).$$

Since $k_s=2$ $(s=1,2)$, we have $q^*: H^*(G/K)\to H^*(G/K^o)$ is an isomorphism by Lemma 3.2.6. Hence $\chi(G/K)=\chi(G/K^o)=0$. Therefore we have $\chi(G/K_s)\neq 0$ from $P(G/K_1;t)=P(G/K_2;t)$ and Lemma 3.4.2. Hence $P(G/K_s;t)$ is an even function from Lemma 3.1.3. Substituting Lemma 3.4.2 for $P(G/K;t)=P(G/K^o;t)=(1+t^3)P(G/K_s;t)$ and comparing the degrees, we have $n=2$, $P(G/K_s;t)=1+t^2+t^4$, and $P(G/K;t)=P(G/K^o;t)=(1+t^3)(1+t^2+t^4)$. Moreover we have $P(G/K_s^o;t)=(1+t^2)(1+t^2+t^4)$ from the equation (12). This result is Theorem 3.1 (v).

3.4.2 The case $\epsilon_1\neq\epsilon_2$.

We will prove this case is also one of Theorem 3.1 (v). In this case we see $n_1\neq n_2$ because $n_1+n_2+1=2n$ (Proposition 3.0.1). We may assume $\epsilon_1=0$ and $\epsilon_2=1$. From (13), for $s=1$,

$$P(G/K^o;t) = (1+t^3)P(G/K_1;t) - P(n_2,n_1;t), \tag{14}$$

moreover for $s = 2$

$$P(G/K^o; t) = (1 + t^3)P(G/K_2; t) - P(n_1, n_2; t) - (1 + t^{-1})t^{2n}. \tag{15}$$

From (14) and (15) we can show the following two equations;

$$\begin{aligned} & (1+t)(1-t+t^2)\{P(G/K_1;t) - P(G/K_2;t)\} \\ = \; & -t^{2n_1+1}(1+t)(1+t^2+\cdots+t^{2(n_2-n_1)-2}) - (1+t)t^{2n-1} \quad (\text{if } n_1 < n_2), \end{aligned} \tag{16}$$

$$\begin{aligned} & (1+t)(1-t+t^2)\{P(G/K_2;t) - P(G/K_1;t)\} \\ = \; & -t^{2n_2+1}(1+t)(1+t^2+\cdots+t^{2(n_1-n_2)-2}) + (1+t)t^{2n-1} \quad (\text{if } n_1 > n_2). \end{aligned} \tag{17}$$

From these equations (16) and (17), we see

$$\chi(G/K_1) - \chi(G/K_2) = m = 3^{-1}(n_2 - n_1 + 1) \in \mathbf{Z} \quad (\text{if } n_1 < n_2), \tag{18}$$

$$\chi(G/K_2) - \chi(G/K_1) = m' = 3^{-1}(n_1 - n_2 - 1) \in \mathbf{Z} \quad (\text{if } n_1 > n_2). \tag{19}$$

Hence if $n_2 > n_1$ then $n_2 - n_1 = 3m - 1$ and if $n_2 < n_1$ then $n_1 - n_2 = 1 + 3m'$.

Now we see $\chi(G/K) = \chi(G/K^o) = 0$ by Lemma 3.2.6, (14) and (15).

Hence we have $\chi(G/K_1) + \chi(G/K_2) = 2n + 2$ by Lemma 3.4.1. Therefore we can easily show $\chi(G/K_s) \neq 0$ $(s = 1, 2)$ by (18) and (19). So we see $\mathrm{rank}(G) = \mathrm{rank}(K_s^o)$ and we have $H^{odd}(G/K_s^o; \mathbf{Q}) = 0$ from [14] Chapter III and Theorem 3.21 in Chapter VII. Consequently we have, by the equation (12),

$$H^{odd}(G/K_s; \mathbf{Q}) = 0.$$

Hence if $n_1 < n_2$ we have from (16),

$$\begin{aligned} P(G/K_2;t) - P(G/K_1;t) &= t^{2n-3m+2}a(3m-2) + t^{2n} \\ t^3(P(G/K_2;t) - P(G/K_1;t)) &= t^{2n-3m+1}a(3m-2) + t^{2n-1}. \end{aligned}$$

Moreover if $n_1 > n_2$ we have from (17),

$$\begin{aligned} P(G/K_1;t) - P(G/K_2;t) &= t^{2n-3m'}a(3m') - t^{2n} \\ t^3(P(G/K_1;t) - P(G/K_2;t)) &= t^{2n-3m'-1}a(3m') - t^{2n-1}. \end{aligned}$$

From the above equations we have

$$\begin{aligned} t^{2n+3m-1} + t^{2n+3m+1} + t^{2n+3} &= t^{2n-3m+1} + t^{2n-3m+3} + t^{2n-1} \quad (\text{if } n_1 < n_2) \\ t^{2n+3m'+1} + t^{2n+3m'+3} - t^{2n+3} &= t^{2n-3m'-1} + t^{2n-3m'+1} - t^{2n-1} \quad (\text{if } n_1 > n_2) \end{aligned}$$

From (18), we see $m \neq 0$. So the case $n_1 < n_2$ does not occur by the above equation. Therefore we see $n_1 > n_2$ and $m' = 0$ by the above equation. From (19) and $2n = n_1 + n_2 + 1$, we have $n_1 = n$ and $n_2 = n - 1$. Hence we have $P(G/K_1; t) = P(G/K_2; t)$ and

$$P(G/K; t) = P(G/K^o; t) = (1 + t^3)P(G/K_s; t) - t^{2n-1} - t^{2n}$$

from (14), (15) and Lemma 3.2.6 where $s = 1$ or 2. Moreover we have $P(\mathrm{Im} f_1^* \oplus f_2^*; t) = a(n) + t^{2n}$ because of the definition of $\mathrm{Im} f_s^*$, $\epsilon_1 = 0$, $\epsilon_2 = 1$, $n_1 = n$ and $n_2 = n - 1$. So we have

$$2P(G/K_s; t) = P(G/K; t) + (1 - t^{2n+1})a(n-1) + 2t^{2n}$$

by Lemma 3.4.1. Therefore we can show $n = 2$, $P(G/K_s; t) = 1 + t^2 + t^4$, $P(G/K_s^o; t) = (1 + t^2)(1 + t^2 + t^4)$ and $P(G/K^o; t) = P(G/K; t) = (1 + t^5)(1 + t^2)$, because of the above two equations and the equation (12). This result is in Theorem 3.1 (v).

Therefore we have Theorem 3.1. Next we will exhibit the pairs of Lie groups (G, U) whose Poincaré polynomial $P(G/U; t)$ satisfies Theorem 3.1.

4 First step to the classification

Let G be a compact connected Lie group and U be its maximal rank closed connected subgroup. The aim of this section is to find pairs (G, U), such that the Poincaré polynomial of the quotient space G/U coincides with a Poincaré polynomial $P(G/K_s^o; t)$ in Theorem 3.1, up to local isomorphism.

4.1 Equivalence relation

We will mention some basic notations. First we define an *essential isomorphism.*

DEFINITION(**essential isomorphism**) Let (G, M) be a pair of a compact Lie group G and a manifold M with G-action. We regard H as an intersection of all isotropy groups $\cap_{x\in M} G_x$ (we call it a *kernel* of (G, M)). Then we call the pair $(G/H, M)$ an *induced effective action* from (G, M). We say that two pairs (G, M) and (G', M') are *essentially isomorphic* if their induced effective actions are equivariantly diffeomorphic.

We will classify (G, M) up to this equivalence relation (essential isomorphism). Next we define an *essential direct product.*

DEFINITION(**essential direct product**) Let $G_1, \cdots, G_k$ be compact Lie groups, and N be a finite normal subgroup of $G^* \simeq G_1 \times \cdots \times G_k$. We say that the factor group $G = G^*/N$ is an *essential direct product* of $G_1, \cdots, G_k$ and denote it $G \simeq G_1 \circ \cdots \circ G_k$.

Note that all compact connected Lie groups are constructed by an essential direct product of some simply connected compact Lie groups and a torus (see [14] Corollary 5.31 in Chapter V). Because we would like to classify up to essential isomorphism, we can assume that

$$G \simeq G_1 \times \cdots \times G_k \times T$$

for some simply connected simple Lie groups G_i and a torus T. Moreover we can assume that G acts almost effectively on M, where we say that G acts *almost effectively* on M if $H = \cap_{x \in M} G_x$ is a finite group. In this case G acts almost effectively on the principal orbit G/K, hence we easily see

Proposition 4.1.1. *K dose not contain any positive dimensional closed normal subgroup of G.*

4.2 Candidates for (G, K_s)

Let G be a simply connected compact simple Lie group and U be its closed connected subgroup of the same rank as G, where the rank of a Lie group means the dimension of a maximal torus subgroup. The purpose of this section is to find the pair (G, U) such that the Poincaré polynomial $P(G/U; t)$ is equal to some Poincaré polynomial in Theorem 3.1.

In Theorem 3.1 we get some even functions $P(G/K_s; t)$ (or $P(G/K_s^o; t)$). If $P(G/K_s; t)$ is an even function, then $\chi(G/K_s) \neq 0$. So we have $\mathrm{rank}G = \mathrm{rank}K_s$ from [14] Chapter III. The following lemma is well known.

Lemma 4.2.1 ([14] Theorem 7.2 in Chapter V)**.** *If $G \simeq G_1 \times \cdots \times G_k \times T$ then the same rank subgroup of G is $G' \simeq G_1' \times \cdots \times G_k' \times T$. Here G_i' is the same rank subgroup of G_i.*

Hence we may only find a simply connected compact simple Lie group G and its same rank closed connected subgroup U such that $P(G/U; t)$ is one of the factors of Poincaré polynomials in Theorem 3.1, that is, since $P(G/K_s^o) = P(G_1/U_1) \cdots P(G_k/U_k)$, we may only find the pair (G_i, U_i).

To find such (G, U), we prepare the following lemma ([14] Theorem 3.21 in Chapter VII).

Lemma 4.2.2 (Hirsch formula)**.** *Let G be a connected compact Lie group and U a same rank connected closed subgroup of G. Suppose $H^*(G; \boldsymbol{Q}) \simeq \Lambda(x_{2s_1+1}, \cdots, x_{2s_l+1})$ and $H^*(U; \boldsymbol{Q}) \simeq \Lambda(x_{2r_1+1}, \cdots, x_{2r_l+1})$ where $l = \mathrm{rank}\ G = \mathrm{rank}\ U$ and x_i is an element of the i-th degree cohomology. Then $P(G/U; t)$ satisfies the equation*

$$P(G/U; t) = \prod_{i=1}^{l} \frac{1 - t^{2s_i}}{1 - t^{2r_i}}.$$

In particular, from this Hirsch formula, we can get $P(G/U; t)$ if we know $H^*(G; \mathbf{Q})$ and $H^*(U; \mathbf{Q})$ only. Let us find (G, U).

If G is a classical simple Lie group, then (G, U) are known ([18] (9.3)). If G is exceptional and U is maximal, then such pairs (G, U) are also known ([14] Chapter V). Hence in these cases we can compute $P(G/U; t)$ by the Hirsch formula. So we may pick up $P(G/U; t)$ which is in the factor of Poincaré polynomials in Theorem 3.1.

Assume G is an exceptional Lie group and U' is not a maximal subgroup, where rank $G =$ rank U'. Now the maximal subgroup U (which has same rank) of G is constructed by the product of the classical Lie groups and a torus, except three cases $(E_7, E_6 \times T^1)$, $(E_8, E_6 \times SU(3))$ and $(E_8, E_7 \times SU(2))$, by [14]. Because U is maximal and U' is not so, they satisfy $G \supset U \supset U'$. Hence, except the above three cases, we can get all $P(G/U'; t) = P(U/U'; t)P(G/U; t)$ by the above same argument. Assume $(G, U) = (E_7, E_6 \times T^1)$, $(E_8, E_6 \times SU(3))$ or $(E_8, E_7 \times SU(2))$. For example we take $U' \subset E_6 \times T^1 \subset E_7 = G$ such that U' is not maximal. Then there is some $V \subset E_6$ such that $U' \subset V \times T^1 \subset E_6 \times T^1$, where V is a maximal subgroup of E_6. Moreover we see such V is constructed by the product of the classical Lie groups and a torus because V is a maximal subgroup of E_6 (see [14]). So we can get $P(G/U'; t) = P(G/(E_6 \times T^1); t)P(E_6/V; t)P(V/U'; t)$ by the same argument. For the other cases we can get $P(G/U'; t)$. Therefore we also have $P(G/U'; t)$ even if G is an exceptional Lie group and U' is not a maximal subgroup. So we may pick up $P(G/U; t)$ which is in the factors of Poincaré polynomials in Theorem 3.1.

From the above argument we get the following propositions. Note that the first three propositions were also known by Uchida (Section 4.2 in [16]).

Proposition 4.2.1. *If $P(G/U; t) = 1 + t^{2a}$, then (G, U) is locally isomorphic to*

$$(SO(2a+1), SO(2a)) \quad or \quad (G_2, SU(3)), a = 3.$$

Proposition 4.2.2. *If $P(G/U; t) = 1 + t^2 + \cdots + t^{2b}$, then (G, U) is locally isomorphic to one of the following.*

$$\begin{aligned}
&(SU(b+1), S(U(b) \times U(1))),\\
&(SO(b+2), SO(b) \times SO(2)), b = 2m+1,\\
&(Sp(\frac{b+1}{2}), Sp(\frac{b-1}{2}) \times U(1)), b = 2m+1,\\
&(G_2, U(2)), b = 5.
\end{aligned}$$

Proposition 4.2.3. *If $P(G/U; t) = (1 + t^{2a})(1 + t^2 + \cdots + t^{2b})$, then (G, U) is locally isomorphic to one of the following.*

$$\begin{aligned}
&(SO(2m+2), SO(2m) \times SO(2)), a = b = m,\\
&(SO(2m+3), SO(2m) \times SO(2)), a = m, b = 2m+1,\\
&(SO(7), U(3)), a = b = 3,\\
&(SO(9), U(4)), a = 3, b = 7,\\
&(SU(3), T^2), a = 1, b = 2,\\
&(SO(10), U(5)), a = 3, b = 7,
\end{aligned}$$

$(SU(5), S(U(2) \times U(3))), a = 2, b = 4,$
$(Sp(3), Sp(1) \times Sp(1) \times U(1)), a = 2, b = 5,$
$(Sp(3), U(3)), a = b = 3,$
$(Sp(4), U(4)), a = 3, b = 7,$
$(G_2, T^2), a = 1, b = 5,$
$(F_4, Spin(7) \circ T^1), a = 4, b = 11,$
$(F_4, Sp(3) \circ T^1), a = 4, b = 11.$

Proposition 4.2.4. *If n is an even number and $P(G/U;t) = 1 + t^n + t^{2n} + t^{3n}$ then $n = 2$ or 4. The case $n = 2$ is in Proposition 4.2.2. If $n = 4$, then (G, U) is locally isomorphic to*

$$(Sp(4), Sp(1) \times Sp(3)).$$

By Theorem 3.1, it is enough to consider the above four cases. Before we start the classification, we outline the proof of the classification.

4.3 Outline of the proof of the classification

We will state the outline for the classification. To classify (G, M), where G is a compact Lie group and M is a rational cohomology complex quadric, we will consider five cases in Theorem 3.1 (i)–(v). Let us recall the following theorem.

Theorem 4.1 (differentiable slice theorem)**.** *Let G be a compact Lie group and M be a smooth G-manifold. Then for all $x \in M$ there is a closed tubular neighborhood U of the orbit $G(x) \cong G/G_x$ and a closed disk D_x, which has an orthogonal G_x-action via the representation $\sigma_x : G_x \to O(D_x)$, such that $G \times_{G_x} D_x \cong U$ as a G-diffeomorphism.*

We call the representation σ_x in this theorem *the slice representation* of G_x at $x \in M$. Since we get candidates of singular isotropy groups in Section 4.2, first we will compute the slice representation of the singular isotropy subgroups K_1 and K_2 from the differentiable slice theorem. Then we will get a candidate for the transformation group G and two tubular neighborhoods $X_1 \cong G \times_{K_1} D^{k_1}$ and $X_2 \cong G \times_{K_2} D^{k_2}$ of two singular orbits G/K_1 and G/K_2.

Next we will construct the G-manifold M up to equivalence by making use of the structure theorem (Theorem 2.1) and the following lemma.

Lemma 4.3.1 ([16] Lemma 5.3.1)**.** *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions is satisfied (where $M(f) = X_1 \cup_f X_2$):*

1. *f is G-diffeotopic to f'.*

2. $f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1.

3. $f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.

From Theorem 2.1, we can put $\partial X_s = G/K$. Hence we may assume the gluing map is in $N(K;G)/K$, because the set of all G-equivariant diffeomorphisms of G/K is isomorphic to $N(K;G)/K$ where $N(K;G)$ is a normalizer group of K in G.

Finally we will compute the cohomology of the manifold which we constructed. Then we can decide whether this manifold is a rational cohomology complex quadric or not. This is a story of the classification.

Let us start to classify (G, M) from the next section.

5 The two singular orbits are non-orientable

In this section, we consider the case two singular orbits are non-orientable. The goal of this section is to prove this case does not occur. By Theorem 3.1 (III), we see $P(G/K_s;t) = 1 + t^2 + t^4$ and $P(G/K_s^o;t) = (1+t^2)(1+t^2+t^4)$. So rank $G =$ rank K_s^o.

5.1 G/K_s^o is indecomposable

A manifold is called *decomposable* if it is a product of positive dimensional manifolds. In this section we consider the case where G/K_s^o is indecomposable. By Proposition 4.2.3 ($a = 1$, $b = 2$), we see $G = SU(3) \times G' \times T^h$ and $K_s^o = T_s^2 \times G' \times T^h$. Here T_s^2 is a maximal torus of $SU(3)$, G' is a product of compact simply connected simple Lie groups and T^h is a torus. First we prove the following lemma.

Lemma 5.1.1. *$G = SU(3)$, $K_1^o = K_2^o = T^2$ and $K_1 = K_2$.*

Proof. Because $k_s = 2$, we see $K_s^o/K^o \cong S^1$. Hence $G' \times T^{h-1} \subset K^o$ from the assumption of G'. Therefore $G' = \{e\}$ and $h = 0$ or 1 from Proposition 4.1.1.

To show $h = 0$, let us consider the slice representation $\sigma_s : K_s \to O(2)$. Since G/K_s is non-orientable, there is an element $g_s \in K_s - K_s^o$ such that

$$\sigma_s(g_s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the centralizer of $\sigma_s(g_s)$ in $O(2)$ is a finite group $\mathbf{Z}_2 \times \mathbf{Z}_2$ and the centralizer of g_s in K_s contains $\{e\} \times T^h$, we see $\{e\} \times T^h \subset \mathrm{Ker}(\sigma_s|_{K_s^o}) = K^o$ where $\sigma_s|_{K_s^o}$ is the restrictions to K_s^o. Hence $h = 0$ from Proposition 4.1.1. Therefore $K_s^o = T_s^2$ which is the maximal torus of $SU(3)$. Moreover $K_1 = K_2$ because $K \subset K_1 \cap K_2$ and $K_s = KK_s^o$. ■

Next we construct the $SU(3)$-manifold. To construct the $SU(3)$-manifold, we will attach two tubular neighborhoods along their boundary. So first we consider two tubular neighborhoods of two singular orbits. Denote the non-trivial slice representation of K_s by $\sigma_s : K_s \to O(2)$ for $s = 1, 2$. Since we can assume

$$T^2 = K_s^o = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} = (u, v, w) \in SU(3) \middle| u, v, w \in U(1), uvw = 1 \right\},$$

the slice representation restricted to T^2 is

$$\sigma_s|_{T^2}((u, v, w)) = \phi(v^m)\phi(w^l) \tag{20}$$

where $\phi : U(1) \to SO(2)$ is a canonical isomorphism and $m, l \in \mathbf{Z}$. Now we can easily check $N(T^2; SU(3))/T^2$ is

$$\left\{ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

This group is isomorphic to the three degree symmetric group S_3. Hence $N(K_s^o; SU(3))/K_s^o \supset K_s/K_s^o \simeq \mathbf{Z}_2$ or S_3 ($K_s^o = T^2$) by non-orientability of $SU(3)/K_s$. We have the following two lemmas.

