GKM manifold

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Abstract. Based on the work of Goresky-Kottwitz-MacPherson, Guilleminn-Zara make a connection between some nice manifolds with torus actions (GKM manifolds) and labeled graphs (GKM graphs). This connection is called the GKM theory. In this article, we give an overview of the GKM theory

1. Introduction

A class of manifolds with torus actions, now referred to as *GKM manifolds*, was first discussed in the seminal work by Goresky-Kottwitz-MacPherson in [**29**] to study the relation between the equivariant cohomology and the ordinary cohomology. GKM is made of the initial letters of the surnames of the authors of [**29**]. The GKM manifolds contain a wide class of manifolds. For example, *non-singular toric varieties* (also called a *toric manifolds*) or compact simply connected maximal rank homogeneous manifolds are GKM manifolds (see Section 2). In particular, in the paper [**29**], Goresky-Kottwitz-MacPherson show that the torus equivariant cohomology of *equivariantly formal GKM manifolds* (see Section 3) can be computed by using the data of its fixed points and one-dimensional orbits only (see Theorem 3.3). Motivated by their work, the concepts of *GKM manifold* and *GKM graph* are introduced by Guillemin-Zara in [**35**] to build a bridge between geometry and combinatorics. This may be regarded as a generalization of the concept of the symplectic toric manifolds and their moment graphs in symplectic geometry (see Section 4). These notions led to a fertile area of mathematics often called *GKM theory* which has been studied from both of geometry and combinatorics, and also applied to other areas (e.g. representation theory, see [**18**]).

Note that there are many works and generalizations about the relations between equivariant topology (or geometry) of GKM manifolds and combinatorics (see References). In this article, we basically survey a part of the original work of Guillemin-Zara [**35**]. In particular, we introduce the connection between GKM manifolds and GKM graphs in Section 4, and how we can compute the equivariant cohomology of toric manifolds by using GKM theory in Section 5.

2. Definition

Because GKM theory is studied from several areas, there are different versions of definitions. In this section, we review two major definitions which are often used in the literature. To define them, we use the following notations: the symbol $T = T^n$ is an n-dimensional torus, i.e., a commutative, compact, connected n-dimensional Lie group, $\mathfrak{t}$ is its Lie algebra, and $\mathfrak{t}^*$ is the dual of $\mathfrak{t}$.

The following is the original definition of the GKM manifold in [**35**].

Definition 2.1 ([**35**]). Let M be a compact $2m$-dimensional manifold with an effective T-action φ. We say that M (or (M^{2m}, φ, T), (M^{2m}, T) if we emphasize the torus action) is a *GKM manifold* if it satisfies that

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(1) the fixed point set M^T is non-empty and finite,
(2) the manifold M has a T-invariant almost complex sturcture,
(3) and for every $p \in M^T$, if $n \geq 2$, the weights

$$\alpha_{i,p} \in \mathfrak{t}^*$$

of the tangential (complex) representation of T on T_pM are pairwise linearly independent, i.e., each pair $\{\alpha_{i,p}, \alpha_{j,p}\}$ for $1 \leq i < j \leq m$ is linearly independent.

REMARK 2.2. When $n = 1$, a GKM manifold is just the complex projective space $\mathbb{C}P^1$.

The existence of the fixed points and the differentiable slice theorem tell us that the inequality $n \leq m$ always holds for every GKM manifold.

EXAMPLE 2.3 (toric manifold). A typical example of GKM manifolds is the complx projective space $\mathbb{C}P^n$ with the standard T^n-action, i.e., every element $(t_1, \dots, t_n) \in T^n$ acts on a point $[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}P^n$ by

$$[z_0 : z_1 : \cdots : z_n] \mapsto [z_0 : t_1 z_1 : \cdots : t_n z_n].$$

In this case, the following $(n+1)$ points are fixed under the T^n-action:

$$(\mathbb{C}P^n)^T = \{[1 : 0 : \cdots : 0], [0 : 1 : \cdots : 0], \dots, [0 : \cdots : 0 : 1]\}.$$

Moreover, the weights of each tangential representation of T^n on $T_p\mathbb{C}P^n$ for $p \in (\mathbb{C}P^n)^T$ can be computed as follows (we compute the weights only for $n = 2$ because it is easy to apply this computation to the general cases). Let $p = [1 : 0 : 0], q = [0 : 1 : 0], r = [0 : 0 : 1] \in (\mathbb{C}P^2)^T$. We identify the open neighborhood of each fixed point with its tangent space by

$$\begin{aligned} U_p &= \{[1 : z_1 : z_2] \mid z_i \in \mathbb{C}\} \simeq \{(z_1, z_2) \mid z_i \in \mathbb{C}\} \simeq T_p\mathbb{C}P^2, \\ U_q &= \{[z_0 : 1 : z_2] \mid z_i \in \mathbb{C}\} \simeq \{(z_0, z_2) \mid z_i \in \mathbb{C}\} \simeq T_q\mathbb{C}P^2, \\ U_r &= \{[z_0 : z_1 : 1] \mid z_i \in \mathbb{C}\} \simeq \{(z_0, z_1) \mid z_i \in \mathbb{C}\} \simeq T_r\mathbb{C}P^2. \end{aligned}$$

