Research on Toric Topology

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Abstract. In this report, we study on toric topology. In Part 1 we classify extended actions of (non-oriented) torus manifolds with codimension 0 and 1 orbits. Moreover we show their topological classification. From this result, we have a negative answer of cohomological rigidity problem for torus manioflds. In Part 2 we study operations and projective bundles of small covers. We prove relations for operations of small covers and a construction theorem of projective bundles of small covers. Finally, in Part 3, we study 6-dimensional quasitoric manifolds.

Keywords: cohomogeneity one action, cohomological rigidity problem, compact group action, complex projective bundle, equivariant surgery, finite group action, homogeneous space, non-singular toric variety, quasitoric manifold, real projective bundle, six-dimensional manifold, small cover, sphere bundle, spin manifold, three-dimensional manifold, three-dimensional simple polytope, torus action, torus manifold, toric topology.

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Contents

Introduction 7

Part 1. Extension problem and cohomological rigidity problem of torus manifolds 9

Chapter 1. Characterization of homogeneous torus manifolds 11
1. Introduction of Chapter 1 11
2. Preliminaries to the characterization 12
2.1. Basic notations and examples 12
2.2. Homogeneous space G/H with finite T-fixed points 13
2.3. Facts from classical Lie theory 14
2.4. Essential isomorphism and a remark for the characterization 14
3. Characterization of homogeneous torus manifolds 14
3.1. Structure of torus manifolds 15
3.2. Characterization of homogeneous torus manifolds 16
4. On other T-manifolds 19

Chapter 2. Classification of quasitoric manifolds with codimension one extended actions 21
1. Introduction of Chapter 2 21
2. Preliminary 23
2.1. Torus manifold 23
2.2. Review of classical Lie theory and the previous chapter 23
2.3. Review of transformation group theory 24
3. Orbits in extended actions 25
3.1. G-orbits on T-fixed points 26
3.2. Differences between quasitoric manifolds and torus manifolds 27
3.3. Singular orbit G/K_1 28
4. Assumptions and structures of two singular orbits 29
5. Tubular neighborhoods of singular orbits 30
5.1. Preparation 30
5.2. Transitive actions on sphere 30
5.3. Slice representations (rough structures) 31
6. Possibility for G, K_1, K_2 32
6.1. Decomposition of G 32
6.2. The cases (3) and (4) 32
6.3. The case (2) 33
6.4. The case (1) and the summary 34
7. Possibility for K and slice representations 34
7.1. Slice representations and K for the case $(1)-(a)$ 34
7.2. Slice representations and K for the case $(1)-(b)$ 36
7.3. Slice representations and K for the case $(2)-(a)$ 37
8. Attaching maps from ∂X_1 to ∂X_2 and constructions of G-manifolds 37
8.1. Preparation 38
8.2. The case $(1)-(a)$ 38
8.3. The case $(1)-(b)$ 40

8.4. The case $(2) - (a)$ 42
9. On moment-angle manifolds 42

Chapter 3. Classification of torus manifolds with codimension one extended actions 45
1. Introduction of Chapter 3 45
2. Review of the previous chapters 47
2.1. Codimension 1 extended G-actions and their singular orbits 47
2.2. Slice representations of G/K_1 47
2.3. Global structures of codimension 1 extended actions 49
3. The case (1): two singular orbits are torus manifolds 50
3.1. Notations and main theorem 50
3.2. Singular isotropy subgroups and images of their slice representations 51
3.3. The case (1)-(a) 51
3.4. The case (1)-(b) 53
3.5. The case (1)-(c) 54
4. Preliminary for the cases (2) and (3) 56
4.1. Structures of G/K_1 and their tubular neighborhoods of the cases (2) and (3) 56
4.2. Important remark for the attaching maps of the cases (2) and (3) 57
5. Main theorem and remarks of the case (2) 57
5.1. Main theorem of the case (2) 58
5.2. Remarks of the case (2) 58
6. The case (2)-(a) 59
6.1. The case (2)-(a)-(i) 60
6.2. The case (2)-(a)-(ii) 61
6.3. The case (2)-(a)-(iii) 63
6.4. The case (2)-(a)-(iv) 64
6.5. Summary of the results from (2)-(a)-(i) to (iv) 65
7. The case (2)-(b) 65
7.1. The case (2)-(b)-(i) 66
7.2. The case (2)-(b)-(ii) 67
7.3. The case (2)-(b)-(iii) 68
7.4. The case (2)-(b)-(iv) 68
7.5. Summary of the results from (2)-(b)-(i) to (iv) 69
8. Main theorem of the case (3) and preparations 69
8.1. Main theorem and notations 69
8.2. Structure of K_2 70
9. The case (3)-(a) 71
9.1. The case $k_2 \geq 3$ 72
9.2. The case $k_2 = 2$ 76
10. The case (3)-(b) 80
10.1. The case (3)-(b)-(i) 80
10.2. The case (3)-(b)-(ii) 82

