Limits of The Burnside Rings and Their Relations

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In Memory of Erkki Laitinen

(April, 1955, Helsinki – August, 1996, Higashihiroshima)

On Occasion of 20th Anniversary of His Death

Problem

$\mathbf{G}$: finite group $\quad \mathbf{V}$: $\mathbb{R}[\mathbf{G}]$-module

$\mathbf{V}^\bullet = \mathbf{S}(\mathbb{R} \oplus \mathbf{V})$: one point compactification of $\mathbf{V}$

$\mathcal{F}$: set of subgroups of $\mathbf{G}$

$\{\mathbf{f_H}\}_{\mathbf{H} \in \mathcal{F}}$ where $\mathbf{f_H} : \mathbf{V}^\bullet \to \mathbf{V}^\bullet$ is $\mathbf{H}$-map

Prob (Globalization)

$\exists_{?}$ $\mathbf{G}$-map $\mathbf{f_G} : \mathbf{V}^\bullet \to \mathbf{V}^\bullet$ such that

$$\mathbf{f_G} \sim_{\mathbf{H}\text{-ht}} \mathbf{f_H} \quad (\forall\ \mathbf{H} \in \mathcal{F}) \tag{1.1}$$

Homotopy Classes of Eqv. Maps

Def (G-homotopy set)

$[V^\bullet, V^\bullet]^G$: the set of G-ht classes of G-maps $V^\bullet \to V^\bullet$

$\mathrm{res}^G_H : [V^\bullet, V^\bullet]^G \to [V^\bullet, V^\bullet]^H$ restriction map

Rem

$$f_G \sim_{H\text{-ht}} f_H \quad (\forall\ H \in \mathcal{F}) \Longleftrightarrow \mathrm{res}^G_H[f_G] = [f_H]\ (\forall\ H \in \mathcal{F})$$

where $[f_G] \in [V^\bullet, V^\bullet]^G, \quad [f_H] \in [V^\bullet, V^\bullet]^H$

Burnside Ring of G

$$A(G) = \{\, [X_1] - [X_2] \mid X_i : \text{ finite } G\text{-sets} \,\}$$

$$= \{\, [X] \mid X : \text{ finite } G\text{-CW} \,\}$$

$$[X] = [Y] \overset{\text{def}}{\Longleftrightarrow} \chi(X^H) = \chi(Y^H) \ (\forall\, H \leq G)$$

G-Homotopy Set and A(G)

Lem (Petrie, tom Dieck, etc.)

If $\mathbf{V} \supset \mathbb{R}[\mathbf{G}] \oplus \mathbb{R}[\mathbf{G}]$ *then* $[\mathbf{V}^\bullet, \mathbf{V}^\bullet]^\mathbf{G} \cong \mathbf{A}(\mathbf{G})$

$$\begin{array}{ccc} [\mathbf{V}^\bullet, \mathbf{V}^\bullet]^\mathbf{G} & \xhookrightarrow{\prod \mathbf{deg_H}} & \prod_{\mathbf{H} \leq \mathbf{G}} \mathbb{Z} \\ \cong \downarrow & \nearrow \prod \chi_\mathbf{H} & \\ \mathbf{A}(\mathbf{G}) & & \end{array}$$

Problem and Burnside Ring

$$\mathrm{res}^{G}_{\mathcal{F}} := \prod_{H\in\mathcal{F}} \mathrm{res}^{G}_{H} : A(G) \to \prod_{H\in\mathcal{F}} A(H)$$

Rem

(For $\{f_H\}_{H\in\mathcal{F}}$) $\exists\ f_G \iff ([f_H])_{H\in\mathcal{F}} \in \mathrm{Im}\left(\mathrm{res}^{G}_{\mathcal{F}}\right)$

Def

For $\{f_H \mid H \in \mathcal{F}\}$,

$$\sigma(\{f_H\}) := [\,([f_H])_{H\in\mathcal{F}}\,]$$
$$\in \mathrm{Coker}\left(\mathrm{res}^{G}_{\mathcal{F}} : A(G) \to \prod_{H\in\mathcal{F}} A(H)\right)$$