Lemma 5.1.2. *If* $\alpha \in K_s$, *then* $\{(\bar{u}^2, u, u) \in SU(3)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$.
If $\beta \in K_s$, *then* $\{(u, u, \bar{u}^2) \in SU(3)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$.
If $\gamma \in K_s$, *then* $\{(u, \bar{u}^2, u) \in SU(3)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$.

Proof. Assume $\alpha \in K_s$. The centralizer of α in K_s contains $\{(\bar{u}^2, u, u) | u \in U(1)\}$. Then the slice representation is $\sigma_s(\bar{u}^2, u, u) = \sigma_s(\alpha(\bar{u}^2, u, u)\alpha^{-1}) \in SO(2)$. On the other hand $\sigma_s(\alpha(\bar{u}^2, u, u)\alpha^{-1}) = \sigma_s(\alpha)\sigma_s(\bar{u}^2, u, u)\sigma_s(\alpha)^{-1} = \sigma_s(\bar{u}^2, u, u)^{-1}$ because $\sigma_s(\alpha) \in O(2) - SO(2)$. This means $\sigma_s(\bar{u}^2, u, u) = \{e\}$ for all $u \in U(1)$.
Similarly we can show other cases. ■

Lemma 5.1.3. $K_s/K_s^o \simeq \boldsymbol{Z}_2$.

Proof. If $K_s/K_s^o \simeq S_3$, then $K_s = N(K_s^o; SU(3))$. Hence $\{\alpha, \beta, \gamma, A, A^{-1}\} \subset K_s$. From Lemma 5.1.2, $\{(\bar{u}^2, u, u), (u, u, \bar{u}^2), (u, \bar{u}^2, u)\} \subset \mathrm{Ker}(\sigma_s|_{K_s^o})$. So we see

$$\{(\bar{u}^2, u, u), (u, u, \bar{u}^2), (u, \bar{u}^2, u)\} \subset K^o.$$

Hence $K^o = T^2$ because K^o is a connected Lie subgroup in $K_s^o = T^2$. This contradicts $K_s^o/K^o \cong S^1$. ■

Because $T^2 \cup \alpha T^2$, $T^2 \cup \beta T^2$ and $T^2 \cup \gamma T^2$ are conjugate, we can consider $K_s = T^2 \cup \alpha T^2$ for $s = 1,\ 2$. We can check $\mathrm{Ker}(\sigma_s|_{K_s^o})/K^o \simeq \mathbf{Z}_m$ as follows. If we put $\mathrm{Ker}(\sigma_1|_{K_1^o})/K^o \simeq \mathbf{Z}_m$ and $\mathrm{Ker}(\sigma_2|_{K_2^o})/K^o \simeq \mathbf{Z}_{m'}$ where $m \neq m'$, then the principal isotropy group of G-action on X_1 is different from the principal isotropy group of G-action on X_2. This contradicts that X_1 and X_2 have a same principal orbit because of $X_1 \cap X_2 = G/K$. Hence we can put $\mathrm{Ker}(\sigma_s|_{K_s^o})/K^o \simeq \mathbf{Z}_m$ for $s = 1,\ 2$. Therefore we can easily see the following lemma from above lemmas and the equation (20).

Lemma 5.1.4. *For $m \in \boldsymbol{N}$, we can consider $\{I, \alpha\} = K_s/K_s^o$, and we have*

$$\begin{aligned} &K^o = \{(\bar{u}^2, u, u)\} \text{ and} \\ &\sigma_s|_{K_s^o}(\overline{uv}, u, v) = \phi(u^m)\phi(v^{-m}). \end{aligned}$$

Moreover we see $\sigma_1|_{T^2} = \sigma_2|_{T^2}$. Hence we get the tubular neighborhood

$$X_s^{(m)} = SU(3) \times_{K_s} D_m^2$$

where K_s acts on the disk D_m^2 by $\sigma_s : K_s \to O(2)$ such that $\mathrm{Ker}(\sigma_s|_{K_s^o})/K^o \simeq \mathbf{Z}_m$.

Next we consider an attaching map from $X_1^{(m)}$ to $X_2^{(m)}$. Since the attaching map f is equivariantly diffeomorphic to G/K, f is in $N(K;G)/K$. Now we have

$$K = \left\{ \begin{pmatrix} \overline{uv} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix}, \begin{pmatrix} -\overline{uv} & 0 & 0 \\ 0 & 0 & u \\ 0 & v & 0 \end{pmatrix} \in SU(3) \,\middle|\, u^m = v^m \right\},$$

for some $m \in \mathbf{N}$ from Lemma 5.1.4.

Hence we see the following lemma.

Lemma 5.1.5. $N(K; SU(3)) = K$.

Hence the attaching map is unique up to equivalence by Lemma 4.3.1 (1.). So we see such an $SU(3)$-manifold exists for each $m \in \mathbf{N}$ and

$$M^{(m)} = SU(3) \times_{K_s} S^2$$

where K_s acts on S^2 via the linear representation $\sigma_s : K_s \to O(2)$ such that $\mathrm{Ker}(\sigma_s|_{K_s^o})/K^o \simeq \mathbf{Z}_m$. From the above argument, we have the following proposition.

Proposition 5.1.1. *Let M be an $SU(3)$-manifold which has codimension one orbits $SU(3)/K$ and two singular orbits $SU(3)/K_s$ $(s = 1, 2)$. Then M is $SU(3)$-equivariant diffeomorphic to $M^{(m)}$ $(m \in \boldsymbol{N})$.*

Finally we show such an $SU(3)$-manifold $M^{(m)}$ is not a rational cohomology complex quadric.

Proposition 5.1.2. $M^{(m)} = SU(3) \times_{K_s} S^2$ *is not a rational cohomology complex quadric.*

Proof. The manifold $N = SU(3) \times_{K_s^o} S^2$ is a double covering of $M^{(m)}$, where K_s^o acts on S^2 by the restricted representation $\sigma_s|_{K_s^o}$. If $M^{(m)}$ is a rational cohomology complex quadric, then $M^{(m)}$ is simply connected. Hence $M^{(m)} \cong N$. Now N is an S^2-bundle over $SU(3)/T^2 = SU(3)/K_s^o$, and $SU(3)/T^2$ is simply connected. Hence $H^*(M^{(m)}; \mathbf{Q}) \simeq H^*(N; \mathbf{Q}) \simeq H^*(S^2; \mathbf{Q}) \otimes H^*(SU(3)/T^2; \mathbf{Q})$ because $H^{odd}(S^2; \mathbf{Q}) = H^{odd}(SU(3)/T^2; \mathbf{Q}) = 0$. Hence $H^*(M^{(m)}; \mathbf{Q}) \not\simeq H^*(Q_4; \mathbf{Q})$. This is a contradiction. ■

Therefore this case does not occur. Next we consider the case G/K_1^o is decomposable.

5.2 G/K_1^o is decomposable

Assume G/K_1^o is decomposable. By Proposition 4.2.1 ($a = 1$), 4.2.2 ($b = 2$), we see that

$$G = SU(2) \times SU(3) \times G' \times T^h,$$
$$K_1^o = T^1 \times S(U(2) \times U(1)) \times G' \times T^h.$$

First we prove the following lemma.

Lemma 5.2.1. $G = SU(2) \times SU(3)$ *and* $K_1^o = T^1 \times S(U(2) \times U(1)) \simeq K_2^o$.

Proof. If G/K_2^o is indecomposable, then we see $K_2^o = SU(2) \times T^2 \times G' \times T^h$. Because $K^o \subset K_1^o \cap K_2^o = T^1 \times T^2 \times G' \times T^h$, we have $\dim K \leq 3 + \dim G' + h$. But we also have $\dim K = 4 + \dim G' + h$ because $K_s^o/K^o \cong S^1$ for $s = 1, 2$. This is a contradiction. So G/K_2^o is decomposable. Hence we have $K_1^o \simeq K_2^o$, $G' = \{e\}$ and $h = 0$ or 1 by Proposition 4.1.1. Moreover we can show $h = 0$ like Lemma 5.1.1. ■

Now we have $N(T^1; SU(2))/T^1 \simeq \mathbf{Z}_2$ and $N(S(U(2) \times U(1)); SU(3)) = S(U(2) \times U(1))$. Because of the non-orientability of G/K_s and Lemma 5.2.1, we get

$$K_1 = N(T^1; SU(2)) \times S(U(2) \times U(1)) \simeq K_2.$$

For the slice representation $\sigma_s : K_s \to O(2)$, there exists $g_s \in K_s - K_s^o$ such that

$$\sigma_s(g_s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here the centralizer of $\sigma_s(g_s)$ in $O(2)$ is a finite group and the centralizer of g_s in K_s contains $\{e\} \times S(U(2) \times U(1))$. Hence $S(U(2) \times U(1)) \subset \mathrm{Ker}(\sigma_s)$. So the slice representation $\sigma_s : K_s \to O(2)$ has a decomposition $\sigma_s : K_s \to N(T^1; SU(2)) \to O(2)$. Moreover $K^o = \{e\} \times S(U(2) \times U(1))$ by $K_s/K \cong S^1$. Therefore there is an equivariant decomposition

$$M \cong \big((SU(2) \times_{N(T^1)} D^2) \cup_\partial (SU(2) \times_{N(T^1)} D^2)\big) \times (SU(3)/S(U(2) \times U(1)))$$

where $N(T^1) = N(T^1; SU(2))$ and ∂ is an attaching map from $\partial(SU(2) \times_{N(T^1)} D^2)$ to itself. As is well known $SU(3)/S(U(2) \times U(1)) \cong P_2(\mathbf{C})$. Hence a G-manifold is $M \cong N \times P_2(\mathbf{C})$, where N is some $SU(2)$-manifold (In fact we easily see $N = SU(2) \times_{N(T^1)} S^2$). However this contradicts M is indecomposable. So this case does not occur.

6 One singular orbit is orientable, the other is non-orientable

The goal of this section is to prove this case is one of the exotic case in Theorem 1.1.

Assume G/K_1 is orientable, G/K_2 is non-orientable. Then $k_1 = 2$ from Lemma 3.2.1. Since $k_1 = 2$, we have $K_1/K \cong S^1$. Let us prove the uniqueness of (G, M).

6.1 Uniqueness of (G, M)

By Theorem 3.1 (II), we see $G/K^o \sim S^{4n-1}$, $G/K_1 \sim P_{2n-1}(\mathbf{C})$, $P(G/K_2^o; t) = (1 + t^n)(1 + t^{2n})$ and $P(G/K_2; t) = (1 + t^{2n})$. Since $P(G/K_1; t) = P(\mathrm{Im} f_1^*; t)$ from Section 3.3, we have G/K_1 is indecomposable. Because $K_1/K \cong S^1$, we get $G = H \times T^h$, $K_1 = H_1 \times T^h$ ($h = 0$ or 1) where H is a simply connected simple Lie group and H_1 is its closed subgroup. First we show the following lemma.

Lemma 6.1.1. *$k_2 = n = 2$ or 4.*

Proof. We see $n = k_2$ from Theorem 3.1. Moreover we have, from Proposition 4.2.2,

$$\begin{aligned}(H, H_1) \approx\ & (SU(2n), S(U(2n-1) \times U(1))),\\ & (SO(2n+1), SO(2n-1) \times SO(2)),\\ & (Sp(n), Sp(n-1) \times U(1)) \text{ or}\\ & (G_2, U(2)),\ n = 3.\end{aligned}$$

Assume $k_2 = n$ is an odd number.

If $(H, H_1) = (SU(2n), S(U(2n-1) \times U(1)))$, then the slice representation $\sigma_1 : K_1 \xrightarrow{\rho} U(1) \xrightarrow{\cong} SO(2)$ is as follows;

$$\rho\left(\begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix}, x\right) = \det(A^{-1})^l x^m \in U(1)$$

where $A \in U(2n-1)$, $x \in T^h$ ($h = 0$ or 1, if $h = 0$ then $x = 1$) and $(l, m) \in \mathbf{Z}^2 - \{(0, 0)\}$. Moreover we see $\mathrm{Ker}(\rho) = K$. Hence we have

$$\begin{aligned} K^o &\simeq SU(2n-1) \text{ if } h = 0 \text{ or}\\ K^o &\simeq U(2n-1) \text{ if } h = 1.\end{aligned}$$

Since $k_2 = n$ is an odd number, $K_2^o/K^o (\cong S^{n-1})$ is an even dimensional sphere. So we see $\mathrm{rank} K_2^o = \mathrm{rank} K^o$ by [14] Chapter III. Hence we get, by the argument in Section 4.1 and Lemma 4.2.1,

$$\begin{aligned}(K_2^o, K^o) &\approx (L_1, SU(2n-1)) \text{ if } h = 0 \text{ or}\\ (K_2^o, K^o) &\approx (L_1 \times L_2, SU(2n-1) \times T^1) \text{ if } h = 1\end{aligned}$$

where L_1 is a simply connected simple Lie group which has a maximal rank subgroup $SU(2n-1)$ and L_2 is a connected Lie group which has a maximal rank subgroup T^1.

Now we have $K_2^o/K^o \cong S^{n-1}$. If $h=0$, then we see $(L_1, SU(2n-1))$ is locally isomorphic to one of the following pairs, by Proposition 4.2.1,

$$\begin{aligned}&(SO(n), SO(n-1)) \text{ or}\\ &(G_2, SU(3)) \text{ if } n=3.\end{aligned}$$

However $SU(2n-1) \not\approx SO(n-1)$ and $SU(5) \not\approx SU(3)$ $(n=3)$. Therefore we have $h=1$. Moreover we have $L_1 = SU(2n-1)$ and (L_2, T^1) is locally isomorphic to one of the above pairs by $K_2^o/K^o \cong L_1/SU(2n-1) \times L_2/T^1$ and Proposition 4.2.1. So we can easily have $n=3$ and $(L_2, T^1) \approx (SO(3), SO(2))$. Therefore we have

$$\begin{aligned}(G, K_1) &= (SU(6)\times T^1, S(U(5)\times U(1))\times T^1) \text{ and}\\ (K_2^o, K^o) &\approx (SU(5)\times SU(2), SU(5)\times T^1).\end{aligned}$$

In the representation ρ, if $l=0$ then we have

$$K^o = (\mathrm{Ker}(\rho))^o = \left\{ \left(\begin{pmatrix} A & 0 \\ 0 & \mathrm{detA}^{-1} \end{pmatrix}, 1 \right) \,\middle|\, A \in U(5) \right\}.$$

Hence $G/K^o \cong P_5(\mathbf{C}) \times T^1$. This contradicts $G/K^o \sim S^{11}$ in Theorem 3.1 (iv). Hence $l \neq 0$ and we have

$$\begin{aligned}K^o &= \left\{ \left(\begin{pmatrix} A & 0 \\ 0 & \mathrm{detA}^{-1} \end{pmatrix}, (\mathrm{det}A^{-1})^{-l/m} \right) \,\middle|\, A \in U(5) \right\} \text{ if } m \neq 0 \text{ or}\\ K^o &= \left\{ \left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, x \right) \,\middle|\, A \in SU(5),\ x \in T^1 \right\} \text{ if } m = 0.\end{aligned}$$

Let $p : G = SU(6)\times T^1 \to T^1$ be a natural projection. Then the restriction map $p|_{K^o}$ is non-trivial homomorphism for all m by the above shape of K^o. Put the natural projection $\pi : (SU(5)\times SU(2), SU(5)\times T^1) \to (K_2^o, K^o)$. Then $q = p|_{K^o} \circ \pi|_{SU(5)\times T^1} : SU(5)\times T^1 \to K^o \to T^1$ is a non-trivial homomorphism. Hence $\hat{q} = p|_{K_2^o} \circ \pi : SU(5)\times SU(2) \to K_2^o \to T^1$ is also a non-trivial homomorphism because $\hat{q}|_{SU(5)\times T^1} = q$. Moreover we see $\hat{q}|_{SU(2)} : SU(2) \to T^1$ is non-trivial. This contradicts that there is no complex one dimensional non-trivial representation of $SU(2)$ (see [20]).

Hence we see $k_2 = n$ is an even number for the case $(H, H_1) = (SU(2n), S(U(2n-1)\times U(1)))$. Also for other cases we see $k_2 = n$ is an even number by the similar argument. Therefore $k_2 = n$ is an even number.

Consequently we see $\mathrm{rank}K_2^o = \mathrm{rank}G$ and we can put $K_2^o = H_2 \times T^h$ such that H_2 is a maximal rank subgroup of H because n is an even number, $P(G/K_2^o; t) = (1+t^n)(1+t^{2n})$ and $G = H \times T^h$. Then $G/K_2^o = H/H_2$ and $P(H/H_2; t) = (1+t^n)(1+t^{2n})$. Therefore we have $k_2 = n = 2$ or 4 by Proposition 4.2.4. ■

We already have $G = H \times T^h$, $K_1 = H_1 \times T^h$. Moreover we have $K_2^o = H_2 \times T^h$ ($h = 0$ or 1) from Lemma 6.1.1, where H is a simply connected simple Lie group and H_s is its connected closed subgroup. By Proposition 4.2.2, 4.2.3 and 4.2.4,

$$\begin{aligned} (H, H_s) &\approx (SU(4), S(U(3) \times U(1))\ (n = 2), \\ &\quad (Sp(2), Sp(1) \times U(1))\ (n = 2) \text{ or} \\ &\quad (SO(5), SO(3) \times SO(2)) \approx (Sp(2), U(2))\ (n = 2), \\ (H, H_1, H_2) &\approx (Sp(4), Sp(3) \times U(1), Sp(1) \times Sp(3))\ (n = 4). \end{aligned}$$

Since G/K_2 is non-orientable, we see $N(K_2^o; G) \neq K_2^o$. Hence $H = Sp(2)$ and $n = 2 = k_2 = k_1$.

Therefore we conclude that this case has just the following three pairs (H, H_1, H_2);

$$\begin{aligned} (H, H_s) &\simeq (Sp(2), Sp(1) \times U(1)), \\ (H, H_s) &\simeq (Sp(2), U(2)) \text{ or} \\ (H, H_s, H_r) &\simeq (Sp(2), Sp(1) \times U(1), U(2)) \end{aligned}$$

for $s + r = 3$. In each case, if $h = 0$ then $\dim K^o = 3$ and if $h = 1$ then $\dim K^o = 4$ by $K_1^o/K^o \simeq S^1 \simeq K_2^o/K^o$. However the above last case $K_1^o \cap K_2^o$ is included in the $(2 + h)$-dimensional maximal torus subgroup of G. So $\dim K^o \leq 2 + h$. This is a contradiction. Hence we have

$$\begin{aligned} (G, K_s^o) &\simeq (Sp(2) \times T^h, Sp(1) \times U(1) \times T^h) \text{ or} \\ &\simeq (Sp(2) \times T^h, U(2) \times T^h) \end{aligned}$$

for $s = 1, 2$. Let us prove the following lemma.

Lemma 6.1.2. *In this case* $G = Sp(2)$, $K_1 = Sp(1) \times U(1)$, $K_2 \simeq Sp(1) \times (U(1)_j \cup U(1)_j\mathbf{i})$ *and* $K \simeq Sp(1) \times \{1, -1, \mathbf{i}, -\mathbf{i}\}$ *where* $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ *is the basis of* $\mathbf{H}$ *and* $U(1)_j = \{a + b\mathbf{j} \mid a^2 + b^2 = 1\}$.