Chapter 4. Topological classification of torus manifolds which have codimension one extended G-actions 85
1. Introduction of Chapter 4 85
2. Notations 86
3. Topological invariants 87
3.1. Topological invariants of N_1 87
3.2. Topological invariants of N_2 and N_3 89
4. Main theorem and Preliminary 91
5. Proof of the main theorem 92
5.1. CASE 1 (1) : $\deg w = 2$ and $\deg f > 4$ 93
5.2. CASE 1 (2) : $\deg w = 2$, $\deg f = 4$ and $\ell > 1$ 93

5.3. CASE 1 (3) : $\deg w = 2$, $\deg f = 4$ and $\ell = 1$ 93
5.4. CASE 2 (1) : $\deg w > 2$ and $\ell \geq 4$ 93
5.5. CASE 2 (2) : $\deg w > 2$ and $\ell = 2,\ 3$ 94
5.6. CASE 2 (3) : $\deg w > 2$ and $\ell = 1$ 95

Part 2. Operations and projective bundles on small covers 97

Chapter 5. Operations on three dimensional small covers 99
1. Introduction of Part 5 99
2. Basics of small covers 100
2.1. Definition of a small cover 100
2.2. Characteristic function and constructions of small covers 100
2.3. Examples 101
2.4. $\mathbb{Z}_2^3$-invariant normal bundle over an invariant S^1 101
3. Operations for 3-dimensional small covers 102
3.1. The equivariant connected sum $\sharp$ 102
3.2. The cutting edge operation $\sharp^e$ 103
3.3. The cutting edge-vertex-edge operation $\sharp^{eve}$ 103
3.4. The equivariant Dehn surgery $\natural$ 104
3.5. The equivariant Dehn surgery $\flat$ 104
3.6. The operation $\sharp^\Delta$ 105
3.7. The coloring change $\sharp^C$ 105
4. Main theorem and corollaries 106
4.1. Main theorem 106
4.2. Constructions of oriented small covers 107
4.3. Constructions of all small covers 107
5. Relation between the Nishimura's theorem and the Lü-Yu's theorem 109
5.1. Relation of Corollary 4.4and the operation $\sharp^C$ 109
5.2. Problem 110
6. On 2-torus manifolds 110
6.1. Basics of 2-torus manifolds 110
6.2. The equivariant surgery $\natural_0$ 110
6.3. Remark on the operations $\sharp^\Delta$ and $\sharp^C$ 111

Chapter 6. On projective bundles over small covers 113
1. Introduction of Part 6 113
2. General results 113
3. Basic results for small covers 114
4. Structures of projective bundles over small covers 115
5. Topological classification of projective bundles over the 1-dimensional small cover 116
6. Construction theorem of 2-dimensional small covers 116
7. Topological classification of projective bundles over $\mathbb{R}P(2)$ and T^2 117
8. New characteristic function of projective bundles over 2-dimensional small covers 118
9. New operations of projective bundles over 2-dimensional small covers 119
10. Construction theorem of projective bundles over 2-dimensional small covers 120

Part 3. Appendix 121

Chapter 7. On 6-dimensional quasitoric manifolds 123
1. Introduction of Chapter 7 123
2. Geometric realization of the basic parts 123
3. Properties of oriented small covers and spin quasitoric manifolds 125
4. Remark on the McGavran's paper 125

Bibliography 127

Acknowledgements 129

List of publications and preprints in postdoctoral period 131
1. ACCEPTED PAPERS 131
2. PREPRINTS 131
3. PROCEEDINGS 131
4. OTHERS 131

Introduction

Let G be a Lie group and X be a smooth manifold. A smooth map $\varphi : G \times X \to X$ is called a (smooth) *G-action on X* if it satisfies the following two properties:

(1) $\varphi(e, x) = x$ for the identity element e of G and all $x \in X$;
(2) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $g, h \in G$ and $x \in X$.

We call a triple (G, X, φ) a (smooth) *transformation group*.