Obstruction Group

Def

$\mathsf{Ob}(G,\mathcal{F}) = \mathrm{Coker}\left(\mathrm{res}^G_{\mathcal{F}} : A(G) \to \prod_{H\in\mathcal{F}} A(H)\right)$

$\sigma(\{f_H\}) = [\,([f_H])_{H\in\mathcal{F}}\,] \in \mathsf{Ob}(G,\mathcal{F})$

Rem

(For $\{f_H\}_{H\in\mathcal{F}}$) $\quad \exists\, f_G \iff \sigma(\{f_H\}) = 0$ in $\mathsf{Ob}(G,\mathcal{F})$

$B(G,\mathcal{F}) := \mathrm{res}^G_{\mathcal{F}}(A(G)) \quad (\subset \prod_{H\in\mathcal{F}} A(H))$

Therefore $\mathsf{Ob}(G,\mathcal{F}) = \left(\prod_{H\in\mathcal{F}} A(H)\right)/B(G,\mathcal{F})$

Inverse Limit of $A(-)$

$\mathcal{F}$: lower-closed, conjugation-invariant

$$H \in \mathcal{F} \Longrightarrow \mathcal{S}(H) \subset \mathcal{F}$$

$$g \in G,\ H \in \mathcal{F} \Longrightarrow gHg^{-1} \in \mathcal{F}$$

Def (Inverse Limit)

$\lim\limits_{\longleftarrow \mathcal{F}} A^*$ is the set of all $(a_H)_{H\in\mathcal{F}} \in \prod_{H\in\mathcal{F}} A(H)$ such that

$$\begin{cases} \mathrm{res}^K_H a_K = a_H & (H \le K \in \mathcal{F}) \\ c(g)^* a_{gHg^{-1}} = a_H & (H \in \mathcal{F},\ g \in G) \end{cases}$$

where $c(g) : H \to gHg^{-1}$; $c(g)(h) = ghg^{-1}$

Basic Facts

$$\mathrm{res}^G_{\mathcal{F}} : A(G) \to \prod_{H\in\mathcal{F}} A(H), \quad B(G,\mathcal{F}) = \mathrm{res}^G_{\mathcal{F}}(A(G))$$

$$\mathrm{rank}_{\mathbb{Z}}\, B(G,\mathcal{F}) = |\mathcal{F}/G\text{-conj}|$$

$$Ob(G,\mathcal{F}) = \left(\prod_{H\in\mathcal{F}} A(H)\right) \Big/ B(G,\mathcal{F})$$

Prop

(1) $B(G,\mathcal{F}) \subset \varprojlim_{\mathcal{F}} A^* \subset \prod_{H\in\mathcal{F}} A(H)$

(2) $Q_A(G,\mathcal{F}) = \varprojlim_{\mathcal{F}} A^*/B(G,\mathcal{F})$ *is finite*

(3) $\prod_{H\in\mathcal{F}} A(H)/\varprojlim_{\mathcal{F}} A^*$ *is* $\mathbb{Z}$*-free*

(4) $Ob(G,\mathcal{F}) \cong Q_A(G,\mathcal{F}) \oplus \left(\prod_{H\in\mathcal{F}} A(H)/\varprojlim_{\mathcal{F}} A^*\right)$

Basic Facts

Prop

$$\mathrm{rank}_{\mathbb{Z}}\, \mathsf{Ob}(G,\mathcal{F}) = \left(\sum_{H\in\mathcal{F}} |\mathcal{S}(H)/H\text{-}\mathit{conj}|\right) - |\mathcal{F}/G\text{-}\mathit{conj}|$$

$$B(G,\mathcal{F}) = \mathrm{Im}[\mathrm{res}^G_{\mathcal{F}} : A(G) \to \prod_{H\in\mathcal{F}} A(H)]$$

$$\overline{B(G,\mathcal{F})} = \left\{ x \in \prod_{H\in\mathcal{F}} A(H) \;\middle|\; mx \in B(G,\mathcal{F})\ (\exists\, m \in \mathbb{N}) \right\}$$