Then, the induced T^2-actions on elements in $U_p, U_q, U_r \subset \mathbb{C}P^2$ are defined by

$$\begin{aligned} &[1 : z_1 : z_2] \mapsto [1 : t_1 z_1 : t_2 z_2], \\ &[z_0 : 1 : z_2] \mapsto [z_0 : t_1 : t_2 z_2] = [t_1^{-1} z_0 : 1 : t_1^{-1} t_2 z_2], \\ &[z_0 : z_1 : 1] \mapsto [z_0 : t_1 z_1 : t_2] = [t_2^{-1} z_0 : t_2^{-1} t_1 z_1 : 1], \end{aligned}$$

respectively. Therefore, each tangential (complex) representation decomposes into the following irreducible representations:

$$\begin{aligned} T_p\mathbb{C}P^2 &\simeq V(\alpha) \oplus V(\beta); \\ T_q\mathbb{C}P^2 &\simeq V(-\alpha) \oplus V(-\alpha + \beta); \\ T_r\mathbb{C}P^2 &\simeq V(-\beta) \oplus V(\alpha - \beta), \end{aligned}$$

where the symbol $V(\lambda)$ represents the complex 1-dimensional T-representation space induced from the representation $(\lambda : T^2 \to S^1) \in \mathrm{Hom}(T, S^1) \simeq \mathbb{Z}^n \simeq \mathfrak{t}^*_{\mathbb{Z}}$ (i.e., the lattice of $\mathrm{Hom}(T, S^1) \otimes \mathbb{R} \simeq \mathbb{R}^n \simeq \mathfrak{t}^*$) and α (resp. β) is the projection from T^2 onto the 1st (resp. 2nd) coordinate. Because α and β are the basis of $\mathfrak{t}^*$, the weights of each tangential representation are linearly independent. This establishes that the complex projective space with the standard torus action is a GKM manifold.

More generally, a *toric manifold* X is defined as an n-dimensional, non-singular, (complete) complex algebraic variety with a $(\mathbb{C}^*)^n$-action having a dense orbit (see [**52**] for details), where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. If we restrict the algebraic torus $(\mathbb{C}^*)^n$-action on X to the topological (compact) torus T^n-action, then a toric manifold is a GKM manifold.

There is an alternative geometric way to state the third condition in Definition 2.1. Assume that M satisfies the first two conditions in Definition 2.1. Then, M satisfies the third condition if and only if each connected component of the set of points $p \in M$ such that its T-orbit $T(p)$ is diffeomorphic to the circle S^1, i.e., $T(p) \cong S^1$, is equivariantly diffeomorphic to $\mathbb{C} \setminus \{0\}$ and its closure is a T-invariant embedded 2-sphere. By this alternative condition, we can define the graph

Γ by the *one-skeleton* of M, i.e., the set of points $p \in M$ such that the orbit $T(p) = \{p\}$ (a fixed point) or $T(p) \cong S^1$. Namely, the vertices $V(\Gamma)$ of Γ are the fixed points of the T-action and the edges $E(\Gamma)$ of Γ are the embedded 2-spheres connecting two fixed points. With this geometric viewpoint in mind, we give the second definition of GKM manifold, which is the generalization of the first one.

DEFINITION 2.4 ([**31**]). Let M be a compact $2m$-dimensional manifold with an effective T-action and $M^T \neq \emptyset$. We say that M is a *GKM manifold* if it satisfies that the set of the 0-dimensional orbits (i.e., the fixed points) in M/T is zero dimensional and the set of 1-dimensional orbits in M/T is one dimensional. In other words, the one-skeleton of M has the structure of a graph.

EXAMPLE 2.5 (torus manifold). A typical example of GKM manifolds in the sense of Definition 2.4 (but not of Definition 2.1) is the $2n$-dimensional sphere for $n \geq 2$ with the standard T^n-action, i.e., every element $(t_1, \ldots, t_n) \in T^n$ acts on a point $(z_1, \ldots, z_n, r) \in S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ such that $|z_1|^2 + \cdots + |z_n|^2 + r^2 = 1$ as follows

$$(z_1, \ldots, z_n, r) \mapsto (t_1 z_1, \ldots, t_n z_n, r).$$

In this case, there are 2 fixed points, i.e., 0-dimensional orbits, which are denoted by

$$(S^{2n})^T = \{(0, \ldots, 0, 1), (0, \ldots, 0, -1)\}.$$

Moreover, the set of 1-dimensional orbits are the disjoint union of n connected components

$$\{(z_1, 0 \ldots, 0, r)\} \sqcup \cdots \sqcup \{(0 \ldots, 0, z_n, r)\}$$

for $r \neq \pm 1$. Namely, the one-skeleton of S^{2n} is the graph which has two vertices and n edges connecting these two vertices. Therefore, the $2n$-dimensional spheres are GKM manifolds in the sense of Definition 2.4. However, it is well-known that there is no almost complex structures on S^{2n} for $n = 2$ and $n \geq 4$ and there is no T^3-invariant almost complex structure on S^6. This shows that (S^{2n}, T^n) for $n \geq 2$ with the above action is not a GKM manifolds in the sense of Definition 2.1. Note that (S^2, T^1) is equivariantly diffeomorphic to $(\mathbb{C}P^1, T^1)$ (cf. Example 2.3) and S^6 has the T^2-invariant almost complex structure induced by the maximal torus T^2-action on $G_2/SU(3) (\cong S^6)$ (cf. Example 2.6), where G_2 is the exceptional Lie group. From this point of view, (S^2, T^1) and (S^6, T^2) are the only cases in $2n$-dimensional spheres which are GKM manifolds in the sense of Definition 2.1.

