GKM manifold - definition

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1. Introduction

A class of manifold, now referred to as *GKM manifold*, was first discussed in the seminal work by Goresky-Kottwitz-MacPherson in [**8**] to study the relation between the equivariant cohomology and the ordinary cohomology. So the letters 'G', 'K' and 'M' stand for the first initials of the authors of [**8**]. Motivated by their work, the concepts of *GKM manifold* and *GKM graph* are introduced by Guillemin-Zara in [**11**] to build a bridge between combinatorics and geometry. These notions led to a fertile area of mathematics often called *GKM theory* which has been studied from both of geometry and combinatorics, and also applied to the other areas (e.g. representation theory, see [**6**]).

2. Definition

Because GKM theory is studied from several areas, there are different versions of definitions. Here, we review two major definitions which are often used in the literature.

The following is the original definition in [**11**].

Definition 2.1 ([**11**]). Let $T = T^n$ be an n-dimensional torus, i.e., a commutative, compact, connected n-dimensional Lie group, $\mathfrak{t}$ be its Lie algebra, and M be a compact $2d$-dimensional manifold with an effective T-action. We say that M is a *GKM manifold* if it satisfies that

(1) the fixed point set M^T is finite,
(2) the manifold M has a T-invariant almost complex sturcture,
(3) and for every $p \in M^T$, if $n \geq 2$, the weights

$$\alpha_{i,p} \in \mathfrak{t}^*$$

of the tangential (complex) representation of T on T_pM are pairwise linearly independent, i.e., each pair $\{\alpha_{i,p}, \alpha_{j,p}\}$ for $1 \leq i < j \leq d$ is linearly independent.

Remark 2.2. When $n = 1$, a GKM manifold is just the complex projective space $\mathbb{C}P^1$.

Example 2.3 (toric manifold). A typical example of GKM manifolds is the complx projective space $\mathbb{C}P^n$ with the standard T^n-action, i.e., every element $(t_1, \ldots, t_n) \in T^n$ acts on a point $[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}P^n$ by

$$[z_0 : z_1 : \cdots : z_n] \mapsto [z_0 : t_1 z_1 : \cdots : t_n z_n].$$

In this case, the following $(n+1)$ points are fixed under the T^n-action:

$$(\mathbb{C}P^n)^T = \{[1 : 0 : \cdots : 0], [0 : 1 : \cdots : 0], \ldots, [0 : \cdots : 0 : 1]\}.$$

Moreover, the weights of each tangential representation of T^n on $T_p\mathbb{C}P^n$ for $p \in (\mathbb{C}P^n)^T$ can be computed as follows (we compute the weights only for $n = 2$ because it is easy to apply this

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computation to the general cases). Let $p = [1:0:0], q = [0:1:0], r = [0:0:1] \in (\mathbb{C}P^2)^T$. We identify the open neighborhood of each fixed point with its tangent space by

$$U_p = \{[1:z_1:z_2] \mid z_i \in \mathbb{C}\} \simeq \{(z_1, z_2) \mid z_i \in \mathbb{C}\} \simeq T_p\mathbb{C}P^2,$$
$$U_q = \{[z_0:1:z_2] \mid z_i \in \mathbb{C}\} \simeq \{(z_0, z_2) \mid z_i \in \mathbb{C}\} \simeq T_q\mathbb{C}P^2,$$
$$U_r = \{[z_0:z_1:1] \mid z_i \in \mathbb{C}\} \simeq \{(z_0, z_1) \mid z_i \in \mathbb{C}\} \simeq T_r\mathbb{C}P^2.$$