Prop

$$\overline{B(G,\mathcal{F})} = \varprojlim_{\mathcal{F}} A^*$$

Prop

$$Q_A(G,\mathcal{F}) = \overline{B(G,\mathcal{F})}/B(G,\mathcal{F})$$

Examples

$\mathcal{F}_G = \{H \mid H < G\}$

Ex 1. $G = C_p$ (p : prime) Then $Q_A(G, \mathcal{F}_G) = 0$

Ex 2. $G = C_p \times C_p$ (p: prime) Then $Q_A(G, \mathcal{F}_G) \cong \mathbb{Z}_p$

Ex 3. $G = C_{p^n}$ (p: prime) Then $Q_A(G, \mathcal{F}_G) \cong \mathbb{Z}_p{}^{\oplus n-1}$

Ex 4. $G = C_p \times C_q$ (p, q: dist. primes)

Then $Q_A(G, \mathcal{F}_G) = 0$

Ex 5. $G = A_4$ Then $Q_A(G, \mathcal{F}_G) = 0$

(Ex 1 – 5 are computed by Y. Hara–M.)

Ex 6 (M. Sugimura). $G = A_5$ Then $Q_A(G, \mathcal{F}_G) = 0$

Nilpotent Case

Thm (Hara–M.)

$\mathbf{G}$: *nilpotent group* *Then*

$$Q_A(G, \mathcal{F}_G) = 0 \iff \begin{cases} G \text{ is cyclic group such that} \\ |G| = p_1 \cdots p_m \\ \text{for some distinct primes } p_i \end{cases}$$

k_G, $G^{\{p\}}$ and G^{nil}

Oliver's number k_G (≥ 1): the product of primes such that

$$p \text{ (prime) divides } k_G \iff \exists\, N \triangleleft G \text{ with } |G/N| = p$$

Dress' subgroup G^p (p prime) : smallest $N \trianglelefteq G$ such that

$$|G/N| \text{ is a power of } p$$

G^{nil} : smallest $N \trianglelefteq G$ such that G/N is nilpotent (E. Laitinen–M.)

Rem

(1) $G^{nil} = \bigcap_p G^p$

(2) $k_G = \displaystyle\prod_{p \text{ prime: } G^p \neq G} p$

Exponent Theorem of $Q_A(-)$

$\mathcal{F}_G = \{H \mid H < G\}$, $\overline{G} = G/G^{\mathrm{nil}}$, $\mathcal{F}_{\overline{G}} = \{H \mid H < \overline{G}\}$

$k_G = \prod p$ where p ranges over primes such that $G^p \neq G$

Thm (Exponent of $Q_A(G, \mathcal{F}_G)$)

$$k_G \, Q_A(G, \mathcal{F}_G) = 0$$

Cor

If G *is* p*-group* *Then* $p \, Q_A(G, \mathcal{F}_G) = 0$

Cor

If G *is* p*-group* *Then* $Q_A(G, \mathcal{F}_G)$ *is elementary abelian* p*-group*

Structure Theorem of $Q_A(-)$

Thm (Structure of $Q_A(G, \mathcal{F}_G)$)

$$Q_A(G, \mathcal{F}_G) \cong \prod_{p|k_G} Q_A(G/G^p, \mathcal{F}_{G/G^p}) \cong Q_A(\overline{G}, \mathcal{F}_{\overline{G}})$$

Cor

$$Q_A(G, \mathcal{F}_G) = 0 \iff \begin{cases} G/G^{\mathrm{nil}} \text{ is a cyclic group} \\ \text{of order } k_G \ (= p_1 \cdots p_m, \\ p_i \text{ distinct primes}) \end{cases}$$

Cor

$G = A_n, S_n$ *Then* $Q_A(G, \mathcal{F}_G) = 0$

Case of p-Group

$\mathcal{F}_G = \mathcal{S}(G) \setminus \{G\}$

p: prime **G**: **p**-group

$G_0 = \bigcap_L L$ (**L** ranges all maximal subgroups of **G**)

$$G/G_0 \cong C_p \times \cdots \times C_p$$

$N \lhd G$

For $N \leq H \leq G$, we have $\mathrm{fix}^H_{H/N} : A(H) \to A(H/N)$;

$$\mathrm{fix}^H_{H/N}([X]) = [X^N]$$

$\{\mathrm{fix}^H_{H/N}\}$ induces

$$\mathrm{fix}^{\mathcal{F}_G}_{\mathcal{F}_{G/N}} : Q_A(G, \mathcal{F}_G) \to Q_A(G/N, \mathcal{F}_{G/N})$$