Proof. Suppose $(G, K_s^o) \simeq (Sp(2) \times T^h, U(2) \times T^h)$. Since G/K_2 is non-orientable, we have $K_2 \simeq N(U(2); Sp(2)) \times T^h$ (K_2 has two components). We can assume $K_1 = U(2) \times T^h$ without loss of generality. Then $K^o = (SU(2) \times \{e\}) \circ \Delta$ (where $\Delta \simeq T^h$) since $K_1/K \cong S^1$. So we have $SU(2) \times T^h \subset K_2^o \subset G = Sp(2) \times T^h$ because $K_2^o \simeq U(2) \times T^h$. Then we easily see $SU(2) \times T^h$ is a normal subgroup of $K_2^o \simeq U(2) \times T^h$. Therefore $K_2^o = U(2) \times T^h$ because we see $K_2^o \subset N(SU(2) \times T^h; G) = N(U(2) \times T^h; G)$. Hence we have $K_2 = N(U(2); Sp(2)) \times T^h$. Because $K \subset K_1 = U(2) \times T^h$, we get $K_2/K \cong N(U(2); Sp(2))/(F \circ SU(2)) \cong S^1 \cup S^1$ (disconnected) where F is a diagonal finite subgroup of $U(2)$. This contradicts $K_2/K \cong S^1$. So this case does not occur.

Therefore $(G, K_s^o) \simeq (Sp(2) \times T^h, Sp(1) \times U(1) \times T^h)$. Assume $h = 0$. Since G/K_1 is orientable and G/K_2 is non-orientable, we have $K_1 = Sp(1) \times U(1) = K_1^o$ and $K_2 =$

$N(K_2^o;G)$. Since $K_s/K \cong S^1$, we have $K = Sp(1) \times F$ where F is a finite subgroup of $U(1)$. If $K_2^o = K_1 = Sp(1)\times U(1)$, then $K_2/K \cong N(U(1);Sp(1))/F \cong S^1 \cup S^1$ (disconnected). This contradicts $K_2/K \cong S^1$. Hence we have $K_2^o = Sp(1) \times gU(1)g^{-1}$ such that $gU(1)g^{-1} \neq U(1)$ for some $g \in \{e\} \times Sp(1) \subset Sp(2)$, because $K_2^o \cap (\{e\} \times Sp(1))$ is a maximal torus in $\{e\} \times Sp(1)$. Moreover we easily have $gU(1)g^{-1} \cap U(1) = \{1,\ -1\}$. Put $N = N(gU(1)g^{-1}; Sp(1))$, then we have $K_2 = N(Sp(1) \times gU(1)g^{-1}; Sp(2)) = Sp(1) \times N$. Because $K_2 \cap (\{e\} \times U(1)) \supset K \cap (\{e\} \times U(1)) = F$, we see $N \cap U(1) \supset F$. Here

$$\mathbf{Z}_2 \simeq K_2/K_2^o \simeq N/gU(1)g^{-1} \supset (N \cap U(1))/(gU(1)g^{-1} \cap U(1)) \supset F/\{1, -1\}.$$

Since $S^1 \cong K_2/K \cong N/F$, we see $F \neq \{1, -1\}$. Hence $\mathbf{Z}_4 \simeq F \subset U(1)$, so we have $F = \{1, -1, \mathbf{i}, -\mathbf{i}\}$. Therefore we can put

$$K_2 = Sp(1) \times (U(1)_j \cup U(1)_j \mathbf{i}).$$

If $h = 1$, then we have $G = Sp(2)\times T^1$, $K_1 = Sp(1)\times U(1)\times T^1$ and $K^o = Sp(1)\times \Delta$ where $\Delta \simeq T^1$ is a subgroup in $U(1) \times T^1$. Let $p_2 : K^o \to \{e\} \times U(1) \times \{e\}$ be a natural projection on the second factor of K_1. Then we see p_2 is a surjective map because of Proposition 4.1.1. So we have $K_2^o = Sp(1) \times U(1) \times T^1 = K_1$ because $K^o \subset K_2^o$, $T^1 \subset K_2^o$ and $K_2^o \simeq Sp(1) \times U(1) \times T^1$. Because G/K_2 is non-orientable, we have $K_2 = Sp(1) \times N(U(1); Sp(1)) \times T^1$. However we have $K_2/K \simeq S^1 \cup S^1$ from $K \subset K_1 = Sp(1) \times U(1) \times T^1$. This contradicts $K_2/K \simeq S^1$. ■

Next we prove the following lemma.

Lemma 6.1.3. *Let $(Sp(2), M)$ be an $Sp(2)$-manifold which has codimension one principal orbits $Sp(2)/Sp(1) \times \{1, -1, \mathbf{i}, -\mathbf{i}\}$ and two singular orbits $Sp(2)/Sp(1) \times U(1)$ and $Sp(2)/Sp(1) \times (U(1)_j \cup U(1)_j\mathbf{i})$. Then this $(Sp(2), M)$ is unique up to essential isomorphism.*

Proof. The slice representations of $K_1 = Sp(1)\times U(1)$ and $K_2 = Sp(1)\times(U(1)_j\cup U(1)_j\mathbf{i})$ decompose as follows:

$$\sigma_1 : K_1 \to U(1) \xrightarrow{\rho_1} O(2),$$
$$\sigma_2 : K_2 \to N(U(1)_j; Sp(1)) = U(1)_j \cup U(1)_j\mathbf{i} \xrightarrow{\rho_2} O(2).$$

Since $\mathrm{Ker}(\rho_1) = F = \{1, -1, \mathbf{i}, -\mathbf{i}\}$, we can assume

$$\rho_1(\exp(\mathbf{i}\theta)) = \begin{pmatrix} \cos(4\theta) & -\sin(4\theta) \\ \sin(4\theta) & \cos(4\theta) \end{pmatrix}$$

up to equivalence. So the slice representation σ_1 is unique up to equivalence. Since $K_2/K \cong S^1$ and $\mathrm{Ker}(\rho_2|_{U(1)_j}) = \{1, -1\}$, we can put

$$\rho_2(\mathbf{i}) = \rho_2(-\mathbf{i}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore the slice representation σ_2 is also unique up to equivalence. Moreover $N(K;G)/K \simeq U(1)/F$ has only one connected component. Hence the attaching map is unique up to equivalence by Lemma 4.3.1 (1). Therefore $(Sp(2), M)$ which satisfies the conditions of this lemma is unique up to essential isomorphism. ■

Consequently the following proposition holds.

Proposition 6.1.1. *Let M be an $Sp(2)$-manifold which satisfies the conditions of Lemma 6.1.3. Then $M \cong S^7 \times_{Sp(1)} P_2(\boldsymbol{C})$.*

Proof. If $M = S^7 \times_{Sp(1)} P_2(\mathbf{C})$ where $S^7 \cong Sp(2)/Sp(1)$, $Sp(2)$ acts naturally on S^7 and $Sp(1)$ acts on $P_2(\mathbf{C}) = P(\mathbf{R}^3 \otimes_{\mathbf{R}} \mathbf{C})$ through the double covering $Sp(1) \to SO(3)$ (see [16] Example 3.2). Then we can easily check this manifold satisfies the conditions of Lemma 6.1.3. From Lemma 6.1.3, we get this proposition. ■

Hence this case has a unique (G, M) up to essential isomorphism.

6.2 Topology of $M = S^7 \times_{Sp(1)} P_2(\mathbf{C})$

In this section, we study the topology of M.

First we show M is a rational cohomology complex quadric. This manifold M is a $P_2(\mathbf{C})$-bundle over $S^7/Sp(1) \cong S^4$. Since $H^{odd}(S^4) = H^{odd}(P_2(\mathbf{C})) = 0$ and S^4 is simply connected, the induced map $p^* : H^*(S^4) \to H^*(M)$ is injective where $p : M \to S^4$ is a projection and $i^* : H^*(M) \to H^*(P_2(\mathbf{C}))$ is surjective where $i : P_2(\mathbf{C}) \cong p^{-1}(w) \to M$ for fixed $w \in S^4$ by [14] Theorem 4.2 in Chapter III. Hence there exists a generator $x \in H^4(M)$ such that $x^2 = 0 \in H^8(M)$ and $c \in H^2(M)$ such that $i^*(c) \in H^2(P_2(\mathbf{C}))$ is a generator of $H^*(P_2(\mathbf{C}))$. Because $i^*(x) = 0$, we see $c^2 \neq x$ in $H^4(M) \simeq \mathbf{Q} \oplus \mathbf{Q}$. Next we assume $S^7 \times P_2(\mathbf{C})$ is a $Sp(1)$-bundle over M. From the Thom-Gysin exact sequence, $H^6(M) \simeq \mathbf{Q}$ is generated by xc and $H^8(M) \simeq \mathbf{Q}$ is generated by xc^2.

Let us show $0 \neq c^3 \in H^6(M)$. The manifold M has an $Sp(2)$-action and the action has codimension one principal orbits from Section 6.1. Therefore we can use the Mayer-Vietoris exact sequence from Theorem 2.1. If we denote the principal orbit by G/K, the orientable singular orbit by G/K_1 and the non-orientable singular orbit by G/K_2, then we have $H^*(G/K) \simeq H^*(S^7)$ and $H^*(G/K_2) \simeq H^*(S^4)$ from Theorem 3.1. Moreover we see, from Section 6.1, the orientable singular orbit G/K_1 is diffeomorphic to $P_3(\mathbf{C})$. Hence the induced homomorphism $j^* : H^2(M) \to H^2(G/K_1)$ is isomorphic. Therefore $j^*(c)$ is a generator in $H^2(G/K_1)$ and $j^*(c^3) = j^*(c)^3 \neq 0$ because $H^*(P_3(\mathbf{C})) \simeq \mathbf{Q}[c]/(c^4)$. Hence M is a rational cohomology complex quadric.

Next we show M does not have a spin structure, we call such a manifold non-spin. It is easy to show if a fibre is non-spin then its total space is also non-spin. Hence M is non-spin because $P_2(\mathbf{C})$ is non-spin, that is, the second Stiefel-Whiteny class $w_2(P_2(\mathbf{C})) \neq 0$. By definition, Q_4 is a degree 2 non-singular algebraic hypersurface in $P_5(\mathbf{C})$. So Q_4 is a spin manifold (see Section 16.5 in [3] or [10]). Therefore M is not diffeomorphic to Q_4.

Hence we get the following proposition.

Proposition 6.2.1. *The 8-dimensional manifold $S^7 \times_{Sp(1)} P_2(\boldsymbol{C})$ is not diffeomorphic to Q_4, but a rational cohomology complex quadric .*

From the next section we will consider the case both singular orbits are orientable.

7 $G/K_1 \sim P_{2n-1}(\mathbf{C}), G/K_2 \sim S^{2n}$

Assume G/K_1, G/K_2 are orientable and $G/K_1 \sim P_{2n-1}(\mathbf{C}), G/K_2 \sim S^{2n}$. The goal of this section is to prove there are three cases (G, M) up to essential isomorphism. In this case G/K_1, G/K_2 are indecomposable. Because of the dimension of G/K_1 and G/K_2, we have $k_1 = 2$ and $k_2 = 2n$ $(n \geq 2)$. Therefore $K_1 = K_1^o$ from Lemma 3.2.1.

Put $G = H \times G" \times T^h$ and $K_1 = H_1 \times G" \times T^h$ such that $H/H_1 \simeq G/K_1 \sim P_{2n-1}(\mathbf{C})$, where $G"$ is semi-simple. Then we have $G" = \{e\}$ and $h = 0$ or 1 because of Proposition 4.1.1. Hence we have $G = H \times T^h$ and $K_1 = K_1^o = H_1 \times T^h$ ($h = 0$ or 1).

By Proposition 4.2.2,

$$\begin{aligned}(H, H_1) \approx\ & (SU(2n), S(U(2n-1) \times U(1))) \text{ or} \\ & (SO(2n+1), SO(2n-1) \times SO(2)) \text{ or} \\ & (Sp(n), Sp(n-1) \times U(1)) \text{ or} \\ & (G_2, U(2)),\ n = 3.\end{aligned}$$

Since $k_1 = 2$, we can use Lemma 3.2.3 and Lemma 3.2.4. So we have

$$H^*(G/K_2^o; \mathbf{Q}) = \mathrm{Im}(q_2^*) + J \cdot \chi + J \cdot \chi^2 \text{ (possibly non direct sum)}$$

where $q_2^* : H^*(G/K_2; \mathbf{Q})(\simeq H^*(S^{2n}; \mathbf{Q})) \to H^*(G/K_2^o; \mathbf{Q})$ is the injective induced homomorphism, $J_k = q_2^* H^k(G/K_2; \mathbf{Q})$ and $J = \oplus_k J_k$. Since $\chi \in H^{2n}(G/K_2^o; \mathbf{Q})$ by $k_2 = 2n$ and $H^i(G/K_2; \mathbf{Q}) = 0$ for $i \neq 0,\ 2n$, we see $H^*(G/K_2^o; \mathbf{Q}) = H^*(S^{2n}; \mathbf{Q})$. Hence $P(G/K_2^o; t) = P(G/K_2; t) = 1 + t^{2n}$.

Therefore we see $(H, H_2) \approx (SO(2n+1), SO(2n))$ or $(G_2, SU(3))$ and $n = 3$ by Proposition 4.2.1, where $K_2^o = H_2 \times T^h$. So we have that

$$\begin{aligned}(H, H_1, H_2) =\ & (Spin(2n+1), Spin(2n-1) \circ T^1, Spin(2n)) \text{ or} \\ & (G_2, U(2), SU(3)) \text{ and } n = 3.\end{aligned}$$

7.1 $G = Spin(2n+1) \times T^h$

Assume $G = Spin(2n+1) \times T^h$. We will prove this case is the one of results. First we show the following lemma.

Lemma 7.1.1. $h = 0$.

Proof. If $h = 1$, then $K_2^o = Spin(2n) \times T^1$. Because G/K_2 is orientable, we get $K_2 = K_2^o$. Since $k_2 = 2n$, we have the slice representation $\sigma_2 : K_2 \to SO(2n)$. From $n \geq 2$, we see the restricted representation $\sigma_2|_{Spin(2n)}$ is a natural projection from $Spin(2n)$ on $SO(2n)$. Hence $\sigma_2(\{e\} \times T^1) \subset C(SO(2n))$ where $C(SO(2n))$ is the center of $SO(2n)$ that is $C(SO(2n)) = \{I_{2n}, -I_{2n}\}$. Hence $\{e\} \times T^1 \subset \mathrm{Ker}(\sigma_2) \subset K$. This contradicts Proposition 4.1.1. So we have $h = 0$. ■

From the above Lemma 7.1.1, we have $G = Spin(2n+1)$ and $K_1 = Spin(2n-1) \circ T^1$. Because G/K_2 is orientable, we have $K_2 = K_2^o = Spin(2n)$. Since $K_1/K \simeq S^1$ and $K_2/K \cong S^{2n-1}$ $(n \geq 2)$, we see $K = K^o = Spin(2n-1)$. Let us prove the following lemma.

Lemma 7.1.2. *Let (G, M) be a G-manifold which has codimension one orbits $G/K = Spin(2n+1)/Spin(2n-1)$, two singular orbits $G/K_1 \simeq Q_{2n-1}$ and $G/K_2 \simeq S^{2n}$ where $G = Spin(2n+1)$, $K = Spin(2n-1)$, $K_1 = Spin(2n-1) \circ T^1$ and $K_2 = Spin(2n)$. Then such (G, M) is unique up to essential isomorphism.*

Proof. Because $n \geq 2$, we can decompose the slice representation $\sigma_1 : K_1 \to O(2)$ into $\sigma_1 : K_1 = Spin(2n-1) \circ T^1 \stackrel{proj}{\to} T^1 \stackrel{\rho}{\to} O(2)$. Since $\mathrm{Ker}(\sigma_1) \subset K$, ρ is an injection. So the slice representation σ_1 is unique up to equivalence. Next we consider the slice representation $\sigma_2 : K_2 = Spin(2n) \to SO(2n) \subset O(2n)$. Now we see $\mathbf{Z}_2 \subset \mathrm{Ker}(\sigma_2) \subset \sigma_2^{-1}(SO(2n-1)) = K$ where $\mathbf{Z}_2$ is a center of K. Hence we have a natural surjective map $K_2 = Spin(2n) \to Spin(2n)/\mathbf{Z}_2 \simeq SO(2n)$. Hence σ_2 decomposes into $\sigma_2 : K_2 = Spin(2n) \stackrel{proj}{\to} SO(2n) \stackrel{\rho}{\to} SO(2n)$. Because $SO(2n)$ acts transitively on S^{2n-1} $(n \geq 2)$, we see that ρ is an isomorphism by [6] Section I. Hence the slice representation σ_2 is unique up to equivalence.

Since $N(K, G)$ has two connected components, for $[y] \in N(K, G)/N(K, G)^o$, we can assume

$$p(y) = \begin{pmatrix} -I_{2n} & 0 \\ 0 & 1 \end{pmatrix}$$

where $p : Spin(2n+1) \to SO(2n+1)$ is the natural projection and y can be an element of the center of $K_2 = Spin(2n)$, which is not in the center $\mathbf{Z}_2$ of $K = Spin(2n-1)$. It suffices to prove that the right translation R_y on G/K is extendable to a G-diffeomorphism on X_2 from Lemma 4.3.1 (3.). Because y is in the center of $K_2 = Spin(2n)$, we have the following commutative diagram

$$\begin{array}{ccc} G \times_{K_2} K_2/K & \longrightarrow & G/K \\ \downarrow R_y \times 1 & & \downarrow R_y \\ G \times_{K_2} K_2/K & \longrightarrow & G/K. \end{array}$$

Here $G \times_{K_2} K_2/K = \partial(G \times_{K_2} D^{2n}) = \partial X_2$. It is clear that $R_y \times 1$ is extendable to a G-diffeomorphism on X_2. ■

Consequently (G, M) is unique up to essential isomorphism. Such an example of (G, M) will be constructed in Section 12.1. This is one of the results in Theorem 1.1.

7.2 $G = G_2 \times T^h$

Assume $G = G_2 \times T^h$. We will prove there are two cases ($h = 0$ and $h = 1$ cases). The exceptional Lie group G_2 is defined by Aut($\mathbf{O}$). Here $\mathbf{O}$ is the Cayley numbers generated by $\mathbf{R}$-basis $\{1, e_1, \cdots, e_7\}$. It is well known that $G_2 \subset SO(7)$ and $SU(3) \simeq \{A \in G_2 | \ A(e_1) = e_1\}$.

Let us consider the cases $h = 0$ and 1.

7.2.1 $h = 0$

Put $h = 0$. In this case $K_1 \simeq U(2), K_2^o \simeq SU(3), K^o \simeq SU(2)$. We can put $K_2^o = \{A \in G_2 | \ A(e_1) = e_1\}$. Then $N(K_2^o, G)$ has two components. Since G/K_2 is orientable and $G_2/SU(3) \cong S^6$, $K_2 = K_2^o$ and $K = K^o$. Also in this case (G, M) is unique by the following lemma.

Lemma 7.2.1. *Let (G_2, M) be a G_2-manifold which has codimension one orbits $G_2/SU(2)$, two singular orbits $G_2/U(2)$ and S^6. Then (G_2, M) is unique up to essential isomorphism.*

Proof. Because $K_2 \simeq SU(3)$ acts transitively on $K_2/K \cong SU(3)/SU(2) \cong S^5$, the slice representation $\sigma_2 : K_2 \simeq SU(3) \to SO(6)$ is unique up to equivalence by [6] Section I. Then we see that $\sigma_2^{-1}(SO(5)) = \{B \in K_2 | \ B(e_2) = e_2\} = K \simeq SU(2)$.

The slice representation σ_1 decomposes into $\sigma_1 : K_1 \simeq U(2) \xrightarrow{\pi} U(1) \xrightarrow{\rho} O(2)$ where ρ is an injection to $SO(2)$ and $\pi(A) = (\det A)^m$ ($m \in \mathbf{N}$), because $\mathrm{Ker}(\sigma_1) = K \simeq SU(2)$. We also have $m = 1$ from $\mathrm{Ker}(\sigma_1) = \mathrm{Ker}(\pi) = K \simeq SU(2)$, and the slice representation σ_1 is unique up to equivalence.

Now $N(K; G)/K \simeq SO(3)$ is known (Section 7.4 in [16]). Consequently (G, M) is unique up to essential isomorphism by Lemma 4.3.1 (1.). ■

Hence, in this case, (G, M) is unique up to essential isomorphism. Such an example of (G, M) will be constructed in Section 12.5. This is one of the results in Theorem 1.1.