More generally, a *torus manifold* is defined as a $2n$-dimensional, compact, oriented manifold which has an effective T^n-action with fixed points. This is defined by Hattori-Masuda in [**39**] as a generalization of a *unitary toric manifold* [**51**] which is a torus manifold with a T-invariant stably complex structure (also see [**54**] about the stably complex structure). Every torus manifold is a GKM manifold in the sense of Definition 2.4 (see [**11, 50**] for details).

EXAMPLE 2.6 (homogeneous space). Let G be a compact, connected Lie group. We call the dimension of a maximal torus of G the *rank* of G. Let H be a closed, connected, maximal rank subgroup of G, i.e., a maximal torus T of H is also a maximal torus of G. Then, every homogeneous space G/H with the standard left T-action is a GKM manifold in the sense of Definition 2.4 (see [**31**] for details). If there is a T-invariant almost complex structure on G/H, then such a homogeneous space is also a GKM manifold in the sense of Definition 2.1. This condition is equivalent to saying that H is a parabolic subgroup of G, that is, G/H is a generalized flag manifold, e.g. the flag manifold $SU(n+1)/T^n$, the complex Grassmannian $U(m+n)/U(m) \times U(n)$, and the 6-dimensional sphere $G_2/SU(3)$, etc.

3. The Goresky-Kottwitz-MacPherson Theorem

One prominent feature of GKM manifolds is that the T-equivariant cohomology can be described in an algebraic-combinatorial fashion. An *equivariantly formal space* X defined in [**29**] is a (possibly singular) space with a compact, connected Lie group G-action such that the Leray-Serre cohomology spectral sequence for the Borel construction

$$X \hookrightarrow EG \times_G X \to BG$$

collapses at E_2-term.

EXAMPLE 3.1. Every T-space X with $H^{odd}(X;\mathcal{R})=0$ is an equivariantly formal space over $\mathcal{R}$-coefficient cohomology, where $\mathcal{R}$ is a coefficient ring.

REMARK 3.2. In [**29**], by using the equivariant derived category introduced by Bernstein-Lunts [**7**], Goresky-Kottwitz-MacPherson systematically study the several conditions to be an equivariantly formal for more general sheaf cohomology.

Let X be a (possibly singular) complex, projective, algebraic variety with an algebraic, complex torus $(\mathbb{C}^*)^n$-action. Suppose that the $(\mathbb{C}^*)^n$-action has finitely many fixed points, say $x_1,\ldots,x_k$, and finitely many complex 1-dimensional orbits, say $E_1,\ldots,E_\ell$. Then, the closure $\overline{E_j}=E_j\cup\{x_{j_0}\}\cup\{x_{j_\infty}\}$ is an embedded 2-sphere for all $j=1,\ldots,\ell$. Let $\alpha_{j_0j_\infty}\in\mathfrak{t}^*\simeq H^2(BT;\mathbb{R})$ be an element which generates the kernel of the isotropy weight of the T-action on E_j. More precisely, the isotropy subgroup on non-fixed points in E_j is the codimension-one subtorus $K\subset T$. Therefore, there is the induced surjective homomorphism $i_K^*:\mathfrak{t}^*\to\mathfrak{k}^*$ on their dual of Lie algebras; *the kernel of the isotropy weight* means the kernel of i_K^*, i.e., $(\alpha_{j_0j_\infty})=\ker i_K^*$. Recall that $H^*(BT;\mathbb{R})$ is isomorphic to the polynomial ring $S(\mathfrak{t}^*)\simeq\mathbb{R}[\alpha_1,\ldots,\alpha_n]$ generated by degree two elements $\alpha_1,\ldots,\alpha_n$, where we may regard α_i, $i=1,\ldots,n$, as the projection from T^n to the ith factor.

THEOREM 3.3 (Goresky-Kottwitz-MacPherson [**29**], see also [**34**]). *Let X be a $(\mathbb{C}^*)^n$-space with the above conditions. Then, the restricted map $H_T^*(X;\mathbb{R})\to H_T^*(X^T;\mathbb{R})\simeq\bigoplus_{i=1}^k H^*(BT;\mathbb{R})$ is injective, and its image is the subalgebra*

$$\{(f_1,\ldots,f_k)\in\bigoplus_{i=1}^k H^*(BT;\mathbb{R})\mid f_{j_0}-f_{j_\infty}\in(\alpha_{j_0j_\infty})\ \text{for}\ 1\le j\le\ell\},$$

where $(\alpha_{j_0j_\infty})$ is the ideal generated by $\alpha_{j_0j_\infty}\in H^*(BT;\mathbb{R})$.