Case of p-Group

Prop 1

$\mathbf{p}$: *prime*, $\mathbf{G}$: $\mathbf{p}$-*group*, $\mathbf{N} \trianglelefteq \mathbf{G}$ *with* $\mathbf{N} \subset \mathbf{G_0}$ *Then*

$$\mathrm{fix}^{\mathcal{F}_G}_{\mathcal{F}_{G/N}} : Q_A(G, \mathcal{F}_G) \to Q_A(G/N, \mathcal{F}_{G/N}) \text{ is surjective}$$

Prop 2 (Hara–M.)

$\mathbf{p}$: *prime*, $\mathbf{n} \geq 2$, $\mathbf{G} = \mathbf{C_p} \times \cdots \times \mathbf{C_p}$ (*$\mathbf{n}$-fold*) *Then*

$$Q_A(G, \mathcal{F}_G) \neq 0$$

Thm 3 (Hara–M.)

$\mathbf{p}$: *prime* $\mathbf{G}$: *nontriv* $\mathbf{p}$-*group* *Then*

$$Q_A(G, \mathcal{F}_G) = 0 \iff |G| = p$$

Computation Results for $G = C_{p^m} \times C_{p^n}$

p: prime, G: p-group Then $Q_A(G, \mathcal{F}_G) \cong \mathbb{Z}_p^k$

Def

$q(m, n)$ is defined by $Q_A(G, \mathcal{F}_G) = {\mathbb{Z}_p}^{q(m,n)}$ for $G = C_{p^m} \times C_{p^n}$

Thm (Y. Hara–M.)

$q(m, 0) = m - 1$ *for* $m \geq 1$

Thm (M. Sugimura)

$q(m, 1) = 1 + (m - 1)\,p$ *for* $m \geq 1$

Thm (M. Sugimura)

$q(m, 2) = p + 1 + (m - 1)\,(p^2 + 1)$ *for* $m \geq 2$

Computation Results for $G = C_{p^m} \times C_{p^n}$

Thm

If $m \geq n \geq 3$ *then*

$$q(m,n) = (p^n + 2p^{n-1}) + \sum_{k=1}^{n-3}(2k+1)p^{n-k-1} + ((2n-4)p + (2n-2)) + (m-n)\left(\sum_{k=0}^{n} p^k - p^{n-1}\right)$$

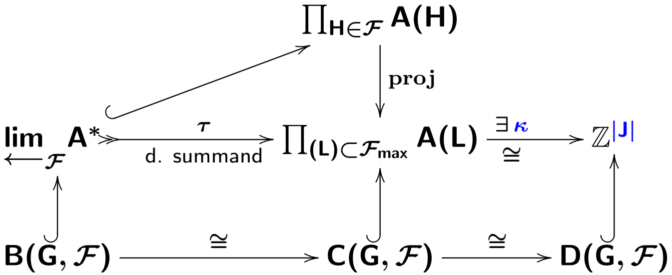
Rem

$$q(1,1) = 1, \quad q(2,2) = p^2 + p + 2$$
$$q(3,3) = p^3 + 2p^2 + 2p + 4$$
$$q(4,4) = p^4 + 2p^3 + 3p^2 + 4p + 6$$

Strategy of Computation of $Q_A(G, \mathcal{F})$

F: $\mathbb{Z}$-free module, **D**: submodule of **F**

Closure $\overline{D}$ of **D** in **F**, $\overline{D} := \{x \in F \mid k\,x \in D \text{ for some } k \in \mathbb{N}\}$



where $C(G, \mathcal{F}) = \tau(B(G, \mathcal{F}))$, $D(G, \mathcal{F}) = \kappa(C(G, \mathcal{F}))$

$$Q_A(G, \mathcal{F}) = \overline{B(G, \mathcal{F})}/B(G, \mathcal{F}) \cong \overline{D(G, \mathcal{F})}/D(G, \mathcal{F})$$