7.2.2 $h = 1$

Put $h = 1$. In this case we have $G = G_2 \times T^1$, $K_1 \simeq U(2) \times T^1$, $K_2 \simeq SU(3) \times T^1$ and $K \simeq (SU(2) \times \{e\}) \circ \Delta$ where $\Delta \simeq T^1$ is a subgroup of $D \times T^1 \subset U(2) \times T^1$ ($D \simeq U(1)$ is a diagonal subgroup of $U(2)$). We can easily show $\Delta \neq D \times \{e\}, \ \{e\} \times T^1$ because of $K_2/K \simeq S^5$ and Proposition 4.1.1. From the following lemma we see this case is unique.

Lemma 7.2.2. *Let $(G_2 \times T^1, M)$ be a $G_2 \times T^1$-manifold which has codimension one orbits $(G_2 \times T^1)/K$ and two singular orbits $G_2/U(2)$ and S^6. Then $(G_2 \times T^1, M)$ is unique up to essential isomorphism.*

Proof. First we consider the slice representations. Let $p : K_2 \simeq SU(3) \times T^1$ be an isomorphism. Then we can put the slice representation as $\sigma_2 = \rho_2 \circ p : K_2 \simeq SU(3) \times T^1 \xrightarrow{\rho_2} O(6)$. Because $K_2/K \simeq S^5$ and $\rho_2(\{e\} \times T^1) \subset C(\rho_2(SU(3) \times \{e\}); SO(6))$, where $C(E; F) = \{b \in F|\ ab = ba \text{ for all } a \in E\}$ for $E \subset F$, the slice representation $\sigma_2 : K_2 \simeq SU(3) \times T^1 \xrightarrow{\rho_2} O(6)$ is as follows

$$\rho_2(A + \mathbf{i}B, \cos\theta + \mathbf{i}\sin\theta) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \cos(m\theta)I_3 & -\sin(m\theta)I_3 \\ \sin(m\theta)I_3 & \cos(m\theta)I_3 \end{pmatrix}$$

for some $m \in \mathbf{N}$ up to equivalence. Hence

$$\begin{aligned} K &= \sigma_2^{-1}(SO(5)) \simeq \rho_2^{-1}(SO(5)) \\ &= \left\{ \left(\begin{pmatrix} e^{-m\mathbf{i}\theta} & 0 \\ 0 & X \end{pmatrix}, e^{\mathbf{i}\theta} \right) \middle|\ \det(X) = e^{m\mathbf{i}\theta} \right\}. \end{aligned}$$

From this equation, we have

$$\begin{aligned} K_1 &\simeq U(2) \times T^1 \\ &= \left\{ \left(\begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & X \end{pmatrix}, e^{\mathbf{i}\phi} \right) \middle|\ 0 \le \theta, \phi \le 2\pi,\ \det(X) = e^{-\mathbf{i}\theta} \right\}. \end{aligned}$$

Moreover we see the slice representation $\sigma_1 : K_1 \simeq U(2) \times T^1 \overset{\rho_1}{\to} U(1) \overset{\simeq}{\to} SO(2)$ is as follows

$$\rho_1 \left(\begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & X \end{pmatrix}, e^{\mathbf{i}\phi} \right) = e^{\mathbf{i}\theta} e^{m\mathbf{i}\phi}$$

because $\mathrm{Ker}(\sigma_1) = K$. Therefore there is a unique pair (σ_1, σ_2) for each $m \in \mathbf{N}$. Since we can assume the action of $\{e\} \times T^1 (\subset G_2 \times T^1 = G)$ on M is effective (up to essential isomorphism), we can put $m = 1$. Hence there are unique slice representations σ_1 and σ_2 up to essential isomorphism.

Next we consider the gluing map. Now we can assume $K \subset SO(7) \times T^1$ as follows:

$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi(z) & 0 \\ 0 & 0 & X \end{pmatrix}, z \right) \middle|\ X \in SU(2) \subset SO(4), \varphi(z) \in SO(2), z \in T^1, \right\},$$

where $\varphi : T^1 \to SO(2)$ is an isomorphism. Because $N(K; G) = N(K; SO(7) \times T^1) \cap (G_2 \times T^1)$, we have

$$N(K; G)/N(K; G)^o \simeq \mathbf{Z}_2.$$

We can take one of the element in $N(K,G)-N(K,G)^o$ as follows

$$w=\left(\begin{pmatrix}-1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&I_4\end{pmatrix},1\right).$$

Put the element

$$(z,X,r)=\left(\begin{pmatrix}1&0&0\\0&\varphi(z)&0\\0&0&X\end{pmatrix},r\right)\in K_1,$$

where $z,r\in T^1$ and $X\in SU(2)\subset SO(4)$. Then we have

$$w\cdot(z,X,r)\cdot w^{-1}=(z^{-1},X,r).$$

So the following diagram is commutative

$$\begin{array}{ccc}G\times_{K_1}K_1/K & \xrightarrow{f} & G/K\\ R_w\times\rho\downarrow & & \downarrow R_w\\ G\times_{K_1}K_1/K & \xrightarrow{f} & G/K,\end{array}$$

where $f([g,kK])=gkK$, $R_w(g)=gw$ ($R_w(kK)=kwK$) and $\rho((z,X,r)K)=(z^{-1},X,r)K$. Now $\rho:K_1/K\to K_1/K$ is the antipodal involution on $K_1/K\simeq S^1$. Hence ρ is extendable to a K_1-equivariant diffeomorphism on D^2. Therefore the G-equivariant diffeomorphism $R_w\times\rho$ is extendable to a G-equivariant diffeomorphism $X_1\to X_1$. From Lemma 4.3.1 (2.), we see $M(R_w)=M(id)$. Consequently (G,M) is unique up to essential isomorphism. ■

Consequently the following proposition holds.

Proposition 7.2.1. *Let M be an $G_2\times T^1$-manifold which has codimension one orbits $(G_2\times T^1)/K$ and two singular orbits $G_2/U(2)$ and S^6. Then $M\cong G_2\times_{SU(3)}P_3(\boldsymbol{C})$.*

Proof. If $M=G_2\times_{SU(3)}P_3(\mathbf{C})$ where $SU(3)$ acts on G_2 naturally and $P_3(\mathbf{C})$ by $\phi:[z_0:\mathbf{z}]\mapsto[z_0:A\mathbf{z}]$, here $A\in SU(3)$ and $[z_0:\mathbf{z}]\in P_3(\mathbf{C})$. We can easily check the $SU(3)$-action on $P_3(\mathbf{C})$ has codimension one principal orbits $SU(3)/S(U(1)\times U(2))$ and two singular orbits $SU(3)/SU(3)$ and $SU(3)/SU(2)$.

This manifold M has an action $\varphi:(G_2\times T^1)\times M\to M$ defined by

$$\varphi((g,t),[g',[z_0:\mathbf{z}]])=[gg',[tz_0:\mathbf{z}]]$$

where $g\in G_2$, $t\in T^1$ and $[g',[z_0:\mathbf{z}]]\in M$. Then this action φ has codimension one orbit $(G_2\times T^1)/(SU(2)\times\{e\})\circ\Delta$ ($\Delta\simeq T^1$) and two singular orbits $(G_2\times T^1)/(SU(3)\times T^1)\cong G_2/SU(3)$ and $(G_2\times T^1)/(SU(2)\times T^1)\cong G_2/SU(2)$. From Lemma 7.2.2, such pair is unique up to essential isomorphism. Hence this proposition holds. ■

We will explain this manifold is diffeomorphic to Q_6 in Section 12.6. Hence this is one of the results in Theorem 1.1.

8 $G/K_s \sim P_n(\mathbf{C})$

Assume G/K_s is orientable and $G/K_s \sim P_n(\mathbf{C})$ $(s = 1, 2)$. The goal of this section is to prove there are two cases up to essential isomorphism, in this case. Because of $k_s = 2n$ $(n \geq 2)$ and Lemma 3.2.1, we have $K_s = K_s^o$.

First we assume that $G = H_1 \times H_2 \times G' \times T^h$, $K_1 = H_{(1)} \times H_2 \times G' \times T^h$, $K_2 = H_1 \times H_{(2)} \times G' \times T^h$ where H_s is a simply connected simple Lie group, $H_{(s)}$ is its closed subgroup, G' is a product of simply connected simple Lie groups and T^h is a torus. Then $K_1 \cap K_2 = H_{(1)} \times H_{(2)} \times G' \times T^h$. So $\dim(G/K_1 \cap K_2) = 4n \leq \dim(G/K)$ because $K \subset K_1 \cap K_2$. This contradicts $\dim G/K = 4n - 1$. Hence we can put

$$
\begin{aligned}
G &= H \times G' \times T^h, \\
K_s &= H_{(s)} \times G' \times T^h.
\end{aligned}
$$

where H is a simply connected simple Lie group and $H_{(s)}$ is its closed subgroup. By Proposition 4.2.2,

$$
\begin{aligned}
(H, H_{(s)}) \approx\ & (SU(n+1), S(U(n) \times U(1))) \text{ or} \\
& (SO(n+2), SO(n) \times SO(2)),\ n = 2m+1 \text{ or} \\
& (Sp(\frac{n+1}{2}), Sp(\frac{n-1}{2})),\ n = 2m+1 \text{ or} \\
& (G_2, U(2)),\ n = 5.
\end{aligned}
$$

Next we prepare the following lemma.

Lemma 8.0.1 (Theorem I' in [11])**.** *Let G_1 and G_2 be two compact connected Lie groups and let $G = (G_1 \times G_2)/N$ where N is a finite normal subgroup of $G_1 \times G_2$. If G acts transitively on S^n then one of the two subgroups of G corresponding to G_1 and G_2 acts transitively on S^n.*

Moreover we easily see the following lemma.

Lemma 8.0.2. *Let H be a subgroup of $G_1 \times G_2$ and $p : G_1 \times G_2 \to G_2$ be a projection. Then the following two conditions are equivalent.*

1. *G_1 acts transitively on $(G_1 \times G_2)/H$.*

2. *$p(H) = G_2$.*

Then we show the following lemma.

Lemma 8.0.3. *$H = SU(n+1)$, $H_{(s)} \simeq S(U(n) \times U(1))$ and $H_{(s)}$ acts on $K_1/K \cong S^{2n-1}$ transitively.*

Proof. If $H_{(1)}$ acts non-transitively on $K_1/K \cong S^{2n-1}$, then $V = G' \times T^h$ acts transitively on K_1/K by Lemma 8.0.1 and $K_1/K \cong V/V'$ where $V' = K \cap V$. So we see $p_1(K) = H_{(1)} = p_1(K_1)$ where $p_1 : G \to H$ by Lemma 8.0.2. Hence $V\backslash M$ is a mapping cylinder of $V\backslash G/K_1 = H/H_{(1)} \cong V\backslash G/K \to V\backslash G/K_2 = H/H_{(2)}$. From the following commutative diagram

$$\begin{array}{ccc} G/K_2 & \longrightarrow & M \\ \downarrow = & & \downarrow p \\ V\backslash G/K_2 = H/H_{(2)} & \xrightarrow{i} & V\backslash M \end{array}$$

where i is a homotopy equivalent map, we get the induced diagram

$$\begin{array}{ccc} H^*(V\backslash M) & \xrightarrow{i^*} & H^*(V\backslash G/K_2) \simeq H^*(H/H_{(2)}) \\ \downarrow p^* & & \downarrow = \\ H^*(M) & \longrightarrow & H^*(G/K_2). \end{array}$$

From this diagram we see p^* is an injective map. Denote the generator by $c \in H^2(V\backslash M) \simeq H^2(H/H_{(2)})$. Then $p^*(c) = u \in H^2(M)$ is a generator. Since $c^{n+1} = 0$, we see $p^*(c)^{n+1} = u^{n+1} = 0$. This is a contradiction to $u^{n+1} \neq 0$ from $H^*(M) = H^*(Q_{2n})$.

So $H_{(s)}$ acts transitively on $K_s/K \simeq S^{2n-1}$. By making use of [6] Section I, we get $(H, H_{(s)}) \simeq (SU(n+1), S(U(n) \times U(1)))$. Hence we can put $G = SU(n+1) \times G' \times T^h$ and $K_s \simeq S(U(n) \times U(1)) \times G' \times T^h$. ■

Consider the slice representation $\sigma_s : K_s \simeq S(U(n) \times U(1)) \times G' \times T^h \xrightarrow{\rho_s} O(2n)$. Because the subgroup of K_s which is isomorphic to $SU(n)$ acts transitively on $K_s/K \cong S^{2n-1}$, we can assume that $\rho_s|_{SU(n)}$ is a natural inclusion up to equivalence. Hence we can assume $\sigma_s : K_s \simeq S(U(n) \times U(1)) \times G' \times T^h \xrightarrow{\rho_s} U(n) \subset O(2n)$ and $\rho_s(\{e\} \times G' \times T^h)$ is in the center of $U(n)$. This implies $G' \subset \mathrm{Ker}(\sigma_s) \subset K$. Hence $G' = \{e\}$ from Proposition 4.1.1. Then we see $\rho_s|_{S(U(n) \times U(1)) \times \{e\}} = \tau_{x_s}$ for some integer x_s where $\tau_{x_s} : S(U(n) \times U(1)) \to U(n)$ is

$$\tau_{x_s}\begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix} = (\det(A^{-1}))^{x_s} A \quad for \quad A \in U(n).$$

Moreover we get $K \simeq (SU(n-1) \times \{e\}) \circ T^{h+1}$ by $K_s/K \cong S^{2n-1}$. From Proposition 4.1.1, we see $h \leq 1$.

Assume $h = 0$. Then we can put $G = SU(n+1)$, $K_1 = S(U(n) \times U(1))$, $K_2 \simeq S(U(n) \times U(1))$ and $K \simeq (SU(n-1) \times \{e\}) \circ T^1$. Because of the slice representation $\sigma_1 : K_1 = S(U(n) \times U(1)) \xrightarrow{\tau_{x_1}} U(n) \subset O(2n)$ and $\tau_{x_1}^{-1}(U(n-1)) = K$, we have

$$K = \left\{ \begin{pmatrix} a^{-x_1} & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & a \end{pmatrix} \in S(U(n) \times U(1)) \;\middle|\; X \in U(n-1),\ \det X = a^{x_1 - 1} \right\}$$

Since we have $K \subset K_2 \simeq S(U(n) \times U(1))$, we easily see the following two cases occur;

1. $K_2 = K_1$ and $x_1 = x_2$ or
2. K_2 is as follows and $x_1 = x_2 = -1$;

$$K_2 = \left\{ \begin{pmatrix} \det(A^{-1}) & 0 \\ 0 & A \end{pmatrix} \;\middle|\; A \in U(n) \right\} = S(U(1) \times U(n)).$$

In each case above $N(K;G)/K$ is connected. Hence the attaching map from X_1 to X_2 is unique up to equivalence by Lemma 4.3.1 (1.). Therefore $(SU(n+1), M)$ is unique in each case above.

If $K_2 = K_1$, we construct a G-manifold as $M = SU(n+1) \times_{S(U(n)\times U(1))} S^{2n}$ where $S(U(n)\times U(1))$ acts on S^{2n} by the representation $\tau_x : S(U(n)\times U(1)) \to U(n)$ $(x = x_1 = x_2)$ ($U(n)$ canonically acts on $S^{2n} \subset \mathbf{C}^n \times \mathbf{R}$). However this manifold $SU(n+1) \times_{S(U(n)\times U(1))} S^{2n}$ is a S^{2n} bundle over $P_n(\mathbf{C})$. Because $H^{odd}(S^{2n}; \mathbf{Q}) = H^{odd}(P_n(\mathbf{C}); \mathbf{Q}) = 0$, we have $c^{n+1} = 0$ for all $c \in H^2(M; \mathbf{Q})$. Hence the cohomology ring of M is not isomorphic to $H^*(Q_{2n}; \mathbf{Q})$. So this case $(K_2 = K_1)$ does not occur.

Consequently this case is $K_2 = S(U(1) \times U(n))$. Such a pair (G, M) will be constructed in Section 12.2.

Next we put $h = 1$. Then we can put $G = SU(n+1) \times T^1$, $K_1 = S(U(n) \times U(1)) \times T^1$, $K_2 \simeq S(U(n)\times U(1))\times T^1$ and $K \simeq (SU(n-1)\times\{e\})\circ T^2$. In this case the slice representation is

$$\sigma_s : K_s \simeq S(U(n) \times U(1)) \times T^1 \xrightarrow{\rho_s} U(n) \subset O(2n).$$

Here the representation ρ_s $(s = 1,\ 2)$ is defined as follows;

$$\rho_s \left(\begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix}, z \right) = \det(A^{-1})^{x_s} z^{m_s} A$$

where $m_s \in \mathbf{Z}$, $A \in U(n)$ and $z \in T^1$. From Proposition 4.1.1, we see $m_s \neq 0$ for $s = 1,\ 2$.

Since $\rho_1^{-1}(U(n-1)) = K$, we have

$$K = \left\{ \left(\begin{pmatrix} a^{-x_1} z^{-m_1} & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & a \end{pmatrix}, z \right) \;\middle|\; z \in T^1,\ X \in U(n-1),\ a^{1-x_1} z^{-m_1} \det X = 1 \right\}.$$

Now we see $K \subset K_2 \simeq S(U(n) \times U(1)) \times T^1$. Hence we easily have the following two cases

$$K_2 = K_1 \text{ and } x_1 = x_2,\ m_1 = m_2 \text{ or}$$
$$K_2 = S(U(1) \times U(n)) \times T^1 \text{ and } x_1 = x_2 = \pm 1,\ m_1 = \pm m_2.$$

Moreover we see if $K_2 = S(U(1) \times U(n)) \times T^1$ and $x_1 = x_2 = -1$ then $m_1 = -m_2 = 0$. This contradicts $m_1,\ m_2 \neq 0$. Hence there are following two cases in this case;

1. $K_2 = K_1$ and $x_1 = x_2,\ m_1 = m_2 \neq 0$ or

2. $K_2 = S(U(1) \times U(n)) \times T^1$ and $x_1 = x_2 = 1,\ m_1 = m_2 \neq 0$.

In all cases above $N(K;G)/K$ is connected. Therefore the pair $(SU(n+1)\times T^1, M)$ is unique in those cases, because of Lemma 4.3.1 (1.).

If $K_2 = K_1$, then we construct such manifold M as $(SU(n+1) \times T^1) \times_{S(U(n)\times U(1))\times T^1} S^{2n}$ where $S(U(n) \times U(1)) \times T^1$ acts S^{2n} by the representation $\rho_1 = \rho_2$. However M is a S^{2n} bundle over $P_n(\mathbf{C})$. This is not a rational cohomology complex quadric by the same argument of the case $h = 0$.

Therefore $K_2 = S(U(1) \times U(n)) \times T^1$, $x_1 = x_2 = 1$ and $m = m_1 = m_2 \neq 0$. Then we have $\{I_{n+1}\} \times \mathbf{Z}_m \subset K \cap (\{I_{n+1}\} \times T^1) \subset \{I_{n+1}\} \times T^1$. Hence $(SU(n+1) \times T^1, M)$ is essentially isomorphism for all $m \in \mathbf{Z} - \{0\}$. Moreover we can assume the pair (G, M) as $(U(n+1), M)$ up to essentially isomorphism because $SU(n+1) \times_{\mathbf{Z}_{n+1}} T^1 \simeq U(n+1)$ and $\mathbf{Z}_{n+1} = \{(zI_{n+1}, z^{-1})|\ z^{n+1} = 1\} \subset K \cap C(SU(n+1) \times T^1)$, where $C(SU(n+1) \times T^1)$ means the center of $SU(n+1) \times T^1$ (remark when $m_1 = m_2 = 2$, then $\mathbf{Z}_{n+1} \subset K$). Hence we get the unique pair $(U(n+1), M)$ in this case and such pair will be constructed in Section 12.2.

9 $P(G/K_1;t) = (1 + t^{k_2-1})a(n)$, k_2 is odd: Preliminary.