REMARK 3.4. Projective toric manifolds and complex flag manifolds are examples which satisfy the conditions of this theorem. Moreover, this theorem corresponds to the piecewise polynomial description [**10, 21**] and Arabia's description [**3**] of the T-equivariant cohomology of those manifolds.

REMARK 3.5. We also note that if the above space X is non-singular and we restrict the $(\mathbb{C}^*)^n$-action to the maximal torus $T\subset(\mathbb{C}^*)^n$ action, then X is a GKM manifold in the sense of Definition 2.1. If X is a non-singular algebraic variety with the above $(\mathbb{C}^*)^n$-action, this is called an *algebraic GKM manifold* in [**47**].

We also note that Theorem 3.3 for T-manifolds (non-singular) can be generalized to the following theorem, by using the Chang-Skjelbred lemma [**12**, Lemma 2.3] (also see [**34, 41**]) and the Mayer-Vietoris exact sequence of T-equivariant cohomology for each invariant 2-sphere.

THEOREM 3.6 (also see [**22**]). *Let M be an equivariantly formal, GKM manifold in the sense of Definition 2.4. Then, the restricted map $H_T^*(M;\mathbb{Q})\to H_T^*(M^T;\mathbb{Q})\simeq\bigoplus_{i=1}^k H^*(BT;\mathbb{Q})$ is injective, and its image is the subalgebra*

$$\{(f_1,\ldots,f_k)\in\bigoplus_{i=1}^k H^*(BT;\mathbb{Q})\mid f_{j_0}-f_{j_\infty}\in(\alpha_{j_0j_\infty})\ \text{for}\ 1\le j\le\ell\},$$

where $(\alpha_{j_0j_\infty})\subset H^*(BT;\mathbb{Q})$ is the kernel of the isotropy weight of the invariant 2-sphere $\overline{E_j}=E_j\cup\{x_{j_0}\}\cup\{x_{j_\infty}\}$ $(1\le j\le\ell)$.

REMARK 3.7. More generally, we can also describe the several kinds of cohomology theories of GKM manifolds under some suitable assumptions by using the similar way of Theorem 3.6. For example, the equivariant cohomology with integer coefficient, the equivariant (complex oriented) generalized cohomology theory (in particular, equivariant K-theory) or some sheaf cohomologies etc (see e.g. [**8, 22, 25, 36**]). This type of description is often called a *GKM description* of cohomology.

Remark 3.8. If M is an equivariantly formal, the following sequence is exact (see [**12**]):

$$H_T^*(M;\mathbb{Q}) \xrightarrow{\iota^*} H_T^*(M^T;\mathbb{Q}) \xrightarrow{\delta_0} H_T^{*+1}(M_1, M^T;\mathbb{Q}),$$

where ι^* is the induced homomorphism from the inclusion $M^T \to M$ and δ_0 is the homomorphism of the relative (equivariant) cohomology of $(M_1, M^T)(= (M_1, M^T, \emptyset))$. Here, M_k, $0 \leq k \leq n$, is the set of points whose dimension of T^n-orbits is less than or equal to k, i.e., $M_0 = M^T$, $M_n = M$ and M_1 is the set of points whose orbit is 0 or 1-dimensional. This sequence is called a *Chang-Skjelbret sequence*. Theorem 3.6 may be regarded as the description of the image of ι^* of the Chang-Skjelbret sequence.

On the other hand, by using the relative cohomology of (M_{k+1}, M_k, M_{k-1}), we can generalize the Chang-Skjelbret sequence to the following long sequence (see [**4, 9**]):

$$\begin{aligned} &H_T^*(M;\mathbb{Q}) \xrightarrow{\iota^*} H_T^*(M^T;\mathbb{Q}) \xrightarrow{\delta_0} H_T^{*+1}(M_1, M^T;\mathbb{Q}) \\ &\xrightarrow{\delta_1} H_T^{*+2}(M_2, M_1;\mathbb{Q}) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-2}} H_T^{*+n-1}(M_{n-1}, M_{n-2}) \xrightarrow{\delta_{n-1}} H_T^{*+n}(M_n, M_{n-1}). \end{aligned}$$

This sequence is called an *Atiyah-Bredon sequence*. In [**2**], Allday-Franz-Puppe studies when the Atiyah-Bredon sequence is exact.

4. GKM graphs and GKM manifolds

The Goresky-Kottwitz-MacPherson Theorem says that the equivariant cohomology ring $H_T^*(X;\mathcal{R})$ (over an appropriate coefficient ring $\mathcal{R}$) of the equivariantly formal GKM manifold X can be computed by the one-skeleton of X and the kernel of the isotropy weights of one-dimensional orbits. This leads us to define the labelled graph (Γ, α) from X, called a *GKM graph*. We first define the GKM graph abstractly and then we introduce the relation between a GKM graph and a GKM manifold in this section. In Section 4 and 5, we assume the coefficient of cohomology is $\mathbb{R}$ for simplicity.