Strategy of Computation of $Q_A(G, \mathcal{F})$

$\mathcal{F}_{max}$: set of maximal elements of $\mathcal{F}$

τ: composition : $\varprojlim_{\mathcal{F}} A^* \hookrightarrow \prod_{H \in \mathcal{F}} A(H) \to \prod_{(L) \subset \mathcal{F}_{max}} A(L)$

$\kappa = \prod_{(L) \subset \mathcal{F}_{max}} \kappa_L$; $\kappa_L : A(L) \to \prod_{(K) \subset \mathcal{S}(L)} \mathbb{Z}$

$\kappa_L = \prod_{(K)_L \subset \mathcal{S}(L)} \kappa_{L,K}$; $\kappa_{L,K} : A(L) \to \mathbb{Z}$

$A(L)$ has basis $\{[L/K] \mid (K)_L \subset \mathcal{S}(L)\}$

Each $x \in A(L)$ has form $x = \sum_{(K)_L \subset \mathcal{S}(L)} \kappa_{L,K}(x)[L/K]$

$J := \{((L), (K)_L) \mid (L) \subset \mathcal{F}_{max}, \ (K)_L \subset \mathcal{S}(L)\}$

Strategy of Computation of $Q_A(G, \mathcal{F}_G)$

Basis of $\mathbf{A(G)}$: $\{[\mathbf{G/H}] \mid (\mathbf{H}) \subset \mathcal{S}(\mathbf{G})\}$

$$\mathbf{I} := \mathcal{S}(\mathbf{G})/\mathbf{G}\text{-conj}$$

$$\mathrm{res}^{\mathbf{G}}_{\mathcal{F}_{\max}} := \mathrm{proj} \circ \mathrm{res}^{\mathbf{G}}_{\mathcal{F}} : \mathbf{A(G)} \to \prod_{(\mathbf{L}) \subset \mathcal{F}_{\max}} \mathbf{A(L)}$$

Def

$\mathbf{M} = (a_{ij})_{i\in I,\ j\in J}$: matrix presentation of $\mathrm{res}^{\mathbf{G}}_{\mathcal{F}_{\max}}$

$$a_{ij} := \kappa_{\mathbf{L},\mathbf{K}}(\mathrm{res}^{\mathbf{G}}_{\mathbf{L}}([\mathbf{G/H}])) \quad (\text{for } \mathbf{i} = (\mathbf{H}),\ \mathbf{j} = ((\mathbf{L}), (\mathbf{K})_{\mathbf{L}}))$$

(coefficient of $[\mathbf{L/K}]$)

Strategy of Computation of $Q_A(G, \mathcal{F}_G)$

For each $\mathbf{i} \in \mathbf{I}$, $\mathbf{a_i := (a_{ij})_{j \in J}} \in \mathbb{Z}^{|J|}$ (row vector of $\mathbf{M}$)

$$\mathbf{D}(\mathbf{G}, \mathcal{F}) = \langle \mathbf{a_i} \mid \mathbf{i} \in \mathbf{I} \rangle_{\mathbb{Z}} \subset \mathbb{Z}^{|J|}$$

$$\overline{\mathbf{D}(\mathbf{G}, \mathcal{F})} = \{\mathbf{x} \in \mathbb{Z}^{|J|} \mid \mathbf{k}\,\mathbf{x} \in \mathbf{D}(\mathbf{G}, \mathcal{F}) \ (\exists\, \mathbf{k} \in \mathbb{N})\}$$

By Elementary Deformations of $\{\mathbf{a_i} \mid \mathbf{i} \in \mathbf{I}\}$,

we can obtain basis $\{\mathbf{b_t} \mid \mathbf{t} \in \mathbf{T}\}$ of $\overline{\mathbf{D}(\mathbf{G}, \mathcal{F})}$ such that

$$\{\mathbf{m_t}\, \mathbf{b_t} \mid \mathbf{t} \in \mathbf{T}\} \text{ is basis of } \mathbf{D}(\mathbf{G}, \mathcal{F}) \quad (\mathbf{m_t} \in \mathbb{N})$$

Then $\mathbf{Q_A}(\mathbf{G}, \mathcal{F}) \cong \overline{\mathbf{D}(\mathbf{G}, \mathcal{F})}/\mathbf{D}(\mathbf{G}, \mathcal{F}) \cong \prod_{\mathbf{t} \in \mathbf{T}} \mathbb{Z}_{\mathbf{m_t}}$