Then, the induced T^2-actions on elements in $U_p, U_q, U_r \subset \mathbb{C}P^2$ are defined by

$$[1:z_1:z_2] \mapsto [1:t_1z_1:t_2z_2],$$
$$[z_0:1:z_2] \mapsto [z_0:t_1:t_2z_2] = [t_1^{-1}z_0:1:t_1^{-1}t_2z_2],$$
$$[z_0:z_1:1] \mapsto [z_0:t_1z_1:t_2] = [t_2^{-1}z_0:t_2^{-1}t_1z_1:1],$$

respectively. Therefore, each tangential (complex) representation decomposes into the following irreducible representations:

$$T_p\mathbb{C}P^2 \simeq V(\alpha_1) \oplus V(\alpha_2);$$
$$T_q\mathbb{C}P^2 \simeq V(-\alpha_1) \oplus V(-\alpha_1 + \alpha_2);$$
$$T_r\mathbb{C}P^2 \simeq V(-\alpha_2) \oplus V(\alpha_1 - \alpha_2),$$

where the symbol $V(\lambda)$ represents the complex 1-dimensional T-representation space induced from the representation $\lambda : T^2 \to S^1 \in \mathrm{Hom}(T, S^1) \simeq \mathfrak{t}^*$ and α_i is the projection onto the ith coordinate $(i = 1, 2)$. Because α_1 and α_2 are the basis of $\mathfrak{t}^*$, the weights of each tangential representation are linearly independent. This establishes that the complex projective space is a GKM manifold.

More generally, a *toric manifold* X is defined as an n-dimensional, non-singular, complete, complex algebraic variety which has a $(\mathbb{C}^*)^n$-action with dense orbit (see [**19**] for details), where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. If we restrict the algebraic torus $(\mathbb{C}^*)^n$-action on X to the topological (compact) torus T^n-action, then a toric manifold is a GKM manifold.

There is an alternative geometric way to state the third condition in Definition 2.1. Assume that M satisfies the first two conditions in Definition 2.1. Then, M satisfies the third condition if and only if each connected component of the set of points $p \in M$ such that $T(p) \simeq S^1$ is equivariantly diffeomorphic to $\mathbb{C} \setminus \{0\}$ and its closure is a T-invariant embedded 2-sphere. By this alternative condition, we can define the graph Γ by the *one-skeleton* of M, i.e., the set of points $p \in M$ such that the orbit $T(p) = \{p\}$ (a fixed point) or $T(p) \simeq S^1$. Namely, the vertices $V(\Gamma)$ of Γ are the fixed points of the T-action and the edges $E(\Gamma)$ of Γ are embedded 2-spheres connecting two fixed points. With this geometric viewpoint in mind, we give the second definition of GKM manifold, which is the generalization of the first one.

Definition 2.4 ([**9**]). Let T be an n-dimensional torus and M be a compact $2d$-dimensional manifold with an effective T-action. We say that M is a *GKM manifold* if it satisfies that the set of the 0-dimensional orbits in M/T is zero dimensional and the set of 1-dimensional orbits in M/T is one dimensional. In other words, the one-skeleton of M has the structure of a graph.

Example 2.5 (torus manifold). A typical example of GKM manifolds in the sense of Definition 2.4 (but not of Definition 2.1) is the $2n$-dimensional sphere for $n \geq 2$ with the standard T^n-action, i.e., every element $(t_1, \ldots, t_n) \in T^n$ acts on a point $(z_1, \ldots, z_n, r) \in S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ such that $|z_1|^2 + \cdots + |z_n|^2 + r^2 = 1$ as follows

$$(z_1, \ldots, z_n, r) \mapsto (t_1z_1, \ldots, t_nz_n, r).$$

In this case, there are 2 fixed points, i.e., 0-dimensional orbits, which are denoted by

$$(S^{2n})^T = \{(0, \ldots, 0, 1), (0, \ldots, 0, -1)\}.$$

Moreover, the set of 1-dimensional orbits are the disjoint union of n connected components

$$\{(z_1, 0 \ldots, 0, r)\} \sqcup \cdots \sqcup \{(0 \ldots, 0, z_n, r)\}$$

for $r \neq \pm 1$. Namely, the one-skeleton of S^{2n} is the graph which has two vertices and n edges which connecting these two vertices. Note that S^2 is equivariantly diffeomorphic to $\mathbb{C}P^1$ (cf. Example 2.3).