Assume G/K_1, G/K_2 are orientable, $P(G/K_1;t) = (1 + t^{k_2-1})a(n)$ and k_2 is odd. The aim of this section is to prove Proposition 9.0.1. Put $G = G' \times G$" and $K_1 = K_1^o = K_1' \times G$" (by Lemma 3.2.1). First we prove the following technical lemma.

Lemma 9.0.1. *Let $V \subset G$ be a subgroup such that*

$$\begin{aligned}
&\pi^* : H^*(V\backslash G/K_s) \longrightarrow H^*(V\backslash G/K) \text{ is injective},\\
&p^* : H^*(V\backslash G/K_r) \longrightarrow H^*(G/K_r) \text{ is injective},\\
&q : V\backslash G/K_r \cong V\backslash G/K
\end{aligned}$$

where $s + r = 3$, $\pi : V\backslash G/K \to V\backslash G/K_s$ and $p : G/K_r \to V\backslash G/K_r$ are projections, $q : V\backslash G/K_r \to V\backslash G/K$ is the inverse of the natural projection $V\backslash G/K \to V\backslash G/K_r$. Then $f^* : H^*(V\backslash M) \to H^*(M)$ is injective where $f : M \to V\backslash M$ is a projection and we have $H^2(V\backslash G/K_s; \boldsymbol{Q}) = 0$.

Proof. Consider a diagram

$$\begin{array}{ccccc}
G/K_s & \xrightarrow{i_s} & M & \xleftarrow{i_r} & G/K_r \\
\downarrow & & f\downarrow & & p\downarrow \\
V\backslash G/K_s & \xrightarrow{j_s} & V\backslash M & \xleftarrow{j_r} & V\backslash G/K_r \\
=\downarrow & & & & q\downarrow\cong \\
V\backslash G/K_s & & \xleftarrow{\pi} & & V\backslash G/K
\end{array}$$

where i_s, i_r, j_s, j_r are natural inclusions. Now $V\backslash M$ is a mapping cylinder of

$$V\backslash G/K_r \cong V\backslash G/K \xrightarrow{\pi} V\backslash G/K_s.$$

Hence j_s is a homotopy equivalent map. So the induced map $j_s^* : H^*(V\backslash G/K_s) \to H^*(V\backslash M)$ is an isomorphic map and the above diagram induces the following commutative diagram;

$$\begin{array}{ccccc}
H^*(G/K_s) & \xleftarrow{i_s^*} & H^*(M) & \xrightarrow{i_r^*} & H^*(G/K_r) \\
\uparrow & & f^* \uparrow & & p^* \uparrow \\
H^*(V\backslash G/K_s) & \xleftarrow{j_s^*} & H^*(V\backslash M) & \xrightarrow{j_r^*} & H^*(V\backslash G/K_r) \\
=\uparrow & & & & q^* \uparrow \simeq \\
H^*(V\backslash G/K_s) & & \xrightarrow{\pi^*} & & H^*(V\backslash G/K).
\end{array}$$

Therefore we have j_r^* is an injection, because of the assumptions (π^* is injective, $q : V\backslash G/K_r \cong V\backslash G/K$) and $q^* \circ \pi^* \circ j_s^* = j_r^*$. Hence f^* is an injection because $i_r^* \circ f^* = p^* \circ j_r^*$ is an injective map by the assumption (p^* is injective).

Assume $H^2(V\backslash G/K_s; \mathbf{Q}) \neq 0$. Then we can take some non-zero element $c' \in H^2(V\backslash G/K_s)$ such that $f^* \circ (j_s^*)^{-1}(c') = c \in H^2(M)$. Hence $c^{2n} = \{f^* \circ (j_s^*)^{-1}(c')\}^{2n} \neq 0$ because $H^*(M) \simeq H^*(Q_{2n})$ where $n \geq 2$. Therefore $0 \neq (c')^{2n} \in H^{4n}(V\backslash G/K_s)$. This contradicts $\dim(V\backslash G/K_s) \leq \dim(G/K_s) \leq \dim(M) - 2 = 4n - 2$. ■

Hence we can prove Proposition 9.0.1.

Proposition 9.0.1. *K_1' acts transitively on K_1/K.*

Proof. If K_1' acts non-transitively on $K_1/K \cong S^{k_1-1}$ then G" acts transitively on K_1/K by Lemma 8.0.1. Hence $p(K) = K_1' = p(K_1)$ by Lemma 8.0.2 where $p : G \to G'$ is the natural projection. Put $p(K_2) = K_2'$. Then K_2'/K_1' is connected, because the induced map $p' : K_2/K (\cong S^{k_2-1}) \to K_2'/K_1'$ from $p : G \to G'$ is continuous. Hence we see K_2' is connected from the fibre bundle $K_1' \to K_2' \to K_2'/K_1'$ and the connectedness of K_1'. Now $K_1' = p(K) \subset p(K_2) = K_2' \subset G'$. Therefore rank K_1' = rank G' = rank K_2'. We also have K_2'/K_1' and G'/K_2' are simply connected, because connected Lie groups K_1', K_2' and G' have same rank. So we get

$$P(G/K_1;t) = (1 + t^{k_2-1})a(n) = P(G'/K_1';t) = P(K_2'/K_1';t)P(G'/K_2';t) \tag{21}$$

by $G/K_1 \cong G'/K_1'$, the fibration $K_2'/K_1' \to G'/K_1' \to G'/K_2'$ and $H^{odd}(K_2'/K_1') = 0 = H^{odd}(G'/K_2')$.

Since $K_2/K \cong K_2^o/K^o$ is an even dimensional sphere S^{k_2-1}, we see rank K_2^o = rank K^o. So rank$(K_1 \cap K_2^o)$ = rank K^o because of $K^o \subset K_1 \cap K_2^o$. We also have $(K_1 \cap K_2^o)/K^o$ is connected, because of the homotopy exact sequence $(\cdots \to \pi_1(K_2'/K_1') \to \pi_0((K_1 \cap K_2^o)/K^o) \to \pi_0(K_2^o/K^o) \to \cdots)$ for the fibration $(K_1 \cap K_2^o)/K^o \to K_2^o/K^o \xrightarrow{p"} K_2'/K_1'$ (where

p" is the induced map from $p : G \to G'$) and the simply connectedness of K_2'/K_1'. Now we have $H^{odd}((K_1 \cap K_2^o)/K^o) = H^{odd}(K_2'/K_1') = 0$. Therefore the equation

$$P(K_2^o/K^o; t) = 1 + t^{k_2-1} = P(K_2'/K_1'; t)P((K_1 \cap K_2^o)/K^o; t) \tag{22}$$

holds by $K_2^o/K^o \cong S^{k_2-1}$ and the fibration $(K_1 \cap K_2^o)/K^o \to K_2^o/K^o \xrightarrow{p"} K_2'/K_1'$. From the equation (22), we have $P(K_2'/K_1'; t) = 1+t^{k_2-1}$ or 1. So we see $H^2(G'/K_2') = H^2(G"\backslash G/K_2) \neq 0$ from the equation (21).

On the other hand we have $G"\backslash G/K = G"\backslash G/K_1 = G/K_1$. Moreover we see $\pi^* : H^*(G'/K_2') \to H^*(G"\backslash G/K) = H^*(G'/K_1')$ is injective by the fibration $K_2'/K_1' \to G'/K_1' \xrightarrow{\pi} G'/K_2'$. So this case satisfies the conditions of Lemma 9.0.1 where $V = G"$, $s = 2$ and $r = 1$. However the fact $H^2(G'/K_2') = H^2(G"\backslash G/K_2) \neq 0$ contradicts Lemma 9.0.1. Therefore K_1' acts transitively on K_1/K. ■

From the next section we will study the case of $P(G/K_1; t) = (1 + t^{k_2-1})a(n)$, k_2 is odd. To classify such case, we will consider two cases where G/K_1 is decomposable or not.

10 $P(G/K_1; t) = (1 + t^{k_2-1})a(n)$, k_2 is odd: G/K_1 is decomposable.

Assume G/K_1, G/K_2 are orientable, $P(G/K_1; t) = (1 + t^{k_2-1})a(n)$, k_2 is odd and G/K_1 is decomposable. The goal of this section is to prove there is a unique (G, M) up to essential isomorphism in this case. In this case we have $K_1 = K_1^o$ because $k_2 > 2$ and Lemma 3.2.1. Because G/K_1 is decomposable, we can put $G = H_1 \times H_2 \times G"$ and $K_1 = H_{(1)} \times H_{(2)} \times G"$ where $H_1/H_{(1)} \sim S^{k_2-1}, H_2/H_{(2)} \sim P_n(\mathbf{C})$. Then $G/K_1 = H_1/H_{(1)} \times H_2/H_{(2)}$. So by Propositions 4.2.1 and 4.2.2,

$$\begin{aligned}
(H_1, H_{(1)}) &= (Spin(k_2), Spin(k_2 - 1)) \text{ or} \\
& (G_2, SU(3)) \ (k_2 = 7). \\
(H_2, H_{(2)}) &= (SU(n + 1), S(U(n) \times U(1))) \text{ or} \\
& (Spin(n + 2), Spin(n) \circ T^1) \ (n \text{ is odd}) \text{ or} \\
& (Sp(\frac{n+1}{2}), Sp(\frac{n-1}{2}) \times U(1)) \ (n \text{ is odd}) \text{ or} \\
& (G_2, U(2)) \ (n = 5).
\end{aligned}$$

10.1 Candidates for (G, K_1)

The goal of this section is to prove $k_1 = 2n - 2$, $k_2 = 3$ and the pair (G, K_1) is one of the following

$$(G, K_1) = \left(Sp(1) \times Sp(\frac{n+1}{2}) \times G", T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times G" \right)$$

or $n = 9$,

$$(G, K_1) = \big(Sp(1) \times Spin(11) \times G\text{''}, T^1 \times Spin(9) \circ T^1 \times G\text{''}\big)$$

or $n = 2$,

$$(G, K_1) = \big(Sp(1) \times SU(3) \times G\text{''}, T^1 \times S(U(2) \times U(1))\big).$$

First we prove the following proposition.

Proposition 10.1.1. *$H_{(2)}$ acts transitively on K_1/K.*

To show Proposition 10.1.1, we prepare some notations.

Let $p_t : G \to H_t$, $p'_t : G \to H_t \times G\text{''}$ be the natural projection, and let $h_t : H_t \to G$, $h'_t : H_t \times G\text{''} \to G$ be the natural inclusion. Put

$$\begin{aligned}
&L_{st} = p_t(K_s),\ L_t = p_t(K),\ L'_{st} = p'_t(K_s),\ L'_t = p'_t(K),\\
&N_{st} = h_t^{-1}(K_s),\ N_t = h_t^{-1}(K),\ N'_{st} = h_t'^{-1}(K_s),\ N'_t = h_t'^{-1}(K).
\end{aligned}$$

Then $N_{st} \triangleleft L_{st},\ N_t \triangleleft L_t,\ N'_{st} \triangleleft L'_{st}$ and $N'_t \triangleleft L'_t$ where $A \triangleleft B$ means a group A is a normal subgroup of B. In particular $L_{1t} = N_{1t} = H_{(t)}$ and $L'_{1t} = N'_{1t} = H_{(t)} \times G\text{''}$ by the equality $K_1 = H_{(1)} \times H_{(2)} \times G\text{''}$.

Let us prove Proposition 10.1.1.

Proof of Proposition 10.1.1. If $H_{(2)}$ does not act transitively on $K_1/K \cong S^{k_1-1}$, then $H_{(1)}$ acts transitively on K_1/K by Lemma 8.0.1 and Proposition 9.0.1. Hence $L_2 = H_{(2)} = L_{12}$ by Lemma 8.0.2.

Put $V = H_1 \times G\text{''}$. Now $L_{22}/H_{(2)}(\cong V\backslash K_2/K)$ is connected because the induced map $p'_2 : K_2/K \to V\backslash K_2/K \cong L_{22}/H_{(2)}$ is continuous. Hence L_{22} is connected by the fibration $H_{(2)} \to L_{22} \to L_{22}/H_{(2)}$. Since $L_2 = H_{(2)} \subset L_{22} \subset H_2$, we have rank $H_{(2)} =$ rank $L_{22} =$ rank H_2 and $H^{odd}(L_{22}/H_{(2)}) = H^{odd}(H_2/L_{22}) = 0$. Because L_{22} is connected and rank $L_{22} =$ rank H_2, we see $H_2/L_{22} \cong V\backslash G/K_2$ is simply connected. Hence the map

$$\pi^* : H^*(H_2/L_{22})(\simeq H^*(V\backslash G/K_2)) \to H^*(H_2/H_{(2)})(\simeq H^*(V\backslash G/K))$$

is injective from the fibration $L_{22}/H_{(2)} \to H_2/H_{(2)} \xrightarrow{\pi} H_2/L_{22}$. Moreover we have $G/K_1 = H_1/H_{(1)} \times H_2/H_{(2)}$ and $V\backslash G/K_1 = H_2/H_{(2)} \cong V\backslash G/K$ where the last diffeomorphism defines by the natural projection. So we have $p^* : H^*(V\backslash G/K_1) \to H^*(G/K_1)$ is injective where $p : G/K_1 \to V\backslash G/K_1$ is a natural projection. Therefore f^* is an injective homomorphism from Lemma 9.0.1 (the case $s = 2$, $r = 1$, $V = H_1 \times G'$), where $f^* : H^*(V\backslash M) \to H^*(M)$ is an induced homomorphism from the natural projection $f : M \to V\backslash M$.

Now we see $V\backslash M$ is a mapping cylinder of $V\backslash G/K \to V\backslash G/K_2 \simeq H_2/L_{22}$. Hence we can consider $H^*(V\backslash M; \mathbf{Q}) \subset H^*(P_n(\mathbf{C}); \mathbf{Q})$ by $H^*(V\backslash M) \simeq H^*(H_2/L_{22}) \xrightarrow{\pi^*} H^*(H_2/H_{(2)}) \simeq$

$H^*(P_n(\mathbf{C}))$. So we can take $(0 \neq)a \in H^{2m}(V\backslash M) \subset H^*(P_n(\mathbf{C}))$ for some $(0 \neq)m \leq n$. If $m \neq n$, then we can put $f^*(a) = \lambda c^m$ for $0 < m < n$ and $(0 \neq)\lambda \in \mathbf{Q}$ where c is a generator in $H^2(M)$. However there is an l such that $n < lm < 2n$ and $f^*(a^l) = \lambda^l c^{lm} \neq 0$ in $H^{2lm}(M)$ because of $H^*(M) \simeq H^*(Q_{2n})$. This contradicts dim $H_2/L_{22} \leq 2n$. Hence $m = n$. Then we have $(H^*(P_n(\mathbf{C})) \supset)H^*(V\backslash M) \simeq H^*(H_2/L_{22}) \simeq H^*(S^{2n})$ and dim $H_2/L_{22} = 2n$. On the other hand, by the fibration $L_{22}/H_{(2)} \to H_2/H_{(2)}(\sim P_n(\mathbf{C})) \to H_2/L_{22}$, we also have $H_{(2)} = L_{22}$. So $H_2/H_{(2)} \cong H_2/L_{22} \sim S^{2n}$. This contradicts $H_2/H_{(2)} \sim P_n(\mathbf{C})$. Consequently $H^{2m}(V\backslash M) \simeq H^{2m}(H_2/L_{22}) = 0$ for all $m \neq 0$, so we have $L_{22} = H_2$. Therefore dim $L_{22}/L_2(= H_2/H_{(2)}) = 2n$ by $L_2 = H_{(2)}$. From the surjection $K_2/K \cong S^{k_2-1} \to V\backslash K_2/K \cong L_{22}/L_2$, we see $k_2 - 1 \geq 2n$. This contradicts $k_1 + k_2 = 2n + 1$ and $k_1 \geq 2$. ■

From Proposition 10.1.1, $H_{(2)}$ acts transitively on K_1/K. Then $H_{(2)}/N_2 \cong K_1/K \cong S^{k_1-1}$. Since $\{pt\} = H_{(2)}\backslash K_1/K \cong (H_{(1)} \times G")/L'_1$, we have the following lemma.

Lemma 10.1.1. *$L'_1 = H_{(1)} \times G"$ and $L_1 = H_{(1)} = L_{11}$.*

Moreover we have the following lemma.

Lemma 10.1.2. dim *$L'_1/N'_1 \leq 3$.*

Proof. Consider the two homomorphisms $K \xrightarrow{q'_1=p'_1|_K} L'_1$ and $K \xrightarrow{q_2=p_2|_K} L_2$. Then we see q'_1 and q_2 are surjective, Ker $q'_1 = (\{e\} \times H_2) \cap K = N_2 = h_2^{-1}(K)$ and Ker $q_2 = (\{e\} \times H_1 \times G") \cap K = N'_1 = (h'_1)^{-1}(K)$ by the definitions. So we have

$$\text{dim } K - \text{dim } L'_1 = \text{dim } N_2, \quad \text{dim } K - \text{dim } L_2 = \text{dim } N'_1$$

Hence dim $L'_1/N'_1 =$ dim L_2/N_2. Since L_2/N_2 $(N_2 \triangleleft L_2 \subset L_{12} = H_{(2)})$ acts freely on $H_{(2)}/N_2 \cong S^{k_1-1}$, we have dim $L_2/N_2 \leq 3$ by [4] 6.2. Theorem in Chapter IV. ■

Let us prove the following lemma.

Lemma 10.1.3. *$L_{21} = H_1$.*

Proof. First we have L_{21} is connected because K_2/K is connected, $H_{(1)} = L_1$ (Lemma 10.1.1) is connected and the map $\bar{p_1} : K_2/K \to L_{21}/L_1 = L_{21}/H_{(1)}$ induced by $p_1 : G \to H_1$ is continuous. Consider the fibration

$$L_{21}/H_{(1)} \longrightarrow H_1/H_{(1)} \longrightarrow H_1/L_{21}.$$

Then we have rank $H_{(1)} =$ rank $L_{21} =$ rank H_1 by $H_{(1)} = L_1 \subset L_{21} \subset H_1$. So we have $H^*(H_1/H_{(1)}) \simeq H^*(S^{k_2-1}) \simeq H^*(H_1/L_{21}) \otimes H^*(L_{21}/H_{(1)})$. Therefore we see $L_{21} = H_{(1)}$ or H_1.

If we put $L_{21} = H_{(1)} = L_1$, then $(H_2 \times G")\backslash M \cong [0,1] \times H_1/H_{(1)}$ by Lemma 10.1.1. Consider the following commutative diagram

$$\begin{array}{ccc} H_1/H_{(1)} \times H_2/H_{(2)} \cong G/K_1 & \xrightarrow{i_1} & M \\ \downarrow q_1 & & \downarrow f \\ H_1/H_{(1)} \cong (H_2 \times G")\backslash G/K_1 & \xrightarrow{j_1} & (H_2 \times G")\backslash M. \end{array}$$

Here j_1 is a homotopy equivalence. Hence the induced homomorphism $q_1^* \circ j_1^*$ is injective. Therefore $f^* : H^*((H_2 \times G")\backslash M) \simeq H^*(S^{k_2-1}) \to H^*(M) \simeq H^*(Q_{2n})$ is injective. Hence $k_2 \geq 2n+1$ by the ring structure of $H^*(Q_{2n})$. But this contradicts $k_1 + k_2 = 2n+1$ and $k_1 \geq 2$. Hence we see $L_{21} = H_1$. ■

Hence we can prove the following lemma.

Lemma 10.1.4. $N_1 \neq H_{(1)}$.