Elementary Deformation of Row Vectors

$\mathbf{R}$: commutative ring $\ni 1$

Set of $\mathbf{R}$-vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$, $\mathcal{B}' = \{\mathbf{b}'_1, \ldots, \mathbf{b}'_m\}$

Each of the following deformations $\mathcal{B} \rightsquigarrow \mathcal{B}'$ is called elementary deformation

1. $\mathbf{b}'_i = \mathbf{b}_i + r\,\mathbf{b}_j$ (for some $r \in \mathbf{R}$, $i \neq j$)
2. $\mathbf{b}'_i = \mathbf{b}_j$ and $\mathbf{b}'_j = \mathbf{b}_i$ (for some $i \neq j$)
3. $\mathbf{b}'_i = r\,\mathbf{b}_i$ (for some $r \in \mathbf{R}^\times$ and i)

"$\rightsquigarrow \cdots \rightsquigarrow$" is written as "$\boldsymbol{\rightsquigarrow}$"

$$\mathcal{B} \boldsymbol{\rightsquigarrow} \mathcal{B}'' \Longrightarrow \langle \mathcal{B} \rangle_R = \langle \mathcal{B}'' \rangle_R$$

Rank of $Q_A(G, \mathcal{F}_G)$ over $\mathbb{Z}_p$

Suppose **G is p-group** (**p** prime)

$M = (a_{ij})_{i \in I, j \in J}$: matrix presentaion of $\mathrm{res}^G_{\mathcal{F}_{max}}$

$$G \neq E \Longrightarrow k_G = p$$

$\varphi : \mathbb{Z} \to \mathbb{Z}_p$ canonical map

Prop

$Q_A(G, \mathcal{F}_G) \cong \mathbb{Z}_p^r$ *with*

$$r = \mathrm{rank}_{\mathbb{Z}} M - \mathrm{rank}_{\mathbb{Z}_p} \varphi M \quad \textit{and} \quad \mathrm{rank}_{\mathbb{Z}} M = |\mathcal{F}/G\text{-}conj|$$

Inverse Limits of M^*

$\mathfrak{S}_G$: the subgroup category of G

$\mathrm{Obj}(\mathfrak{S}_G) = \{H \mid H \le G\}$

$\mathrm{Mor}(\mathfrak{S}_G) = \{(H, g, K) \mid gHg^{-1} \le K\}$ $(H,\ K \le G,\ g \in G)$

$\mathfrak{A}$: the cat of abel groups,

$M^* : \mathfrak{S}_G \to \mathfrak{A}$ contravariant functor

Def (Inverse Limit)

$\varprojlim_{\mathcal{F}} M^* = \{(x_H)\} \subset \prod_{H\in\mathcal{F}} M^*(H)$ such that
$(H, g, K)^* x_K = x_H$ for (H, g, K) with H, $K \in \mathcal{F}$

$$Q_M(G, \mathcal{F}) = \mathrm{Coker}\left(\mathrm{res}^G_{\mathcal{F}} : M^*(G) \to \varprojlim_{\mathcal{F}} M^*\right)$$

Direct Limits of M_*

$M_* : \mathfrak{S}_G \to \mathfrak{A}$ covariant functor

Def (Direct limit)

$\varinjlim_{\mathcal{F}} M_* = \left(\bigoplus_{H \in \mathcal{F}} M_*(H) \right) \Big/ S$, where

$$S = \left\langle a_H - a_K \;\middle|\; \begin{array}{c} a_H \in M_*(H),\ a_K \in M_*(K) \\ (H, g, K)_* a_H = a_K \\ (H, g, K) \text{ with } H,\ K \in \mathcal{F} \end{array} \right\rangle$$

Induction homomorphism $\mathrm{ind}_{\mathcal{F}}^{G} : \varinjlim_{\mathcal{F}} M_* \to M_*(G)$

$$\mathrm{ind}_{\mathcal{F}}^{G} \left(\sum_{H \in \mathcal{F}} [x_H] \right) = \sum_{H \in \mathcal{F}} \mathrm{ind}_{H}^{G} x_H \quad (x_H \in M(H))$$