4.1. Abstract GKM graph. To define a GKM graph abstractly, we prepare some notation. Let $\Gamma = (V(\Gamma), E(\Gamma))$ (or (V, E) when the graph Γ is clear from the context) be an *(abstract) graph* comprising a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of oriented edges; denote the initial vertex of $e \in E(\Gamma)$ by $i(e)$ and its terminal vertex by $t(e)$. The symbol $\overline{e} \in E(\Gamma)$ represents the edge e with its orientation reversed, i.e., $i(e) = t(\overline{e})$ and $t(e) = i(\overline{e})$. In this article, we assume that there are no loops in $E(\Gamma)$, i.e., for any $e \in E(\Gamma)$, $i(e) \neq t(e)$, and Γ is connected and finite, i.e., $V(\Gamma)$ and $E(\Gamma)$ are finite sets. Put the subset of all out-going edges as

$$E_p(\Gamma)(= E_p) = \{e \in E(\Gamma) \mid i(e) = p\} \subset E(\Gamma).$$

An abstract graph Γ is called an *m-valent graph* if $|E_p| = m$ for all $p \in V$, where the symbol $|X|$ represents the cardinality of a finite set X.

Let Γ be an m-valent graph. In order to define a GKM graph, we need to introduce a label $\alpha : E \to H^2(BT;\mathbb{R})$ on edges of Γ, often denoted by (Γ, α), where BT is a classifying space of an n-dimensional torus T. Because the cohomology ring of BT is isomorphic to the polynomial ring

$$H^*(BT) \simeq \mathbb{R}[\alpha_1, \ldots, \alpha_n],$$

with degree two generators α_i $(i = 1, \ldots, n)$, its degree 2 part $H^2(BT)$ is isomorphic to $\mathbb{R}^n$. Take any function $\alpha : E(\Gamma) \to H^2(BT)$ and set

$$\alpha(E_p) = \{\alpha(e) \mid e \in E_p\} \subset H^2(BT).$$

We call a function $\alpha : E \to H^2(BT^n) \setminus \{0\}$ for $n \leq m$ an *axial function* on Γ if it satisfies the following three conditions:

- **(1):** $\alpha(e) = -\alpha(\overline{e})$;
- **(2):** for each vertex $p \in V$, the set $\alpha(E_p)$ spans $H^2(BT^n)$ and it is *pairwise linearly independent*, i.e., each pair of elements in $\alpha(E_p)$ is linearly independent in $H^2(BT)$;
- **(3):** for each edge $e \in E$, there exists a bijective map $\nabla_e : E_{i(e)} \to E_{t(e)}$ such that
  - (1) $\nabla_{\overline{e}} = \nabla_e^{-1}$,
  - (2) $\nabla_e(e) = \overline{e}$, and

(3) for each $e' \in E_{i(e)}$, $\alpha(\nabla_e(e')) - \alpha(e') \equiv 0 \mod \alpha(e) \in H^2(BT)$; this equation is called a *congruence relation*.

The collection $\nabla = \{\nabla_e \mid e \in E\}$ is called a *connection* on the labelled graph (Γ, α); we denote the labelled graph with connection as (Γ, α, ∇).

DEFINITION 4.1 (GKM graph [**35**]). If an m-valent graph Γ is labeled by an axial function $\alpha : E(\Gamma) \to H^2(BT^n)$ for some $n \leq m$, then such labeled graph is said to be an (abstract) *GKM graph* (see [**35**]), denoted as (Γ, α, ∇) or (Γ, α) (if the connection ∇ is obviously determined).

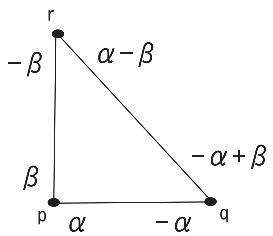

FIGURE 1. An example of a GKM graph, where $\alpha,\ \beta$ are generators of $H^2(BT^2)$.

The following combinatorial fact says that connections of some of GKM graphs are uniquely determined (see [**35, 50**]).

PROPOSITION 4.2. *Let (Γ, α) be a GKM graph. If $\alpha(E_p)$ is three independent (i.e., each three tuple of elements in $\alpha(E_p)$ is linearly independent) for all vertices $p \in V$, then the connection ∇ on (Γ, α) is unique.*

REMARK 4.3. In the paper [**50**], Maeda-Masuda-Panov introduce the notion of a *torus graph* which is the GKM graph analogue for torus manifolds (see Example 2.5).

For the abstract GKM graph (Γ, α), we can define its "equivariant cohomolgy" motivated by the GKM theorem as follows:

$$H^*(\Gamma, \alpha) = \left\{ f : V \to H^*(BT) \;\middle|\; f(p) - f(q) \equiv 0 \mod \alpha(pq) \right\}$$

By using the $H^*(BT)$-algebra structure on $\oplus_{p \in V} H^*(BT)$, $H^*(\Gamma, \alpha)$ may be regarded as an $H^*(BT)$-algebra; in particular, $H^*(\Gamma, \alpha)$ has a ring structure. Note that this does not depend on taking a connection on (Γ, α).