More generally, a *torus manifold* is defined as a $2n$-dimensional, compact, oriented manifold which has an effective T^n-action with fixed points. This is defined by Hattori-Masuda in [**12**] as a generalization of a *unitary toric manifold* [**16**] which is a torus manifold with a T-invariant stably complex structure. Every torus manifold is a GKM manifold in the sense of Definition 2.4 (see [**15, 17**] for details). In the case of a symplectic (or projective) toric manifold M, the one-skeleton of M is nothing but the one-skeleton of the simple convex polytope which is identified with M/T by the moment map.

Example 2.6 (homogeneous space). Let G be a compact, connected Lie group. We call the dimension of a maximal torus of G the *rank* of G. Let H be a closed, connected, maximal rank subgroup of G, i.e., a maximal torus T of H is also a maximal torus of G. Then, every homogeneous space G/H with the standard left T-action is a GKM manifold in the sense of Definition 2.4 (see [**9**] for details). If there is a T-invariant almost complex structure on G/H, then such a homogeneous space is also a GKM manifold in the sense of Definition 2.1. This condition is equivalent to saying that H is a parabolic subgroup of G, that is, G/H is a generalized flag manifold, e.g. the flag manifold $SU(n+1)/T^n$, the complex Grassmannian $U(m+n)/U(m)\times U(n)$, and the 6-dimensional sphere $G_2/SU(3)$, etc.

3. The Goresky-Kottwitz-MacPherson Theorem

One prominent feature of GKM manifolds is that the T-equivariant cohomology can be described in an algebraic-combinatorial fashion. An *equivariantly formal space* X defined in [**8**] is a (possibly singular) space with a compact, connected Lie group G-action such that the Leray-Serre cohomology spectral sequence for the Borel construction

$$X \hookrightarrow EG \times_G X \to BG$$

collapses at E_2-term.

Example 3.1. Every T-space X with $H^{odd}(X) = 0$ is an equivariantly formal space.

Let X be a (possibly singular) complex, projective, algebraic variety with an algebraic, complex torus $(\mathbb{C}^*)^n$-action. Suppose that the $(\mathbb{C}^*)^n$-action has finitely many fixed points, say $x_1, \ldots, x_k$, and finitely many 1-dimensional orbits, say $E_1, \ldots, E_\ell$. Then, the closure $\overline{E_j} = E_j \cup \{x_{j_0}\} \cup \{x_{j_\infty}\}$ is an embedded 2-sphere for all $j = 1, \ldots, \ell$. Let $\alpha_{j_0 j_\infty} \in \mathfrak{t}^* \simeq H^2(BT; \mathbb{R})$ be the weight of the isotropy T-action on $\overline{E_j}$. Recall that $H^*(BT; \mathbb{R})$ is isomorphic to the polynomial ring $S(\mathfrak{t}^*) \simeq \mathbb{R}[\alpha_1, \ldots, \alpha_n]$ generated by degree two elements $\alpha_1, \ldots, \alpha_n$.

Theorem 3.2 (Goresky-Kottwitz-MacPherson [**8**], see also [**10**]). *Let X be a $(\mathbb{C}^*)^n$-space with the above conditions. Then, the restricted map $H_T^*(X; \mathbb{R}) \to H_T^*(X^T; \mathbb{R}) \simeq \oplus_{i=1}^k H^*(BT; \mathbb{R})$ is injective, and its image is the subalgebra*

$$\{(f_1, \ldots, f_k) \in \oplus_{i=1}^k H^*(BT; \mathbb{R}) \mid f_{j_0} - f_{j_\infty} \in (\alpha_{j_0 j_\infty}) \text{ for } 1 \leq j \leq \ell\},$$

where $(\alpha_{j_0 j_\infty})$ is the ideal generated by $\alpha_{j_0 j_\infty} \in H^*(BT; \mathbb{R})$.