Homomorphisms from Inv-Lim to Co-Lim

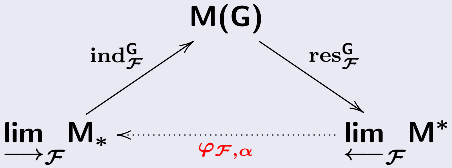
$\mathbf{F} = (\mathbf{F}^*, \mathbf{F}_*)$ Green ring functor on $\mathbf{G}$

$\mathbf{M} = (\mathbf{M}^*, \mathbf{M}_*)$ Mackey functor on $\mathbf{G}$, Green module over $\mathbf{F}$

$\mathbf{F}(\mathbf{G}, \mathcal{F}) := \sum_{\mathbf{H} \in \mathcal{F}} \mathrm{ind}_{\mathbf{H}}^{\mathbf{G}} \mathbf{F}(\mathbf{H})$ $(\subset \mathbf{F}(\mathbf{G}))$

Lem

Each $\alpha = \sum_{\mathbf{H} \in \mathcal{F}} \mathrm{ind}_{\mathbf{H}}^{\mathbf{G}} \mathbf{a}_{\mathbf{H}} \in \mathbf{F}(\mathbf{G}, \mathcal{F})$ $\quad (\mathbf{a}_{\mathbf{H}} \in \mathbf{F}(\mathbf{H}))$
gives $\varphi_{\mathcal{F},\alpha} : \varprojlim_{\mathcal{F}} \mathbf{M}^* \to \varinjlim_{\mathcal{F}} \mathbf{M}_*$
by $\varphi_{\mathcal{F},\alpha}((\mathbf{x}_{\mathbf{H}})_{\mathbf{H} \in \mathcal{F}}) = \sum_{\mathbf{H} \in \mathcal{F}} [\mathbf{a}_{\mathbf{H}} \cdot \mathbf{x}_{\mathbf{H}}]$ *(note* $\mathbf{x}_{\mathbf{H}} \in \mathbf{M}(\mathbf{H})$*)*



$\varphi_{\mathcal{F},\alpha}$, Operation $\star$, and $\mathrm{res}^G_{\mathcal{F}}$

Def

$\alpha\star : \varprojlim_{\mathcal{F}} M^* \to M(G)$ is defined by

$\alpha \star (x_H) = \sum_{H\in\mathcal{F}} \mathrm{ind}_H^G(a_H \cdot x_H) \quad (a_H \in F(H),\ x_H \in M(H))$

Prop (Naturality)

$$\mathrm{res}^G_{\mathcal{F}}(\alpha \star (x_H)_{H\in\mathcal{F}}) = \Big((\mathrm{res}^G_H\alpha)\cdot x_H\Big)_{H\in\mathcal{F}} \quad \textit{in } \varprojlim_{\mathcal{F}} M^*$$

where $\mathrm{res}^G_H\alpha \in F(H)$, $x_H \in M(H)$

Quasi-Commutativity

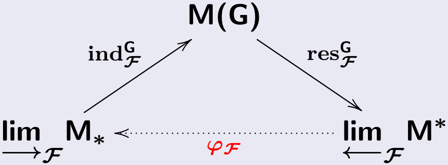
Lem (Quasi-Commutativity)

$\mathbf{M} = (\mathbf{M}^*, \mathbf{M}_*)$, $\mathbf{F} = (\mathbf{F}^*, \mathbf{F}_*)$, $\alpha \in \mathbf{F}(\mathbf{G}, \mathcal{F})$ *as above*, $\mathbf{k} \in \mathbb{N}$

Suppose $\mathrm{res}^{\mathbf{G}}_{\mathbf{H}} \alpha = \mathbf{k}\, 1_{\mathbf{H}}$ *in* $\mathbf{F}(\mathbf{H})$ $\forall\, \mathbf{H} \in \mathcal{F}$

Then $\exists$ *homomorphism* $\varphi_{\mathcal{F}} : \varprojlim_{\mathcal{F}} \mathbf{M}^* \to \varinjlim_{\mathcal{F}} \mathbf{M}_*$ *satisfying*