4.2. GKM graphs induced from GKM manifolds. Let M be a $2m$-dimensional GKM manifold with T^n-action in the sense of Definition 2.1, i.e., there is an invariant almost complex structure, and we fix the invariant almost complex structure. Then, the GKM graph $(\Gamma_M, \alpha_M, \nabla_M)$ is defined by the following way. The abstract graph Γ_M is the orbit space of one-skeleton of M. Namely, the set of vertices V is the set of fixed points M^T, and an edge connecting $p, q \in V$ is an embedding 2-sphere connecting $p, q \in M^T$. The axial function $\alpha_M : E \to H^2(BT^n)$ is defined by using the tangential representations around fixed points. More precisely, using an invariant almost complex structure on M and the differentiable slice theorem, every tangent space T_pM on $p \in M^T$ decomposes into the following irreducible complex T^n-representation spaces:

$$T_pM = \oplus_{i=1}^m V(\alpha_{i,p}), \tag{4.1}$$

where $V(\alpha_{i,p})$ is the complex one-dimensional representation space with $\alpha_{i,p} \in \mathrm{Hom}(T^n, S^1) \simeq \mathbb{Z}^n \subset \mathbb{R}^n = H^2(BT^n)$. From this fact, we also know that the graph Γ_M is an m-valent graph. Now $V(\alpha_{i,p})$ may be regarded as the complex line bundle of the embedded 2-sphere $e_{i,p}$ (this 2-sphere is equivariantly diffeomorphic to $\mathbb{C}P^1$ with the standard T^1-action, see Example 2.3). Therefore, for $E_p = \{e_{1,p}, \ldots, e_{m,p}\}$, the axial function is defined as $\alpha_M(e_{i,p}) = \alpha_{i,p} \in H^2(BT^n)$. The connection ∇_M is determined by the splitting of the restricted tangent bundle TM of M

to the embedded 2-sphere $e \in E$. More precisely, because e is isomorphic to $\mathbb{C}P^1$, its restricted tangent bundle $TM|_e$ split into the equivariant complex line bundles:

$$TM|_e = \oplus_{i=1}^m \mathbb{L}_{i,e},$$

where $\mathbb{L}_{i,e}$ is the complex line bundle over $\mathbb{C}P^1$ for all $i = 1, \ldots, m$. Recall that the complex line bundle $\mathbb{L}$ over $\mathbb{C}P^1$ can be classified by its 1st Chern class $c_1(\mathbb{L}) \in \mathbb{Z} \subset H^2(\mathbb{C}P^1; \mathbb{R})$. Therefore, the complex line bundle $\mathbb{L}_{i,e}$ is isomorphic to

$$\mathbb{L}_{i,e} \equiv S^3 \times_{S^1} \mathbb{C}_{\rho_i},$$

where S^1 acts on $S^3 \subset \mathbb{C}^2$ by the scaler multiplication and on $\mathbb{C}_{\rho_i}$ by the representation $\rho_i : S^1 \to S^1$. Note that this representation is determined by $\rho_i(z) = z^{a_i}$ for the integer $a_i = c_1(\mathbb{L}_{i,e})$. Then, each line bundle $\mathbb{L}_{i,e}$ satisfies that $\mathbb{L}_{i,e}|_p = V(\alpha_{i,p})$ and $\mathbb{L}_{i,e}|_q = V(\alpha_{i,q})$ for two fixed point $p, q \in V$ connected by the 2-sphere $e \in E$. Therefore, the bijective map $(\nabla_M)_e : E_p \to E_q$ is defined by $(\nabla_M)_e(e_{i,p}) = e_{i,q}$. Consequently, by definition of GKM manifold, $(\Gamma_M, \alpha_M, \nabla_M)$ is a GKM graph. This labeled graph $(\Gamma_M, \alpha_M, \nabla_M)$ is called an *induced GKM graph* from a GKM manifold M.

Example 4.4 (symplectic toric manifolds and their moment graphs). The induced GKM graph from $\mathbb{C}P^2$ is nothing but the GKM graph in Figure 1 (also see Example 2.3). More generally, for a symplectic toric manifold with the Hamiltonian torus action, the moment graph is the GKM graph if we attach the label on each edge by the tangential representation (see [**35**]).

5. Combinatorial formula of equivariant cohomology ring

In Section 4, we introduce the equivariant cohomology $H^*(\Gamma, \alpha)$ of a GKM graph (Γ, α). However, the definition of $H^*(\Gamma, \alpha)$, i.e., the GKM description, does not say anything about the generators and relations as a ring or an algebra. So, for given (Γ, α), finding generators and relations of $H^*(\Gamma, \alpha)$ is the natural question. In this section, we introduce the generators and relations of $H^*(\Gamma, \alpha)$ of a GKM graph (Γ, α) induced from a complete non-singular projective toric variety (also see [**50**] for more general cases, i.e., the torus manifolds with $H^{odd}(M) = 0$). This theorem may be regarded as the GKM theoretical interpretation of the well-known results in toric geometry: the equivariant cohomology ring of complete non-singular projective toric varieties is isomorphic to the face ring of its orbit polytope, proved by Danilov-Jurkiewicz (see [**23**]).