A transformation group naturally appears in mathematics as the group of automorphisms of a manifold with a geometric structure such as a riemaniann metric, for instance the group of affine transformations on a Euclidean space or the group of rotations on a standard sphere. The first mathematician who recognized the importance of a transformation group from the geometrical point of view was Felix Klein. He proposed in his Erlangen program (in 1872) that geometry is the study of structures invariant under a group action. Since then, the theory of transformation groups has become one of the main research areas in mathematics.

The most manageable group G is the compact abelian group $T^n = (S^1)^n$ (*torus*) and $\mathbb{Z}_2^n$ (*2-torus* or *real torus*). Moreover, in the compact (topological) transformation group theory, to consider its maximal abelian group is often essential. For example, the *equivariant cohomology* (one of the most important invariant of topological transformation group) of compact Lie group $H_G^*(M;\ \mathbb{Q})(= H^*(EG \times_G M;\ \mathbb{Q}))$ can be described by its maximal torus subgroup $T \subset G$, that is, $H_G^*(M;\ \mathbb{Q}) \simeq H_T^*(M;\ \mathbb{Q})^W$, where $H_T^*(M;\ \mathbb{Q})^W$ is the invariant part of $H_T^*(M;\ \mathbb{Q})$ by the Weyl group $W = N(T, G)/T$ action. So, to consider the abelian group action is one of the most essential study in compact (topological) transformation group theory.

On the other hand, *toric geometry* has independently developed in algebraic geometry since 1970's. Toric geometry is the study of *toric varieties* defined as the n-dimensional normal algebraic variety which has the algebraic torus $(\mathbb{C}^*)^n$ with the dense orbit. The toric variety corresponds with the combinatorial object *fan*. So, algebraic geometers have studied toric varieties by using both of the algebraic method and the combinatorial method. Here, we remark that the group $(\mathbb{C}^*)^n$ has T^n as its maximal compact subgroup. Moreover, there was the big application from algebraic geometry to combinatorial theory, like the g-theorem proved by Stanley in [**48**] (he proved this theorem by using toric geometry). Therefore, toric geometry is the bridge between algebraic geometry and combinatorial theory.

Motivated by toric geometry, several mathematicians (Davis-Januszkiewicz in [**11**], Masuda in [**29**]) studied toric variety from topological point of view since 1990's, and they found that toric theory can be also carried out in the topology as more generalized theory. Recently, this theory is called *toric topology*. Roughly speaking, toric topology is the study of algebraic, combinatorial, differential, geometric, and homotopy theoretic aspects of a particular class (not only manifolds) of torus actions, whose quotients are highly structured. For example, the orbit space of restricted T^n-actions on non-singular, projective toric varieties is the simple convex polytope (highly structured space). So, toric varieties include research objects in toric topology.

Toric topology is the bridge between topology and combinatorial theory such as toric geometry. For example, the equivariant cohomology of *torus manifold* which is the generalization of toric manifolds can be described by the *face ring* of its orbit space by [**33**]. This is the generalization of the result that the equivariant cohomology of toric manifold can be described by the *Stanley-Reisner ring* of its orbit space in toric geometry (see [**44**]). The equivariant cohomology is the geometric object (topological object) and the face ring or the Stanley-Reisner ring are combinatorial objects. On the other hand, in [**30**], Masuda proved the necessary and sufficient condition of h-vectors of simplicial, cell decompositions on spheres by using toric topology. This result corresponds with the g-theorem for h-vectors of simplicial convex polytopes in toric geometry. Toric topology is developed very quickly in recent years; and simultaneously, toric topology yields new interesting problems. In this report, we study toric topology, in particular, the *extension problem* of torus actions on torus manifolds and operations on *small covers*.

The first Part 1 of the report deals with the extension problem of torus actions on torus manifolds. This extension problem is motivated by the following two problems:

(1) the beginning of toric geometry, i.e, the study of automorphism groups of toric variety by Demazure (see [**12, 44**]);
(2) the cohomological rigidity problem of certain torus manifolds (see [**35**]).