Remark 3.3. Projective toric manifolds and complex flag manifolds are examples which satisfy the conditions of this theorem. Moreover, this theorem corresponds to the piecewise polynomial description [**3**] and Arabia's description [**1**] of the T-equivariant cohomology of those manifolds.

Remark 3.4. We also note that if the above space X is non-singular and we restrict the $(\mathbb{C}^*)^n$-action to the maximal torus $T \subset (\mathbb{C}^*)^n$ action, then X is a GKM manifold in the sense of Definition 2.1. If X is a non-singular algebraic variety with the above $(\mathbb{C}^*)^n$-action, this is called an *algebraic GKM manifold* in [**18**].

Therefore, the equivariant cohomology ring $H_T^*(X; \mathbb{R})$ of the GKM manifold X with the above conditions can be computed by the one-skeleton of X and the (dual of) isotropy weights for each E_j. This leads us to define the labelled graph (Γ, α) from X as follows: Γ is the one-skeleton of X and a label $\alpha : E(\Gamma) \to \mathfrak{t}^*$ by $\alpha(e_j) = \alpha_{j_0 j_\infty}$ for each edge e_j corresponding to E_j, where $E(\Gamma)$ is the set of oriented edges. This labelled graph (Γ, α) is called a *GKM graph* in [**11**] (also see [**14, 20**]). Moreover, in [**11**], Guillemin-Zara define an abstract *GKM graph* (Γ, α) (might not be induced from a GKM manifold) and a *cohomology ring* of GKM graph $H(\Gamma, \alpha)$ motivated by Theorem 3.2. Using these notions, they solve the deformation problem of an edge-reflecting polytope Δ in $\mathbb{R}^n$ in [**11**], i.e., how to deform Δ so that the directions of its edges are unchanged (also see [**2**]). This is an application of GKM theory to solve a problem in combinatorics.

Remark 3.5. Let (Γ, α) be a GKM graph (defined in [**11**]) induced from a GKM manifold in the sense of Definition 2.1. Then, the label on each edge $e \in E(\Gamma)$ satisfies that

$$\alpha(e) = -\alpha(\bar{e}),$$

where $\bar{e} \in E(\Gamma)$ represents the edge e with its orientation reversed. On the other hand, in [**15**], Maeda-Masuda-Panov define a labelled graph induced from the (dual of) isotropy weights of a unitary toric manifold (or more generally an *omnioriented torus manifold*), called *torus graph* (also see [**5**]). Because the label on each edge $e \in E(\Gamma)$ of torus graph (Γ, α) satisfies

$$\alpha(e) = \pm\alpha(\bar{e}),$$

this gives a slightly generalized GKM graph.

Finally, we also note that Theorem 3.2 for T-manifolds can be generalized to the following theorem, by using the Chang-Skjelbred lemma [**4**, Lemma 2.3] (also see [**10, 13**]) and the Mayer-Vietoris exact sequence of T-equivariant cohomology for each invariant 2-sphere.

Theorem 3.6 (see [**7**]). *Let M be an equivariantly formal, GKM manifold in the sense of Definition 2.4. Then, the restricted map $H_T^*(M; \mathbb{Q}) \to H_T^*(M^T; \mathbb{Q}) \simeq \oplus_{i=1}^k H^*(BT; \mathbb{Q})$ is injective, and its image is the subalgebra*

$$\{(f_1, \ldots, f_k) \in \oplus_{i=1}^k H^*(BT; \mathbb{Q}) \mid f_{j_0} - f_{j_\infty} \in (\alpha_{j_0 j_\infty}) \text{ for } 1 \leq j \leq \ell\},$$

where $(\alpha_{j_0 j_\infty}) \subset H^*(BT; \mathbb{Q})$ is the ideal generated by the dual of isotropy weight of the T-action on the invariant 2-sphere $\overline{E_j} = E_j \cup \{x_{j_0}\} \cup \{x_{j_\infty}\}$ $(1 \leq j \leq \ell)$.

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