In this section, we assume that Γ is an n-valent graph and the axial function is $\alpha : E(\Gamma) \to H^2(BT^n)$. In this case, by Proposition 4.2, the connection ∇ is uniquely determined.

5.1. Thom class of a GKM subgraph. Let (Γ, α) be a GKM graph. To describe the combinatorial formula of $H^*(\Gamma, \alpha)$, we need the notion of a Thom class of a GKM subgraph of (Γ, α). Let Γ' be an $(n-h)$-valent subgraph of Γ for $0 \le h \le n$, and ∇ be the connection on (Γ, α). We call Γ' a *GKM subgraph*, if Γ' is closed under the connection ∇, i.e., for all $e \in E(\Gamma')$ with $i(e) = p, t(e) = q \in V(\Gamma')$, the restricted bijection $\nabla_e|_{E_p(\Gamma')} : E_p(\Gamma') \to E_q(\Gamma')$ is well-defined. In this case, the restricted labeled graph, say $(\Gamma', \alpha_{\Gamma'})$, is again a GKM graph. Put $N_p(\Gamma') = E_p(\Gamma) \setminus E_p(\Gamma')$, i.e., the set of all normal edges of Γ' on $p \in V(\Gamma')$. Because Γ' is an $(n-h)$-valent graph, $|N_p(\Gamma')| = h$. Then, we define the function $\tau' : V(\Gamma) \to H^{2h}(BT)$ as follows:

$$\tau'(p) = \begin{cases} \prod_{e \in N_p(\Gamma')} \alpha(e) & p \in V(\Gamma') \\ 0 & p \notin V(\Gamma') \end{cases}$$

By definition of GKM subgraph, it is easy to check that $\tau' \in H^{2h}(\Gamma, \alpha)$. We call this element τ' a *Thom class* of Γ'. Figure 2 shows examples of Thom classes of GKM subgraphs in Figure 1. Note that we formally define $\tau_\Gamma = 1 \in H^0(\Gamma, \alpha)$, i.e., $\tau_\Gamma(p) = 1$ for all $p \in V(\Gamma)$, and $\tau_\emptyset = 0 \in H^0(\Gamma, \alpha)$.

Remark 5.1. From geometric point of view, the equivariant Thom class of a codimension $2h$ GKM submanifold X of a GKM manifold M with an almost complex structure can be defined by the following way (see e.g. [**35, 51**]). Let ν be the normal bundle of X and $Th(\nu)$ be its Thom space, i.e, $Th(\nu) = M/(D(\nu))^c$ is the collapsing space of M on $(D(\nu))^c$, where $D(\nu)^c$ is the complement of the unit disk bundle of ν embedded into M. Because ν has the induced orientation

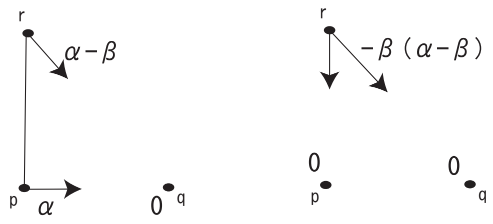

FIGURE 2. Thom classes of some torus subgraphs of torus graphs in Figure 1.

from the almost complex structure (in fact, this becomes a complex h-dimensional vector bundle), there is the following isomorphism, called the Thom isomorphism:

$$H_T^*(X) \to H_T^{*+2h}(Th(\nu)).$$

On the other hand, there is the induced homomorphism $H^*(Th(\nu)) \to H_T^*(M)$ from the collapsing map $M \to Th(\nu)$. Therefore, by taking the composition of these homomorphisms, we have the following homomorphism:

$$\varphi_X : H_T^0(X) \to H_T^{2h}(M).$$

Then, we can define the equivariant Thom class of the codimension $2h$ GKM submanifold X by $\varphi_X(1) = \tau_X \in H^{2h}(M)$. Note that $\iota_X^*(\tau_X) = e^T(\nu)(= c_h^T(\nu)) \in H^{2h}(X)$, where $\iota_X^* : H_T^*(M) \to H_T^*(X)$ is the induced homomorphism from the inclusion $\iota_X : X \to M$ and $e^T(\nu)$ (resp. $c_h^T(\nu)$) is the equivariant Euler class (resp. top Chern class) of ν. Because of the definition of a GKM graph, the GKM subgraph Γ' is induced from some GKM submanifold X; therefore, the Thom class τ' of Γ' is the combinatorial interpretation of the Thom class τ_X of X.