Proof. Suppose $N_1 = H_{(1)}$. Then $H_{(1)} = N_1 \subset N_{21} \triangleleft L_{21} = H_1$ by Lemma 10.1.3. Since H_1 is a simple Lie group, we see $N_{21} = H_1$. Hence we can put $K_2 = H_1 \times X$ and $K = H_{(1)} \times X$ where $X < H_2 \times G"$, because of $N_1 = H_{(1)} = L_1$ (by Lemma 10.1.1). Therefore $H_1\backslash M$ is a mapping cylinder of $H_1\backslash G/K = (H_2 \times G")/X \to H_1\backslash G/K_1 = H_2/H_{(2)}$. Because of the following commutative diagram

$$\begin{array}{ccc} H_1/H_{(1)} \times H_2/H_{(2)} \cong G/K_1 & \longrightarrow & M \\ \downarrow q_2 & & \downarrow p \\ H_2/H_{(2)} \cong H_1\backslash G/K_1 & \xrightarrow{i} & H_1\backslash M \end{array}$$

where i is a homotopy equivalent map, we have the following induced diagram

$$\begin{array}{ccc} H^*(H_1\backslash M) & \xrightarrow{i^*} & H^*(H_2/H_{(2)}) \\ \downarrow p^* & & \downarrow q_2^* \\ H^*(M) & \longrightarrow & H^*(H_1/H_{(1)}) \otimes H^*(H_2/H_{(2)}). \end{array}$$

Hence $p^* : H^*(H_1\backslash M) \to H^*(M)$ is an injection by the injectivity of $q_2^* \circ i^*$. This contradicts $H^*(M) \simeq H^*(Q_{2n})$ and $H^*(H_1\backslash M) \simeq H^*(H_2/H_{(2)}) \simeq H^*(P_n(\mathbf{C}))$. ■

Next we show the following proposition.

Proposition 10.1.2. *$k_1 = 2n-2$, $k_2 = 3$ and $(H_1, H_{(1)}) = (Sp(1), T^1)$.*

Proof. Let us recall,

$$(H_1, H_{(1)}) = (Spin(k_2), Spin(k_2-1))) \text{ or } (G_2, SU(3)) : k_2 = 7.$$

If the odd number $k_2 > 6$, then $H_{(1)}$ is a simple Lie group. We have N_1' is a normal subgroup of $L_1' = H_{(1)} \times G"$ and dim $L_1'/N_1' \leq 3$ by Lemma 10.1.2. Hence $N_1' = H_{(1)} \times X$ where X

is a normal subgroup of G". Therefore $N_1 = H_{(1)}$. This contradicts Lemma 10.1.4. Hence $k_2 = 3$ or 5.

If $k_2 = 5$, then $(H_1, H_{(1)}) = (Spin(5), Spin(4))$. Because of dim $L'_1/N'_1 \leq 3$ (Lemma 10.1.2) and $L'_1 = Spin(4) \times G$" (Lemma 10.1.1), we have dim $N_1 \neq 0$. So dim $N_{21} \geq$ dim $N_1 > 0$. Now H_1 is a simple Lie group and $N_{21} \triangleleft L_{21} = H_1$ from Lemma 10.1.3. Hence $N_{21} = H_1$. This implies $K_2 = H_1 \times Y$ where Y is a subgroup of $H_2 \times G$". Because $K_1 = H_{(1)} \times H_{(2)} \times G$", we see $K \subset K_1 \cap K_2 = H_{(1)} \times (Y \cap (H_{(2)} \times G")) \subset K_2$. Consider the fibration $(K_1 \cap K_2)/K \to K_2/K \to K_2/(K_1 \cap K_2)$ that is

$$(H_{(1)} \times (Y \cap (H_{(2)} \times G")))/K \to K_2/K \to K_2/(H_{(1)} \times (Y \cap (H_{(2)} \times G"))).$$

Because $K_2/K \simeq S^{k_2-1} \simeq H_1/H_{(1)}$, $K_2 = H_1 \times Y$ and H_1 acts on K_2/K non-trivially (because of the relation $K \subset K_1 \cap H_1 = H_{(1)}$), we have $Y \cap (H_{(2)} \times G") = Y$ and $K = K_1 \cap K_2 = H_{(1)} \times Y$. Hence $N_1 = H_{(1)}$. This also contradicts Lemma 10.1.4. Consequently $k_2 = 3$. Hence $k_1 = 2n - 2$ by $k_1 + k_2 = 2n + 1$, and $(H_1, H_{(1)}) = (Spin(3), Spin(2))$. In particular we can consider $(H_1, H_{(1)}) = (Sp(1), T^1)$ by $(Spin(3), Spin(2)) \approx (Sp(1), T^1)$. ■

So $H_{(2)}$ acts transitively on $K_1/K \simeq S^{2n-3}$ from Proposition 10.1.1 and 10.1.2. Hence by Proposition 4.2.2 and [6] Section I, we have the following three cases where $k_1 = 2n - 2$, $k_2 = 3$,

$$\begin{aligned} G &= Sp(1) \times Sp(\frac{n+1}{2}) \times G", \\ K_1 &= T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times G", \end{aligned}$$

and $n = 9$,

$$\begin{aligned} G &= Sp(1) \times Spin(11) \times G", \\ K_1 &= T^1 \times Spin(9) \circ T^1 \times G", \end{aligned}$$

and $n = 2$,

$$\begin{aligned} G &= Sp(1) \times SU(3) \times G", \\ K_1 &= T^1 \times S(U(2) \times U(1)) \times G". \end{aligned}$$

So we see the above three cases occur in this case.

In the above two cases $K_2 = K_2^o$ because n is an odd number and Lemma 3.2.1. Hence $K = K^o$ because $K_2/K \cong S^2$ is simply connected.

In next three sections we will discuss slice representations and attaching maps in each case.

10.2 $G = Sp(1) \times Sp(\frac{n+1}{2}) \times G"$

If $G = Sp(1) \times Sp(\frac{n+1}{2}) \times G"$, then $K_1 = T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times G"$. Now $Sp(\frac{n-1}{2}) \times U(1)$ acts transitively on $K_1/K \cong S^{2n-3}$ because of Proposition 10.1.1. So we can assume the restricted slice representation $\sigma_1|_{Sp(\frac{n-1}{2})}$ is a natural inclusion to $SO(2n-2)$ for $n \geq 3$, because $Sp(\frac{n-1}{2})$ acts transitively on $K_1/K \cong S^{2n-3}$ through $\sigma_1|_{Sp(\frac{n-1}{2})}$. Then we have $\sigma_1(T^1 \times \{e\} \times U(1) \times G") \subset C(\sigma_1(Sp(\frac{n-1}{2})); SO(2n-2)) \simeq Sp(1)$ where $C(E;F) = \{g \in F|\ gk = kg \text{ for all } k \in E\}$. Therefore we have

$$G" = Sp(1),\ T^1,\ \text{or } \{e\}$$

by Proposition 4.1.1 and we can assume the slice representation as

$$\sigma_1 : K_1 \xrightarrow{\varphi} Sp(1) \times Sp(\frac{n-1}{2}) \xrightarrow{\rho} SO(2n-2) \subset O(2n-2)$$

such that $\varphi|_{Sp(\frac{n-1}{2})} : Sp(\frac{n-1}{2}) \to \{e\} \times Sp(\frac{n-1}{2})$ is isomorphic, $\varphi(T^1 \times U(1) \times G") \subset Sp(1) \times \{e\}$, where ρ is a canonical representation induced by $Sp(1) \times Sp(\frac{n-1}{2})$-action on $\mathbf{H}^{\frac{n-1}{2}} (\simeq \mathbf{R}^{2n-2})$ for $n \geq 3$, that is $\rho|_{\{e\} \times Sp(\frac{n-1}{2})}$ is the natural inclusion.

Moreover we have the following lemma.

Lemma 10.2.1. *$G" = \{e\}$ or T^1 and we can assume the slice representation as*

$$\sigma_1 : K_1 \xrightarrow{\varphi} U(1) \times Sp(\frac{n-1}{2}) \xrightarrow{\rho} SO(2n-2) \subset O(2n-2)$$

where $\varphi|_{Sp(\frac{n-1}{2})} : Sp(\frac{n-1}{2}) \to \{e\} \times Sp(\frac{n-1}{2})$ is isomorphic, $\varphi(T^1 \times \{e\} \times U(1) \times G") \subset U(1) \times \{e\}$.

Proof. Suppose $G" = Sp(1)$. Then the restricted representation $\varphi|_{T^1 \times U(1) \times G"}$ is $r : T^1 \times \{e\} \times U(1) \times G" \to Sp(1)$. Because $Sp(1)$ is a simple Lie group, $r|_{Sp(1)}$ is an isomorphism or a trivial map. If $r|_{Sp(1)}$ is an isomorphism, then we have $\mathrm{Ker}(r) = T^1 \times \{e\} \times U(1) \times \{e\}$ because $C(r(Sp(1)); Sp(1)) = \{1, -1\}$. Since $\mathrm{Ker}(r) \subset K$, we have $H_{(1)} = T^1 \subset K$. This contradicts the fact $H_{(1)} = T^1 \not\subset K$ from Lemma 10.1.4. So we see $r|_{Sp(1)}$ is trivial and $Sp(1) \subset \mathrm{Ker}(r) \subset K$. But this contradicts Proposition 4.1.1. ■

Assume $G" = T^1$. Then we can define the representation $\varphi : K_1 = T^1 \times Sp(\frac{n-1}{2}) \times U(1) \times T^1 \to U(1) \times Sp(\frac{n-1}{2})$ as follows;

$$\varphi\left(x, \begin{pmatrix} A & 0 \\ 0 & y \end{pmatrix}, z\right) \mapsto (x^p y^q z^r, A)$$

where p, q, r are in $\mathbf{Z}$. Now we can assume the $U(1) \times Sp(\frac{n-1}{2})$-action ρ on $S^{2n-3} \subset \mathbf{H}^{\frac{n-1}{2}}$ as $\rho((t, X), \mathbf{h}) = X\mathbf{h}\bar{t}$ $(n \geq 3)$. Hence we have

$$K = \left\{ \left(x, \begin{pmatrix} x^p y^q z^r & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix}, z \right) \middle| \, B \in Sp(\frac{n-3}{2}), x, y, z \in T^1 \right\}$$

where $p \neq 0$ by $N_1 \neq T^1$ (by Lemma 10.1.4) because of $\sigma_1^{-1}(SO(2n-3)) = (\rho \circ \varphi)^{-1}(SO(2n-3)) = K$. Moreover we can assume $p > 0$ up to equivalence for the slice representation $\sigma_1 : K_1 \to O(2n-2)$.

Since $K_2/K \cong S^2$, $p > 0$ and $L_{21} = Sp(1)$ (by Lemma 10.1.3), we have

$$K_2 = \left\{ \left(h, \begin{pmatrix} h & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix}, z \right) \middle| \, B \in Sp(\frac{n-3}{2}), h \in Sp(1), y, z \in T^1 \right\},$$

that is $q = r = 0$. Therefore we have $G" = T^1 \subset \mathrm{Ker}(\sigma_2) \subset K$ by the slice representation $\sigma_2 : K_2 \to SO(3)$. This contradicts Proposition 4.1.1. Hence we have $G" = \{e\}$.

Moreover, from the same argument, we can put $p = 1,\ q = 0$ and we have

$$\begin{aligned}
K_1 &= T^1 \times Sp(\frac{n-1}{2}) \times U(1) \\
K_2 &= \left\{ \left(h, \begin{pmatrix} h & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| \, B \in Sp(\frac{n-3}{2}), h \in Sp(1), y \in T^1 \right\}, \\
K &= \left\{ \left(x, \begin{pmatrix} x & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| \, B \in Sp(\frac{n-3}{2}), x, y \in T^1 \right\}
\end{aligned}$$

and

$$\varphi\left(x, \begin{pmatrix} A & 0 \\ 0 & y \end{pmatrix} \right) \mapsto (x, A).$$

We also see the slice representation $\sigma_2 : K_2 \to SO(3)$ is unique up to equivalence.

Next we see

$$N(K; G)/K \simeq (N(\Delta; Sp(1) \times Sp(1))/\Delta) \times (N(U(1); Sp(1))/U(1)),$$

where $\Delta \simeq T^1$ is a diagonal subgroup in $Sp(1) \times Sp(1)$. If we denote by a the generator of $N(\Delta; Sp(1) \times Sp(1))/(N(\Delta; Sp(1) \times Sp(1)))^o \simeq N(\Delta; Sp(1) \times Sp(1))/(T^1 \times T^1) \simeq \mathbf{Z}_2$, then $xa = a\bar{x}$ for all $x \in T^1$. Hence we can consider the following diagram

$$\begin{array}{ccc}
G \times_{K_2} K_2/K & \xrightarrow{f} & G/K \\
\downarrow 1 \times R_\alpha & & \downarrow R_\alpha \\
G \times_{K_2} K_2/K & \xrightarrow{f} & G/K.
\end{array}$$

Here $f([g, kK]) = gkK$ and

$$\alpha = \left(a, \begin{pmatrix} a & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \in N(K; K_2).$$

We have $gkK\alpha = gk\alpha K$ for all $g \in G$ and $k \in K_2$. So this diagram is commutative. In this case R_α is the antipodal involution on $K_2/K \cong S^2$. Hence R_α is extendable to a K_2-equivariant diffeomorphism on D^3. Hence $M(R_\alpha) \cong M(id)$ from Lemma 4.3.1 (3.). Since $N(U(1); Sp(1))/U(1) \simeq \mathbf{Z}_2$, there are just two manifolds up to essential isomorphism. Hence we get the following proposition.

Proposition 10.2.1. *Let (G, M) be a G-manifold which has codimension one orbit G/K and two singular orbit G/K_1 and G/K_2 where $G = Sp(1) \times Sp(\frac{n+1}{2})$, $K_1 = T^1 \times Sp(\frac{n-1}{2}) \times U(1)$,*

$$\begin{aligned} K_2 &= \left\{ \left(h, \begin{pmatrix} h & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| \ B \in Sp(\frac{n-3}{2}), h \in Sp(1), y \in T^1 \right\} \text{ and} \\ K &= \left\{ \left(x, \begin{pmatrix} x & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & y \end{pmatrix} \right) \middle| \ B \in Sp(\frac{n-3}{2}), x, y \in T^1 \right\}. \end{aligned}$$

Then there are just two such (G, M) up to essential isomorphism which are $M = Q_{2n}$ and $M = (Sp(1) \times Sp(k+1)) \times_{Sp(1)\times Sp(k)\times U(1)} S^{4k+2}$ where $k = \frac{n-1}{2}$.

Proof. By the above argument, this case has just two types up to essential isomorphism. If $M = Q_{2n}$, then this case will be realized in Section 12.3. If $M = (Sp(1) \times Sp(k+1)) \times_{Sp(1)\times Sp(k)\times U(1)} S^{4k+2}$ such that $k = \frac{n-1}{2}$ and $S^{4k+2} \subset \mathbf{R}^3 \times \mathbf{H}^k$ has the trivial $U(1)$-action, the canonical $Sp(1)$-action on $\mathbf{R}^3$ and the canonical $Sp(1) \times Sp(k)$-action on $\mathbf{H}^k$. Then this manifold has the $Sp(1) \times Sp(k+1)$-action. We can easily check this manifold satisfies the assumption of this proposition. ∎

$M = (Sp(1) \times Sp(k+1)) \times_{Sp(1)\times Sp(k)\times U(1)} S^{4k+2}$ is the fibre bundle over $Sp(k+1)/U(1) \times Sp(k) \cong P_{2k+1}(\mathbf{C})$ with the fibre S^{4k+2}. We see easily check $H^{odd}(P_{2k+1}(\mathbf{C})) = H^{odd}(S^{4k+2}) = 0$ and $P_{2k+1}(\mathbf{C})$ is simply connected. Hence $p^* : H^*(P_{2k+1}(\mathbf{C})) \to H^*(M)$ is injective where $p : M \to P_{2k+1}(\mathbf{C})$ is a projection. Hence the $2k+2$ times cup product of $c \in H^2(M)$ is vanishing in $H^{4k+4}(M)$. Hence this is not a rational cohomology complex quadric. So this case is unique up to essential isomorphism and such (G, M) will be constructed in Section 12.3.

10.3 $G = Sp(1) \times Spin(11) \times G$"

If $G = Sp(1) \times Spin(11) \times G$", then we have

$$K_1 = T^1 \times Spin(9) \circ T^1 \times G\text{"}$$

and $G" = \{e\}$ or T^1. Let $\sigma_1 : K_1 \to O(16)$ be the slice representation. Then the restricted representation $\sigma_1|_{Spin(9)}$ is the spin representation to $SO(16)$ and we can easily show $C(\sigma_1(Spin(9)); SO(16))$ is a finite group. So we have $\sigma_1(T^1 \times \{e\}) = \{I_{16}\}$ because $T^1 \times \{e\} \subset C(Spin(9); K_1)$, where $e \in Spin(9) \circ T^1 \times G"$ and $I_{16} \in O(16)$ are identity elements. Therefore we see $K \supset \mathrm{Ker}(\sigma_1) \supset T^1 \times \{e\}$. So $N_1 = h_1^{-1}(K) = T^1 = H_{(1)}$, recall h_1 denotes the natural inclusion $H_1 \to G$. This contradicts Lemma 10.1.4. Hence this case does not occur.

10.4 $G = Sp(1) \times SU(3) \times G"$

If $G = Sp(1) \times SU(3) \times G"$, then we have

$$K_1 = T^1 \times S(U(2) \times U(1)) \times G"$$

and $G" = \{e\}$ or T^1. Put the element in K_1 by

$$\left(x, \begin{pmatrix} yA & 0 \\ 0 & y^{-2} \end{pmatrix}\right) = (x, yA) \text{ for } h = 0, \quad \left(x, \begin{pmatrix} yA & 0 \\ 0 & y^{-2} \end{pmatrix} z\right) = (x, yA, z) \text{ for } h = 1$$

where $x, y \in T^1$, $A \in SU(2)$ and $z \in T^1$ for $h = 1$. We can assume the slice representation $\sigma_1 : K_1 \xrightarrow{\kappa} T^1 \to O(2)$ by

$$\kappa(x, yA) = x^p y^{2q} \text{ for } h = 0, \quad \kappa(x, yA, z) = x^p y^{2q} z^r \text{ for } h = 1.$$

Because of Proposition 10.1.1, we have $q \neq 0$. Especially we can assume $q > 0$ up to equivalence. When $h = 1$, we see $r \neq 0$ from Proposition 4.1.1.

Now $K = \mathrm{Ker}(\sigma_1)$. So we have

$$K = \{(x, yA) |\ x^p y^{2q} = 1\} \ (h = 0) \ \mathit{or} \ \{(x, yA, z) |\ x^p y^{2q} z^r = 1\} \ (h = 1)$$

and $K^o \simeq SU(2) \circ T^{h+1}$. Moreover we see $K_2^o \simeq SU(2) \circ X \circ T^h$ where $(X, T^1) \approx (Sp(1), T^1)$ because of $K_2^o / K^o \cong S^2$. Hence $p_2(X) \simeq SO(3)$, $SU(2)$ or $\{e\}$ where $p_2 : G \to SU(3)$.

If $p_2(X) \neq \{e\}$ then we see $SU(2) \circ p_2(X) = p_2(K_2^o) \subset p_2(G) = SU(3)$. Hence we have $p_2(X) \simeq (p_2(X) \circ SU(2))/SU(2) \subset N(SU(2); SU(3))/SU(2) \simeq T^1$. But this contradicts $\dim(p_2(X)) = 3$.

Therefore $p_2(X) = \{e\}$. Consequently we have $X = Sp(1)$, $K_2 = X \times K_2'$ and $K = T^1 \times K_2' = H_{(1)} \times K_2'$, where $K_2' \subset SU(3) \times G"$. However $N_1 = T^1 = H_{(1)}$ contradicts Lemma 10.1.4. Hence this case does not occur.

11 $P(G/K_1; t) = (1 + t^{k_2 - 1}) a(n)$, k_2 is odd: G/K_1 is indecomposable.

Assume G/K_1, G/K_2 are orientable, $P(G/K_1; t) = (1 + t^{k_2 - 1}) a(n)$, k_2 is odd and G/K_1 is indecomposable. In this case $K_1 = K_1^o$ by $k_2 > 2$ and Lemma 3.2.1. Because G/K_1 is indecomposable, we can put $G = G' \times G"$ and $K_1 = K'_1 \times G"$ where G' is a simple Lie group and $G"$ is a direct product of some simple Lie groups and a toral group. The pair (G', K'_1) which satisfies

$$P(G/K_1; t) = P(G'/K'_1; t) = (1 + t^{2a})(1 + t^2 + \cdots + t^{2b})$$

where $2a = k_2 - 1$ and $b = n$ is locally isomorphic to one of the pairs in Proposition 4.2.3.