1. $\mathrm{res}^{\mathbf{G}}_{\mathcal{F}} \circ \mathrm{ind}^{\mathbf{G}}_{\mathcal{F}} \circ \varphi_{\mathcal{F}} = \mathbf{k}\, \mathbf{id}_{\varprojlim_{\mathcal{F}} \mathbf{M}^*}$ *and*
2. $\varphi_{\mathcal{F}} \circ \mathrm{res}^{\mathbf{G}}_{\mathcal{F}} \circ \mathrm{ind}^{\mathbf{G}}_{\mathcal{F}} = \mathbf{k}\, \mathbf{id}_{\varinjlim_{\mathcal{F}} \mathbf{M}_*}$



$\mathcal{F}_G = \{H \mid H < G\}$

Lem (Oliver, Kratzer–Thévenaz)

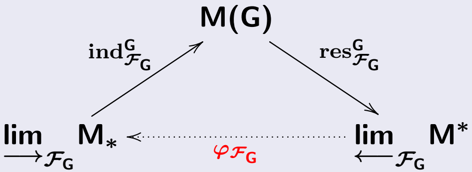
$\exists\ \alpha_G \in A(G, \mathcal{F}_G)$ *such that* $\mathrm{res}^G_H \alpha_G = k_G\, 1_H\ \forall\ H \in \mathcal{F}_G$

Thm (Quasi-Isomorphism)

$\exists$ *homomorphism* $\varphi_{\mathcal{F}_G} : \varprojlim_{\mathcal{F}_G} M^* \to \varinjlim_{\mathcal{F}_G} M_*$ *such that*

(1) $\mathrm{res}^G_{\mathcal{F}_G} \circ \mathrm{ind}^G_{\mathcal{F}_G} \circ \varphi_{\mathcal{F}_G} = k_G\, \mathrm{id}_{\varprojlim_{\mathcal{F}} M^*}$ *and*

(2) $\varphi_{\mathcal{F}_G} \circ \mathrm{res}^G_{\mathcal{F}_G} \circ \mathrm{ind}^G_{\mathcal{F}_G} = k_G\, \mathrm{id}_{\varinjlim_{\mathcal{F}} M_*}$



Exponent of $Q_M(G, \mathcal{F}_G)$

$$M = (M^*, M_*) : \mathfrak{S}_G \to \mathfrak{A} \quad \text{Mackey functor}$$

$$k_G \varprojlim_{\mathcal{F}_G} M^* \subset \mathrm{Im}[\mathrm{res}^G_{\mathcal{F}_G} : M(G) \to \varprojlim_{\mathcal{F}_G} M^*]$$

Thm (Exponent of $Q_M(-)$)

$$k_G\, Q_M(G, \mathcal{F}_G) = 0$$

Cor

$$Q_M(G, \mathcal{F}_G) = \prod_{p \mid k_G} Q_M(G, \mathcal{F}_G)_{(p)}$$

$$p\, Q_M(G, \mathcal{F}_G)_{(p)} = 0$$

Application of Exponent

$\mathbf{X}$: finite CW-comp. with trivial $\mathbf{G}$-action

Prop

1. $k_G \varprojlim_{\mathcal{F}_G} \mathrm{KO}_\bullet(\mathbf{X}) \subset \mathrm{res}^{\mathbf{G}}_{\mathcal{F}_G}(\mathrm{KO}_{\mathbf{G}}(\mathbf{X}))$
2. $k_G \varprojlim_{\mathcal{F}_G} \omega^n_\bullet(\mathbf{X}) \subset \mathrm{res}^{\mathbf{G}}_{\mathcal{F}_G}(\omega^n_{\mathbf{G}}(\mathbf{X}))$

Thm (Case of G-vector bundle)

$\{\xi_H\}_{H\in\mathcal{F}_G}$, ξ_H : *real* $\mathbf{H}$*-vt bdl over* $\mathbf{X}$
Suppose $\{\xi_H\}$ *is compatible w.r.t. restrictions and conjugations*

Then $\exists$ *real* $\mathbf{G}$*-vt bdl* ξ_G *over* $\mathbf{X}$ *and real* $\mathbf{G}$*-module* $\mathbf{V}$ *such that*

$$\xi_G \cong_H {\xi_H}^{\oplus k_G} \oplus \varepsilon_X(\mathbf{V})$$

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