5.2. The combinatorial formula of $H^*(\Gamma, \alpha)$ of some class of GKM graphs. Let (Γ, α) be a GKM graph which satisfies the following conditions: Γ is the one-skelton of an n-dimensional simple convex polytope P^n, i.e., an n-dimensional convex polytope whose vertices consist of the intersection of exactly n codimension one faces (note that this is an n-valent graph); the axial function is $\alpha : E(\Gamma) \to H^2(BT^n)$; and the set $\{\alpha(e) \mid e \in E_p(\Gamma)\}$ spans $H^2(BT^n)$ for all vertices $p \in V(\Gamma)$. In this case, we can find the generators and relations of $H^*(\Gamma, \alpha)$. Let $\mathbb{R}[\tau_H \mid H \subset \Gamma]$ be the polynomial ring generated by Thom classes of all $(n-1)$-valent GKM subgraphs $H \subset \Gamma$. Assume that $\deg \tau_H = 2$. Then, the following theorem holds.

THEOREM 5.2. *Let (Γ, α) be a GKM graph with the condition as above. Then, its equivariant cohomology $H^*(\Gamma, \alpha)$ is isomorphic to the following ring as the graded ring:*

$$\mathbb{R}[\Gamma, \alpha] := \mathbb{R}[\tau_H \mid H \subset \Gamma]/\langle \tau_J \tau_K | K \cap J = \emptyset \rangle. \quad (5.1)$$

Note that $\mathbb{R}[\Gamma, \alpha]$ is isomorphic to the face ring of P^n. The ring $\mathbb{R}[\Gamma, \alpha]$ also has the structure of a graded $H^*(BT)$-algebra by the following fact (also see [**51**]): for all $\alpha \in H^2(BT)$, there exist real numbers $k_1, \ldots, k_m$ such that the following map induces an injective homomorphism $\pi^* : H^*(BT) \to \mathbb{R}[\Gamma, \alpha]$:

$$\pi^* : \alpha \mapsto \alpha (:= \alpha \tau_\Gamma) = \sum_{i=1}^{m} k_i \tau_i$$

where m is the number of all $(n-1)$-valent GKM subgraphs $H_1, \ldots, H_m$ in (Γ, α), τ_i is the Thom class of H_i, for all $i = 1, \ldots, m$ and $\tau_\Gamma = 1 \in H^0(\Gamma, \alpha)$. We also note the following fact:

PROPOSITION 5.3. *The isomorphism $H^*(\Gamma, \alpha) \simeq \mathbb{R}[\Gamma, \alpha]$ in Theorem 5.2 is also an $H^*(BT)$-algebra isomorphism.*

The above facts recover the well-known following fact by Danilov-Jurkiewicz (see e.g. [**52**])

COROLLARY 5.4. *Let M be a projective toric manifold. Then, $H_T^*(M) \simeq \mathbb{R}[\Gamma_M, \alpha_M]$, where (Γ_M, α_M) is the induced GKM graph from M.*

EXAMPLE 5.5. Let $(\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2})$ be the GKM graph induced from the T^2-action on $\mathbb{C}P^2$ (see the graph in Figure 1). Then, the codimension-1 graphs of $\Gamma_{\mathbb{C}P^2}$ are nothing but the three edges pq, pr, qr. It is easy to check that $pq \cap pr \cap qr = \emptyset$. Therefore, if we put these Thom classes as τ_{pq}, τ_{pr}, τ_{qr}, it follows from Theorem 5.2 and Corollary 5.4 that

$$H_T^*(\mathbb{C}P^2; \mathbb{R}) \simeq H^*(\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2}) \simeq \mathbb{R}[\tau_{pq}, \tau_{pr}, \tau_{qr}]/\langle \tau_{pq}\tau_{pr}\tau_{qr} \rangle.$$

Moreover, by definition of Thom classes, the $H^*(BT)$-algebra structure $\pi^* : H^*(BT; \mathbb{R}) \to H_T^*(\mathbb{C}P^2; \mathbb{R})$ is given by

$$\pi^*(\alpha) = \tau_{pr} - \tau_{qr}; \quad \pi^*(\beta) = \tau_{pq} - \tau_{qr}.$$

In addition, because $H^{odd}(\mathbb{C}P^2) = 0$, the ordinary cohomology $H^*(\mathbb{C}P^2)$ is isomorphic to $H_T^*(\mathbb{C}P^2)/\mathrm{Im}\pi^{>0}$. Therefore, we also know the following well-known fact from the above computations:

$$H^*(\mathbb{C}P^2; \mathbb{R}) \simeq H_T^*(\mathbb{C}P^2; \mathbb{R})/\langle \alpha, \beta \rangle \simeq \mathbb{R}[\tau_{pq}]/\langle \tau_{pq}^3 \rangle.$$

REMARK 5.6. The above fact also holds for the integer coefficient.

For several classes of GKM graphs (e.g., induced from flag manifolds), the ring structures of their equivariant cohomology are studied by e.g. [**20, 23, 55**]. In particular, Knutson and Tao [**46**] apply the GKM theory to describe the equivariant cohomology of the flag manifolds; this may be the first and the most notable application of GKM theory to the flag manifolds.

Moreover, for the torus manifolds with $H^{odd}(M) = 0$, their equivariant cohomology ring (over integer coefficient) is completely determined by using the combinatorial structure of their GKM graphs (called torus graphs, see [**50**]).

We also remark that we do not need to use the connection to compute the equivariant cohomology of GKM graphs (see the definition of $H^*(\Gamma, \alpha)$).

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