In the beginning, we will find the candidates for (G', K'_1).

11.1 Candidates for (G', K'_1)

The goal of this section is to prove the pair (G', K'_1) is one of the following

$$\begin{array}{l} (Spin(9), Spin(6) \circ T^1)\ (k_1 = 8, k_2 = n = 7) \text{ or} \\ (SU(3), T^2)\ (k_1 = 2, k_2 = 3, n = 2). \end{array}$$

Now $k_1 \geq 2$ and $k_1 + k_2 = 2n + 1$. So we can easily see the following three cases in Proposition 4.2.3 do not satisfy $k_1 = 2(b - a) \geq 2$.

$$\begin{array}{l} (SO(2n + 2), SO(2n) \times SO(2)), a = b = n, \\ (SO(7), U(3)), a = b = 3, \\ (Sp(3), U(3)), a = b = 3. \end{array}$$

Moreover we see the following six cases in Proposition 4.2.3 contradict Proposition 9.0.1 by the paper [6] Section I.

$$\begin{array}{l} (SO(k_2 + 2), SO(k_2 - 1) \times SO(2)), a = (k_2 - 1)/2, b = k_2,\ (k_2 \neq 7) \\ (SO(10), U(5)), a = 3, b = 7, \\ (Sp(3), Sp(1) \times Sp(1) \times U(1)), a = 2, b = 5, \\ (G_2, T^2), a = 1, b = 5, \\ (F_4, Spin(7) \circ T^1), a = 4, b = 11, \\ (F_4, Sp(3) \circ T^1), a = 4, b = 11. \end{array}$$

Therefore in this case we have that

$$\begin{array}{rcl} (G', K'_1) & = & (Spin(9), Spin(6) \circ T^1) \approx (SO(9), U(4))\ (k_1 = 8, k_2 = n = 7) \text{ or} \\ & & (SU(3), T^2)\ (k_1 = 2, k_2 = 3, n = 2) \text{ or} \\ & & (SU(5), S(U(3) \times U(2)))\ (k_1 = 4, k_2 = 5, n = 4) \text{ or} \\ & & (Sp(4), U(4))\ (k_1 = 8, k_2 = n = 7) \end{array}$$

by Proposition 4.2.3.

If $(G', K_1') = (SU(5), S(U(3) \times U(2)))$, then $k_1 = 4$. Hence $K_1/K \cong S^3$. Since $U(2)$ $(\subset K_1')$ acts transitively on K_1/K by Proposition 9.0.1, we can assume the slice representation as $\sigma_1 : K_1 \to U(2) \to SO(4)$. Therefore we see $G" = T^h$ $(h \leq 1)$ and $K \simeq S(U(3) \times \{e\}) \circ T^{h+1}$ by Proposition 4.1.1 and Proposition 9.0.1. In particular we see $K_2 \supset K \supset SU(3)$. Since $K_2/K \cong S^4$, $(K_2, K) = (A \circ N, B \circ N)$ where $(A, B) \approx (SO(5), SO(4))$ by Proposition 4.2.1. So $K \simeq SU(3) \times T^{h+1}$ contains $SO(4) \approx Sp(1) \times Sp(1)$ as a normal subgroup. But this is a contradiction. Hence this case does not occur.

If $(G', K_1') = (Sp(4), U(4))$, then $k_1 = 8$ and $K_1/K \cong S^7$. From Proposition 9.0.1, we can assume the slice representation as $\sigma_1 : K_1 \to U(4) \to SO(8)$. So $G" = \{e\}$ or T^1 by Proposition 4.1.1. Since $K_2/K \cong S^6$ and $K_1 = U(4)$ or $U(4) \times T^1$, we have $(K_2, K) \approx (G_2 \circ T^1, SU(3) \circ T^1)$ or $(G_2 \circ T^2, SU(3) \circ T^2)$ by Proposition 4.2.1. Therefore we get $Sp(4) \supset G_2$. However the following proposition holds.

Proposition 11.1.1. *$Sp(4) \not\supset G_2$.*

Proof. Assume $Sp(4) \supset G_2$. Let V be the $Sp(4)$-$\mathbf{C}$ irreducible 8-dimensional representation space (complex dimensional). Then we can consider $Sp(4)$ acts effectively on V by the natural representation $\rho : Sp(4) \to U(8)$. We see the restricted representation to G_2 $\rho|_{G_2}$ is not trivial. As is well known the least dimension of non-trivial complex representation of G_2 is 7, and there is no 8-dimensional irreducible representation of G_2 (by Section 5 in [20]; the representation ring of G_2 is $\mathbf{Z}[\lambda_1, \lambda_2]$ where $\dim\lambda_1 = 7$, $\dim\lambda_2 = 21$). Since V is an 8-dimensional space, there is an irreducible decomposition $V = V^7 \oplus W$ where V^7 is a complex seven dimensional G_2-space which has a representation $\rho|_{G_2}$ and W is a complex one dimensional space which has trivial G_2-action. Then V has the structure map $J : V \to V$ such that J is an $Sp(4)$-map, $J^2(v) = -v$ and $J(zv) = \bar{z}J(v)$ for $z \in \mathbf{C}$ and $v \in V$ (see [1] 3.2). Moreover $J(w) \in W$ for $w \in W$ because J is a $G_2(\subset Sp(4))$ map. However W is a complex one dimensional space, so this contradicts W does not have such map. Therefore we see $Sp(4) \not\supset G_2$. ■

Hence the following two cases remain.

11.2 $(G', K_1') = (Spin(9), Spin(6) \circ T^1)$

If $(G', K_1') = (Spin(9), Spin(6) \circ T^1)$, then $k_1 = 8$. So $K_1/K \cong S^7$, hence $G" = T^h$ $(h \leq 1)$ from Proposition 4.1.1 and Proposition 9.0.1.

Assume $h = 1$. Since $K_2/K \cong S^6$, we see $(K_2, K) = (G_2 \circ T^2, SU(3) \circ T^2)$. Consider the slice representation $\sigma_2 : G_2 \circ T^2 \to SO(7)$. Because K_2 acts transitively on $K_2/K \cong S^6$, the restricted representation $\sigma_2|_{G_2}$ is a natural inclusion. So $C(\sigma_2(G_2); SO(7)) = \{e\}$ where $C(E; F) = \{g \in F |\ gk = kg \text{ for all } k \in E\}$. Therefore $G" \subset \mathrm{Ker}(\sigma_2) = T^2 \subset K$. Now $G" = T^1$ is a normal subgroup of G. This contradicts Proposition 4.1.1. Hence $h = 0$.

We get $G" = \{e\}$ and $(G, K_1) = (Spin(9), Spin(6) \circ T^1)$. Since $h = 0$ and $K_2/K \cong S^6$, we see $(K_2, K) = (G_2 \circ T^1, SU(3) \circ T^1)$. Hence we can easily show that the slice representation $\sigma_2 : K_2 \to SO(7)$ is unique up to equivalence (especially $\sigma_2|_{T^1}$ is trivial) and the slice representation $\sigma_1 : K_1 \to SO(8)$ is decomposable as follows

$$K_1 = Spin(6) \circ T^1 \xrightarrow{\varphi} U(4) \xrightarrow{c} SO(8),$$

where c is a canonical inclusion and $\varphi(Spin(6)) = SU(4)$ ($\varphi|_{Spin(6)}$ is isomorphism). Then there are two slice representations σ_1 where are $\varphi|_{T^1}$ is trivial or non-trivial. If $\varphi|_{T^1}$ is non-trivial then we see $\varphi(T^1) = \Delta$ where $\Delta \simeq T^1$ is a diagonal scaler matrix in $U(4)$ because $\varphi(Spin(6)) = SU(4)$ and $C(SU(4); U(4)) = \Delta$. So we have $\sigma_1^{-1}(SO(7)) = K = SU(3) \circ T^1 \subset Spin(6) \circ T^1 (\simeq SU(4) \circ T^1)$. Let V be the $Spin(9)$-$\mathbf{R}$ irreducible 9-dimensional representation space. Then we can consider $Spin(9)$ acts on V by the natural representation $p : Spin(9) \to SO(9)$. So we see the restricted $SU(4) \circ T^1$-representation $s_1 = p|_{SU(4) \circ T^1}$ is non-trivial and $s_1 : SU(4) \circ T^1 \to SO(8) \subset SO(9)$ is the natural inclusion. Moreover from the restricted $SU(3) \circ T^1$-representation $s_1|_{SU(3) \circ T^1}$ we have an irreducible decomposition $V = V^6 \oplus W^3$, where $V^6 \simeq \mathbf{C}^3$ is a $SU(3) \circ T^1$-irreducible 6-dimensional space and W^3 is a 3-dimensional space whose $SU(3) \circ T^1$-action is trivial. On the other hand from the restricted $G_2 \circ T^1$-representation $s_2 = p|_{G_2 \circ T^1}$, we have the decomposition $V = X^7 \oplus Y^2$ where X^7 is a G_2-irreducible 7-dimensional space and Y^2 is a T^1-irreducible 2-dimensional space. Hence from the restricted $SU(3) \circ T^1$-representation, we have the decomposition $V = X'^6 \oplus \mathbf{R} \oplus Y^2$. Since $K \subset K_1 \cap K_2 = SU(4) \circ T^1 \cap G_2 \circ T^1$, we see $s_1|_{SU(3) \circ T^1} = s_2|_{SU(3) \circ T^1}$. However two decompositions $V^6 \oplus W^3$ and $X'^6 \oplus \mathbf{R} \oplus Y^2$ are different decompositions because the former one has trivial W^3 and the other has trivial $\mathbf{R}$. Hence $\sigma_1|_{T^1}$ is trivial.

Moreover we see

$$\begin{aligned} & N(K;G)/N(K;G)^o \\ = \; & N(SU(3) \circ T^1; Spin(9))/N(SU(3) \circ T^1; Spin(9))^o \\ \simeq \; & N(SU(3) \times SO(2); SO(9))/N(SU(3) \times SO(2); SO(9))^o \\ \simeq \; & \mathbf{Z}_2 \oplus \mathbf{Z}_2. \end{aligned}$$

Here we can put $\mathbf{Z}_2 \oplus \mathbf{Z}_2 = \{I, \ \alpha, \ \beta, \ \alpha\beta\}$ where

$$p(\alpha) = \begin{pmatrix} I_6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad p(\beta) = \begin{pmatrix} 0 & I_3 & 0 & 0 \\ I_3 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for the natural projection $p : Spin(9) \to SO(9)$. Then α satisfies $\alpha[A, t] = [A, t^{-1}]\alpha$ for an element $[A, t]$ in $K_1 = Spin(6) \circ T^1$ ($A \in Spin(6)$ and $t \in T^1$). Hence the diffeomorphism

$$R_\alpha \times id : G \times_{K_1} K_1/K \longrightarrow G \times_{K_1} K_1/K$$

defined by $R_\alpha \times id([g,[A,1]K]) = [g\alpha,[A,1]K]$ is well-defined (remark $[A,t]K = [A,1]K$ by the relation $T^1 \subset K \subset K_1$ where $1 \in T^1$ is the identity element). Now the following diagram is commutative;

$$\begin{array}{ccc} G \times_{K_1} K_1/K & \xrightarrow{f} & G/K \\ \downarrow R_\alpha \times id & & \downarrow R_\alpha \\ G \times_{K_1} K_1/K & \xrightarrow{f} & G/K \end{array}$$

where $f(g,kK) = gkK$ and $R_\alpha : G/K \to G/K$ is defined by $R_\alpha(gK) = g\alpha K$. Therefore $R_\alpha : (\partial(G \times_{K_1} D^8) =)G/K \to G/K$ is extendable to $\widetilde{R_\alpha} : G \times_{K_1} D^8 \to G \times_{K_1} D^8$ because $id : K_1/K \cong S^7 \to S^7 \cong K_1/K$ is extendable to $id : D^8 \to D^8$. So we see two manifolds constructed by attaching maps I and α are equivariantly diffeomorphic by Lemma 4.3.1. We also have two manifolds constructed by attaching maps β and $\alpha\beta$ are equivariantly diffeomorphic, because $\alpha\beta \cdot \beta = \alpha$ and the above R_α is extendable to $\widetilde{R_\alpha}$. Hence in this case there are just two G-manifolds M up to essential isomorphism. Hence the following proposition holds.

Proposition 11.2.1. *Let $(Spin(9), M)$ be a $Spin(9)$-manifold which has codimension one orbits $Spin(9)/SU(3) \circ T^1$ and two singular orbits $Spin(9)/K_1$ and $Spin(9)/K_2$ where $K_1 = Spin(6) \circ T^1$ and $K_2 = G_2 \circ T^1$. Then there are just two such $(Spin(9), M)$ up to essential isomorphism, that is, $M = Q_{14}$ and $M = Spin(9) \times_{Spin(7)\circ T^1} S^{14}$.*

Proof. From the above argument this case has just two such $(Spin(9), M)$ up to essential isomorphism. If $M = Q_{14}$, then we will be constructed in Section 12.4. Put $M = Spin(9) \times_{Spin(7)\circ T^1} S^{14}$ such that T^1 acts $S^{14} \subset \mathbf{R}^8 \times \mathbf{R}^7$ trivially and $Spin(7)$ acts canonically on $\mathbf{R}^7$ and acts on $\mathbf{R}^8$ through the spin representation $Spin(7) \to SO(8)$. Then this manifold has a canonical $Spin(9)$ action and satisfies the assumption of this case. ■

But $M = Spin(9) \times_{Spin(7)\circ T^1} S^{14}$ is the fibre bundle over $Spin(9)/Spin(7) \circ T^1 \cong Q_7(\sim P_{14}(\mathbf{C}))$ with the fibre S^{14}. Hence this is not a rational cohomology complex quadric. So this case is unique up to essential isomorphism and such (G, M) will be constructed in Section 12.4.

11.3 $(G', K_1') = (SU(3), T^2)$

If $(G', K_1') = (SU(3), T^2)$, then $k_1 = 2$. Hence $G" = T^h$ and $h \leq 1$. From $K_2/K \cong S^2$ and Proposition 4.2.1, we have $K_2^o = A \circ N$ and $K^o = A' \circ N$ such that A, N are connected normal subgroups of K_2^o and $(A, A') \approx (SU(2), T^1)$.

If $h = 0$ then we have $N = \{e\}$ and $K^o \simeq T^1$ because $K_1/K^o \cong S^1$. Therefore we have $K_2^o \simeq SU(2)$ or $SO(3)$ by $(K_2^o, K^o) = (A, A') \approx (SU(2), T^1)$.

Assume $A = SO(3)$. Because the representation of $SO(3)$ to $\mathbf{C}^3$ is unique up to conjugation, we can consider $SO(3)(= K_2^o \subset SU(3))$ by the canonical subgroup of $SU(3)$. Then $N(SO(3); SU(3)) = \mathbf{Z}_3 \times SO(3)$ where $\mathbf{Z}_3$ is the center of $SU(3)$. Hence $K_2 = SO(3)$ or $\mathbf{Z}_3 \times SO(3)$. Moreover we can easily show the slice representation $\sigma_2 : K_2 = (\mathbf{Z}_3\times)SO(3) \to SO(3)$ is canonical where $\sigma_2(\mathbf{Z}_3) = \{I_3\}$. So we have

$$
\begin{aligned}
K &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \middle| \, X \in SO(2) \right\} = SO(2) \text{ or} \\
& \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi X \end{pmatrix} \middle| \, \xi \in \mathbf{Z}_3, \ X \in SO(2) \right\} = \mathbf{Z}_3 \times SO(2).
\end{aligned}
$$

Since $K_1 \cap K_2 \supset K$, we can put K_1 as follows;

$$
K_1 = \left\{ \begin{pmatrix} t^{-2} & 0 \\ 0 & tX \end{pmatrix} = (t, X) \,\middle|\, t \in T^1, \ X \in SO(2) \right\} = T^1 \circ SO(2).
$$

So we have the slice representation $\sigma_1 : K_1 \to SO(2) \subset O(2)$. Since Ker $\sigma_1 = K$ and we can identify σ_1 up to conjugate in $O(2)$, we have $K_2 = SO(3)$ or $K_2 = \mathbf{Z}_3 \times SO(3)$. Let us construct a manifold. Because $N(K; SU(3))/N(K; SU(3))^o = \mathbf{Z}_2$ and a generator of $\mathbf{Z}_2$ can be taken from K_2, two manifolds constructed by two attaching maps in $\mathbf{Z}_2$ are diffeomorphic by the similar argument of Section 10.2. Hence $(G, \ M)$ with codimension one orbits G/K and two singular orbits G/K_1, G/K_2 is unique for each $K_2 = \mathbf{Z}_3 \times SO(3)$ and $K_2 = SO(3)$. So the following proposition holds.

Proposition 11.3.1. *Let $(SU(3), M)$ be a $SU(3)$-manifold which has codimension one orbits $SU(3)/K$ and two singular orbits $G/K_1 = SU(3)/(T^1 \circ SO(2))$ and $SU(3)/K_2$.*

If $(K_2, K) = (\boldsymbol{Z}_3 \times SO(3), \boldsymbol{Z}_3 \times SO(2))$, then $(SU(3), M)$ is essential isomorphic to $(SU(3), \Delta\backslash G_3(\boldsymbol{R}^6))$ where $SU(3) \subset U(3)$ and the diagonal subgroup $(S^1 \simeq)\Delta \subset U(3) \subset SO(6)$ (Δ is the center of $U(3)$) are commutative and $SU(3)$ acts on

$$
\Delta\backslash G_3(\boldsymbol{R}^6) \cong \Delta\backslash SO(6)/SO(3) \times SO(3)
$$

by the canonical representation $SU(3) \to SO(6)$.

If $(K_2, K) = (SO(3), SO(2))$, then $(SU(3), M)$ is essential isomorphic to the natural induced $SU(3)$-action on the threefold branched covering manifold $\widetilde{N^8}$ of $\Delta\backslash G_3(\boldsymbol{R}^6)$, that is, there exists an $SU(3)$-equivariant map $p : \widetilde{N^8} \to \Delta\backslash G_3(\boldsymbol{R}^6)$ such that the restricted map $p|_{G/K_1}$ is isomorphic and the restricted map $p|_{\widetilde{N^8}-G/K_1}$ is threefold covering.

Proof. Assume $(K_2, K) = (\mathbf{Z}_3 \times SO(3), \mathbf{Z}_3 \times SO(2))$. Because the uniqueness of $(SU(3), M)$ has been proved before this proposition, we may only find such example. Now $U(3)$ acts on $G_3(\mathbf{R}^6) = SO(6)/(SO(3)\times SO(3))$ by the natural representation $U(3) \to SO(6)$ and this action has codimension one orbits and two singular orbits $U(3)/SO(3)$ and $U(3)/T^2$

where T^2 does not contain the diagonal subgroup in $U(3)$. Let $\Delta \subset U(3)$ be the diagonal subgroup. Then Δ commutes with $SU(3) \subset U(3)$ and acts on $G_3(\mathbf{R}^6)$ freely. So we have the 8-dimensional manifold $\Delta\backslash G_3(\mathbf{R}^6)$ and the $SU(3)$-action with codimension one principal orbits $SU(3)/\mathbf{Z}_3 \times SO(2)$, two singular orbits $SU(3)/\mathbf{Z}_3 \times SO(3)$, $SU(3)/T^2$. Hence this $(SU(3), \Delta\backslash G_3(\mathbf{R}^6))$ is the case $(K_2, K) = (\mathbf{Z}_3 \times SO(3), \mathbf{Z}_3 \times SO(2))$.

Assume $(K_2, K) = (SO(3), SO(2))$. Because the uniqueness of $(SU(3), M)$ has been proved before this proposition, we only need to find such example. Put $M = X_1 \cup X_2 = \widetilde{N^8}$ where X_1 and X_2 are tubular neighborhoods of $SU(3)/T^2$ and $SU(3)/SO(3)$. Then we can easily show that $\widetilde{N_8}$ is the threefold branched covering manifold $\widetilde{N^8}$ of $\Delta\backslash G_3(\mathbf{R}^6)$ along $SU(3)/T^2$. Therefore the case $(K_2, K) = (SO(3), SO(2))$ were proved.

Hence we get this proposition. ■

Now we can easily prove $H^2(\Delta\backslash G_3(\mathbf{R}^6); \mathbf{Q}) \simeq \mathbf{Q} \oplus \mathbf{Q}$ and $\dim \Delta\backslash G_3(\mathbf{R}^6) = 8$. Hence $\Delta\backslash G_3(\mathbf{R}^6)$ is not a rational cohomology complex quadric. Let $p : \widetilde{N^8} \to \Delta\backslash G_3(\mathbf{R}^6)$ be a natural projection. Then we can prove that $p^* : H^2(\Delta\backslash G_3(\mathbf{R}^6); \mathbf{Q}) \to H^2(\widetilde{N^8}; \mathbf{Q})$ is an injective homomorphism by two Mayer-Vietoris exact sequences for tubular neighborhoods of G/K_1, G/K_2 in $\Delta\backslash G_3(\mathbf{R}^6)$ and $\widetilde{N^8}$ and the five lemma. Hence $\widetilde{N^8}$ is also not a rational cohomology complex quadric. Therefore we have $A = SU(2)$.

Now we can put

$$K_1 = \left\{ \begin{pmatrix} x^{-1}y^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} = (x, y) \;\middle|\; x,\ y \in T^1 \right\}$$

and the slice representation $\sigma_1 : K_1 = T^2 \xrightarrow{\tau} T^1 \xrightarrow{\rho_1} O(2)$ is

$$\tau(x, y) = x^p y^q$$

where $\rho_1 : T^1 \to O(2)$ is a natural inclusion and $q \neq 0$ without loss of generality. Then $\mathrm{Ker}\ \tau = K = \{(x, y) \in T^2 \mid x^p y^q = 1\}$. Let us consider the restricted slice representation $\sigma_2|_{K_2^o} : K_2^o \simeq SU(2) \xrightarrow{\rho_2} O(3)$. Then we see $\rho_2 : SU(2) \to SO(3) \subset O(3)$ is a natural homomorphism and $\sigma_2|_{K_2^o}^{-1}(SO(2)) = K^o$. So we have $K^o = \{(x, x^{-1}) \in T^2\} \subset K = \{(x, y) \in T^2 \mid x^p y^q = 1\}$. Therefore we get $p = q$ $(p \neq 0)$.

Hence we have the slice representation $\sigma_1^q : K_1 = T^2 \xrightarrow{\tau_q} T^1 \xrightarrow{\rho_1} O(2)$, such that $\tau_q(x, y) = x^q y^q$, is unique for each $q \neq 0$. Since it is easy to show σ_1^q and σ_1^{-q} are equivalent representation, we can assume $q > 0$ up to equivalence. Because $p = q > 0$ and $\sigma_2|_{K_2^o}^{-1}(SO(2)) = K^o$, we have

$$K_2 = \left\{ \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & A \end{pmatrix} \;\middle|\; A \in U(2),\ \det A = \lambda \in \mathbf{Z}_q \right\} \simeq \mathbf{Z}_{2q} \times_{\mathbf{Z}_2} SU(2) \text{ and}$$
$$K = \{(x, x^{-1}\lambda) \mid x \in T^1,\ \lambda \in \mathbf{Z}_q\} \simeq \mathbf{Z}_{2q} \times_{\mathbf{Z}_2} T^1.$$

Here $\mathbf{Z}_{2q} \times_{\mathbf{Z}_2} SU(2) \simeq K_2 \subset S(U(1) \times U(2)) \simeq T^1 \times_{\mathbf{Z}_2} SU(2)$ and $\mathbf{Z}_p = \{x \in T^1 \mid x^p = 1\}$. Put such a slice representation as $\sigma_2^q : K_2 \simeq \mathbf{Z}_{2q} \times_{\mathbf{Z}_2} SU(2) \to O(3)$. Then we see $\sigma_2^q(\{e\} \times SU(2)) = \rho_2(SU(2)) = SO(3)$. For the generator a of $\mathbf{Z}_{2q}$, we have $\sigma_2^q(a) \in O(2)$ because $a \in K \simeq \mathbf{Z}_{2q} \times_{\mathbf{Z}_2} T^1$. Moreover $\sigma_2^q(a) = I_3$ because $\sigma_2^q(a)$ commutes with $\sigma_2^q(SU(2))$. Hence $\sigma_2^q(\mathbf{Z}_{2q} \times \{I_2\}) = \{I_3\}$. So σ_2^q is unique for each $q > 0$.

Moreover we can put $N(K;G)/N(K;G)^o = \mathbf{Z}_2 = \{I_3, [\alpha]\}$ and

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since we can take $\alpha \in K_2$, the $SU(3)$-manifold M is unique up to essential isomorphism by Lemma 4.3.1 for each $q > 0$.

Put the quotient manifold $M = SU(3) \times_{S(U(1)\times U(2))} S^4$ by the $S(U(1) \times U(2))$-action on $S^4 \subset \mathbf{C} \times \mathbf{R}^3$ as follows

$$\begin{pmatrix} t^{-2} & 0 \\ 0 & tX \end{pmatrix} \cdot (z, \mathbf{a}) = (t^{2q}z, \rho(X)\mathbf{a})$$

where $\rho : SU(2) \to SO(3)$ is a natural projection, $X \in SU(2)$, $t \in T^1$ and $(z, \mathbf{a}) \in S^4 \subset \mathbf{C} \times \mathbf{R}^3$. Now $SU(3)$ acts on M by the canonical $SU(3)$-action on $SU(3)$ and it has codimension one principal orbits $SU(3)/K$ and two singular orbits $SU(3)/K_1$ and $SU(3)/K_2$. However this manifold M is a S^4-bundle over $P_2(\mathbf{C})$. Hence this is not a rational cohomology complex quadric.

So we have $h = 1$, $G = SU(3) \times T^1$ and $K_1 = T^2 \times T^1$. Moreover we see $N = T^1$, $K_2^o = A \circ T^1$ and $K^o = A' \circ T^1$ because $K_1/K \cong S^1$ where $(A, A') \approx (SU(2), T^1)$.

Now we can put

$$K_1 = \left\{ \left(\begin{pmatrix} x^{-1}y^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}, z \right) = (x, y, z) \middle| x,\ y,\ z \in T^1 \right\}$$

and the slice representation $\sigma_1 : K_1 = T^2 \times T^1 \xrightarrow{\tau} T^1 \xrightarrow{\rho} O(2)$ is

$$\tau(x, y, z) = x^p y^q z^r$$

where $\rho : T^1 \to O(2)$ is a natural inclusion. Since we have $\mathrm{Ker}\tau = K$, we can assume $r > 0$ up to equivalence by Proposition 4.1.1. Hence we have

$$K = \left\{ \left(\begin{pmatrix} x^{-1}y^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix},\ x^{\frac{-p}{r}} y^{\frac{-q}{r}} \lambda \right) \middle| x,\ y \in T^1,\ \lambda \in \mathbf{Z}_r \right\}.$$

Therefore we have $p_1(K_2^o) = A \circ T^1 \subset SU(3)$, where $p_1 : G = SU(3) \times T^1 \to SU(3)$ is a natural projection. Assume $A = SO(3)$. Then we see $N(SO(3); SU(3)) = \mathbf{Z}_3 \times SO(3)$. However this is a contradiction, because all elements in $T^1 \subset p_1(K_2^o)$ and $A \subset p_1(K_2^o)$ commute. Hence we have $A = SU(2)$. So we can put the singular isotropy group K_2^o is as follows

$$K_2^o = \left\{ \left(\begin{pmatrix} t^{-2} & 0 \\ 0 & tX \end{pmatrix}, t^{-m} \right) \middle| X \in SU(2),\ t \in T^1 \right\},$$

for some $m \in \mathbf{Z}$. Since $K_1 \cap K_2^o \supset gK^o g^{-1} \simeq K^o$ for some $g \in G$, we have

$$\begin{aligned} K_1 \cap K_2^o &= \left\{ \left(\begin{pmatrix} t^{-2} & 0 & \\ 0 & ts & 0 \\ 0 & 0 & ts^{-1} \end{pmatrix}, t^{-m} \right) \middle| t,\ s \in T^1 \right\} \\ \simeq\ K^o &= \left\{ \left(\begin{pmatrix} x^{-1}y^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix},\ x^{\frac{-p}{r}} y^{\frac{-q}{r}} \right) \middle| x,\ y \in T^1 \right\} \end{aligned}$$

(a conjugation $K_1 \cap K_2^o \simeq K^o$ is known by their dimensions). Hence we can put

$$\begin{aligned} K^o = K_1 \cap K_2^o &= \left\{ \left(\begin{pmatrix} x^{-1}y^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix},\ x^{\frac{-p}{r}} y^{\frac{-q}{r}} \right) \middle| x,\ y \in T^1 \right\} \\ &= \left\{ \left(\begin{pmatrix} t^{-2} & 0 & \\ 0 & ts & 0 \\ 0 & 0 & ts^{-1} \end{pmatrix}, t^{-m} \right) \middle| t,\ s \in T^1 \right\} \end{aligned}$$

without loss of generality. Since $x = ts,\ y = ts^{-1}$, we have $p = q,\ m = \frac{2p}{r}$. Now the slice representation $\sigma_2|_{K_2^o}$ decomposes into $\sigma_2|_{K_2^o} : K_2^o \xrightarrow{\pi} SU(2) \xrightarrow{\rho'} SO(3)$ where

$$\pi(\left(\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}, 1 \right)) = X$$

and ρ' is a canonical double covering, and we have $\sigma_2|_{K_2^o}^{-1}(SO(2)) = K^o$. Consequently we have

$$\begin{aligned} K &= \left\{ \left(\begin{pmatrix} t^{-2} & 0 & 0 \\ 0 & ts & 0 \\ 0 & 0 & ts^{-1} \end{pmatrix}, t^{-m}\lambda \right) \middle| t,\ s \in T^1,\ \lambda \in \mathbf{Z}_r \right\} \simeq K^o \times \mathbf{Z}_r \text{ and} \\ K_2 &= \left\{ \left(\begin{pmatrix} t^{-2} & 0 \\ 0 & tX \end{pmatrix}, t^{-m}\lambda \right) \middle| X \in U(2),\ \lambda \in \mathbf{Z}_r \right\} \simeq K_2^o \times \mathbf{Z}_r. \end{aligned}$$

Moreover we have $m \neq 0$ because of Proposition 9.0.1, and $\sigma_2(\{I_3\}\times\mathbf{Z}_r) \subset \{I_3,\ -I_3\} \subset O(3)$ because of $\sigma_2(K_2^o) = SO(3)$. Because $\sigma_2^{-1}(O(2)) = K$ and $C(\sigma_2(SU(2)); O(3)) \cap O(2) = \{I_3, -I_3\} \cap O(2) = \{I_3\}$, we also have $\{I_3\} \times \mathbf{Z}_r \subset \mathrm{Ker}\sigma_2$. Since we classify up to essential isomorphism and $\{I_3\} \times \mathbf{Z}_r \subset \mathrm{Ker}\sigma_i$ for $i = 1, 2$, we can put $r = 1$ that is $K_2 = K_2^o$ and $K = K^o$. Therefore there exists unique (σ_i, K_i, K) $(i = 1, 2)$ for the integer $m \neq 0$. Then we have $N(K; G)/N(K; G)^o \simeq \mathbf{Z}_2 = \{I,\ \alpha\}$. Since we can take $\alpha \in K_2$, this case is unique up to essential isomorphism.

Put the quotient manifold $M = (SU(3) \times T^1) \times_{(S(U(1)\times U(2))\times T^1)} S^4$ by the $(S(U(1) \times U(2)) \times T^1)$-action on $S^4 \subset \mathbf{C} \times \mathbf{R}^3$ as follows

$$\left(\begin{pmatrix} t^{-2} & 0 \\ 0 & tX \end{pmatrix},\ z\right) \cdot (w, \mathbf{a}) = (t^m zw, \rho(X)\mathbf{a})$$

where $\rho : SU(2) \to SO(3)$ is a natural projection, $X \in SU(2)$, $t \in T^1$ and $(w, \mathbf{a}) \in S^4$. Now $SU(3) \times T^1$ acts on M by the canonical $(SU(3) \times T^1)$-action on $SU(3) \times T^1$ and it has codimension one principal orbits $(SU(3) \times T^1)/K$ and two singular orbits $(SU(3) \times T^1)/K_1$ and $(SU(3) \times T^1)/K_2$. However this manifold M is a S^4-bundle over $P_2(\mathbf{C})$. Hence this is not a rational cohomology complex quadric.

12 Compact transformation groups on rational cohomology complex quadrics with codimension one orbits.

All the pairs (G, M) which have codimension one principal orbits are exhibited in this last section.

12.1 $(SO(2n+1), Q_{2n})$

In this case $M = Q_{2n}$ and $SO(2n+1)$ acts on M through the canonical representation to $SO(2n+2)$. Then there are two singular orbits S^{2n} and Q_{2n-1}. The principal orbit type is $\mathbf{R}V_{2n+1,2} \cong SO(2n+1)/SO(2n-1)$.

Remark that we can easily show the pair $(Spin(2n+1),\ M)$ in Section 7.1 and the above example $(SO(2n+1), Q_{2n})$ are essentially isomorphic and we also have the following proposition by this example and [16]

Proposition 12.1.1. *For $n \geq 3$, $Q_n/\boldsymbol{Z}_2 \cong P_n(\boldsymbol{C})$.*

Proof. Put $\mathbf{Z}_2 = \left\{I_{n+2}, \begin{pmatrix} -1 & 0 \\ 0 & I_{n+1} \end{pmatrix} \in O(n+2)\right\}$. This group canonically acts on $Q_n \simeq SO(n+2)/SO(n) \times SO(2)$ and commutes with the action of $SO(n+1) \simeq$

$\{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in SO(2n+1)\}$. The pair $(SO(n+1), Q_n/\mathbf{Z}_2)$ has two singular orbits $P_{2n}(\mathbf{R})$ and Q_{n-1} and the principal orbit is $\mathbf{R}V_{n+1,2}/\mathbf{Z}_2$. From [16] Section 9.6, such manifold $(SO(n+1), M)$ is unique up to essential isomorphism that is we can regard $(SO(n+1), M)$ as $(SO(n+1), P_n(\mathbf{C}))$. Hence we get this proposition. ■

12.2 $(SU(n+1), Q_{2n})$

In this case $M = Q_{2n}$ and $SU(n+1)$ acts by the natural representation of $SO(2n+2)$ that is

$$SU(n+1) \ni A + B\mathbf{i} \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(2n+2).$$

Then there are two singular orbits, both orbit types are $P_n(\mathbf{C})$. The principal orbit type is $SU(n+1)/(SO(2) \times SU(n-1))$.

For $G = U(n+1)$ we get a similar result.

12.3 $(Sp(1) \times Sp(m), Q_{4m-2})$, $m \geq 2$

In this case $M = Q_{4m-2}$ $(n = 2m-1)$ and the action of $Sp(1) \times Sp(m)$ on $\mathbf{H}^m$ is defined by $A\mathbf{x}\bar{h}$ where $(h, A) \in Sp(1) \times Sp(m)$ and $\mathbf{x} \in \mathbf{H}^m$. So there is a natural representation $\rho : Sp(1) \times Sp(m) \to SO(4m)$. Hence we have an action of $Sp(1) \times Sp(m)$ on Q_{4m-2} through the representation ρ. Then there are two singular orbits $S^2 \times P_m(\mathbf{C})$ and $Sp(m)/(Sp(m-2) \times U(1))$. The principal orbit type is $Sp(1) \times_{T^1} Sp(m)/(Sp(m-2) \times U(1))$.

12.4 $(Spin(9), Q_{14})$

In this case $M = Q_{14}$. It is well known that $Spin(9)$ acts on S^{15} transitively by the spin representation $\rho : Spin(9) \to SO(16)$ ([20]). Hence $Spin(9)$ acts on Q_{14} through this representation. Then the principal orbit type is $Spin(9)/SU(3) \circ T^1$ and two singular orbits are $Spin(9)/Spin(6) \circ T^1$ and $Spin(9)/G_2 \circ T^1$.

12.5 (G_2, Q_6)

In this case $M = Q_6$ and the exceptional Lie group G_2 acts through the canonical representation to $SO(7)$. Then there are two singular orbits S^6 and $G_2/S(U(1) \times U(2))$. The principal orbit type is $\mathbf{R}V_{7,2} \cong G_2/SU(2)$.

12.6 $(G_2 \times T^1, G_2 \times_{SU(3)} P_3(\mathbf{C}))$

In this case $M = G_2 \times_{SU(3)} P_3(\mathbf{C})$ and $G_2 \times T^1$ acts by $\varphi : (G_2 \times T^1) \times M \to M$ as follows,

$$\varphi((g,t),[g',[z_0 : \mathbf{z}]]) = [gg',[tz_0 : \mathbf{z}]]$$

where $g \in G_2$, $t \in T^1$ and $[g',[z_0 : \mathbf{z}]] \in M$. The manifold M is a quotient manifold of $G_2 \times P_3(\mathbf{C})$ by the action $SU(3)$ where $SU(3)$ acts on G_2 canonically and on $P_3(\mathbf{C})$ by $\phi : [z_0 : \mathbf{z}] \mapsto [z_0 : A\mathbf{z}]$, here $A \in SU(3)$ and $[z_0 : \mathbf{z}] \in P_3(\mathbf{C})$. Then the action φ has codimension one orbit $(G_2 \times T^1)/(SU(2) \times \{e\}) \circ \Delta$ ($\Delta \simeq T^1$) and two singular orbits $(G_2 \times T^1)/(SU(3) \times T^1) \cong S^6$ and $(G_2 \times T^1)/(SU(2) \times T^1) \cong G_2/SU(2)$.

Moreover we have the following proposition.

Proposition 12.6.1. $G_2 \times_{SU(3)} P_3(\boldsymbol{C}) \cong Q_6$.

Proof. Consider the restricted G_2-action on $G_2 \times_{SU(3)} P_3(\mathbf{C})$. Then it has codimension one principal orbits $G_2/S(U(1) \times U(2))$ and two singular orbits $G_2/SU(3)$ and $G_2/SU(2)$. Hence we have $G_2 \times_{SU(3)} P_3(\mathbf{C}) \cong Q_6$ because of Lemma 7.2.1 and Section 12.5. ■

12.7 $(Sp(2), S^7 \times_{Sp(1)} P_2(\mathbf{C}))$

In this case $M = S^7 \times_{Sp(1)} P_2(\mathbf{C})$ and $Sp(2)$ canonical acts on $S^7 \cong Sp(2)/Sp(1)$. The manifold M is a quotient manifold of $S^7 \times P_2(\mathbf{C})$ by the action $Sp(1)$ where $Sp(1)$ acts on $S^7 \cong Sp(2)/Sp(1)$ canonically and on $P_2(\mathbf{C})$ by the double covering $Sp(1) \to SO(3)$. Then the $Sp(1)$ action on $P_2(\mathbf{C})$ has codimension one principal orbits $Sp(1)/\{1,-1,\mathbf{i},-\mathbf{i}\}$ and two singular orbits $Sp(1)/U(1)$ and $Sp(1)/U(1)_j \cup U(1)_j\mathbf{i}$ where $U(1)_j = \{a + b\mathbf{j} |\ a^2 + b^2 = 1\}$. Hence the $Sp(2)$ action on M has codimension one principal orbits $Sp(2)/Sp(1) \times \{1,-1,\mathbf{i},-\mathbf{i}\}$ and two singular orbits $Sp(2)/Sp(1) \times U(1)$ and $Sp(2)/Sp(1) \times (U(1)_j \cup U(1)_j\mathbf{i})$.

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