The automorphism group of toric variety contains its original $(\mathbb{C}^*)^n$-action; therefore, we can regard the Demazure's study as the extension problem of $(\mathbb{C}^*)^n$-action. Thus, our study corresponds with the Demazure's study (in algebraic geometry) from topological point of view. In the first chapter (Chapter 1), we classify the homogeneous torus manifolds, that is, the torus T^n action on the torus manifold M^{2n} is extendable to a transitive G-action such that G contains T as its maximal torus. In the second and third chapters (Chapter 2, 3), we classify extended actions with codimension one principal orbits. For technical reasons, we classify more general torus manifolds (not only oriented case but also non-oriented case). The *cohomological rigidity problem* is originally the following problem: does the cohomology ring distinguish (quasi)toric manifold? This problem is motivated by the Masuda's theorem: the $H^*(BT)$-algebra of equivariant cohomology distinguishes the equivariant type of (quasi)toric manifold in [**31**]. Then, Masuda-Suh proposed the problem about the non-equivariant case in [**35**]. Because torus manifolds are generalization of (quasi)toric manifolds, they also proposed the problem about the torus manifolds whose orbit spaces are homotopy cell in [**35**]. In the forth chapter (Chapter 4), we classify the topological types of our classification results in Chapter 1, 2, 3. Then, we know that their topological types are determined by not only their cohomology rings but also characteristic classes. Moreover, the orbit space of our torus manifolds which appear in the result of Chapter 1, 2, 3 is a homotopy cell. This gives the negative answer of the cohomological rigidity problem of torus manifolds.

The next Part 2 of the report deals with operations of *small covers* and projective bundles over small covers. Here, the small cover is the real version of quasitoric manifolds introduced by Davis-Januszkiewicz in [**11**]. For example, the *geometric operations* (e.g., equivariant connected sums or equivariant surgeries) are described by the *combinatorial operations* (e.g., connected sum along vertex or bistellar moves). Several mathematicians have studied the operations on 3-dimensional small covers (Izmestiev in [**19**], Nishimura in [**42**], Lü-Yu in [**28**]). In Chapter 5, we prove some relations on operations on 3-dimensional small covers. From this result, we can improve the construction theorem of Nishimura in [**42**] (for oriented 3-dimensional small covers) and Lü-Yu in [**28**] (for 3-dimensional small covers). In Chapter 6, we study on projective bundles over small covers and prove the construction theorem of projective bundles over 2-dimensional small covers, i.e., all projective bundles over 2-dimensional small covers can be constructed by using some operations from the basic projective bundles over small covers. In order to show this theorem, we induce the new characteristic functions and new operations.

In the final Part 3, we give the appendix for 6-dimensional quasitoric manifolds. Small covers and quasitoric manifolds behave similarly in some case; for example, their cohomology rings are described by their Stanley-Reisner rings of their orbit spaces (coefficients are $\mathbb{Z}_2$ for small cover and $\mathbb{Z}$ for quasitoric manifolds). However, in some case, their behaviors are quite different. One of the most typical examples is the construction theorem for 3-dimensional small covers and 6-dimensional quasitoric manifolds (compare [**28**] and [**25, 36**]). We remark 6-dimensional quasitoric manifolds in this last part.

Main results (15 theorems and corollaries) are referred in each introduction of chapters.

Part 1

Extension problem and cohomological rigidity problem of torus manifolds

CHAPTER 1

Characterization of homogeneous torus manifolds

1. Introduction of Chapter 1

A *torus manifold* is an even dimensional oriented manifold M^{2n} acted on by a half-dimensional torus T^n with non-empty fixed point set: typical examples are the complex projective space $\mathbb{C}P(n)$ and the even dimensional sphere S^{2n} equipped with the natural T^n-actions. As is well known, the natural T^n-action on $\mathbb{C}P(n)$ is induced from the transitive $U(n+1)$-action (or $PU(n+1)$-action) on $\mathbb{C}P(n)$, that is, this T^n-action extends to a $U(n+1)$-action or $PU(n+1)$-action (see Example 2.2). Moreover, there is a similar property for the T^n-action on S^{2n} (see Example 2.3). So we can naturally ask which torus manifolds possess such *extended actions* (the exact definition is in Section 2.1). In the first three chapters, we focus on this extension problem of torus actions on torus manifolds.

This problem is reminiscent of the study of automorphism groups of *toric varieties* by Demazure in [**12**], where here a toric variety is a normal algebraic variety V on which an algebraic torus $(\mathbb{C}^*)^n$ acts with a dense orbit (see [**14**]). We note that a compact non-singular toric variety is a torus manifold by restricting its $(\mathbb{C}^*)^n$-action to a T^n-action ($(\mathbb{C}^*)^n$ contains the topological torus $T^n = (S^1)^n$ as its maximal compact subgroup). The automorphism group $Aut(V)$ of V contains $(\mathbb{C}^*)^n$ and the action of $Aut(V)$ restricted to $(\mathbb{C}^*)^n$ coincides with the original $(\mathbb{C}^*)^n$-action on V. Hence, we can regard Demazure's study as the study of the extension problem of $(\mathbb{C}^*)^n$-actions on toric varieties. In fact the notion of torus manifold (or *unitary toric manifold* in the earlier terminology) was introduced by Hattori and Masuda in [**17, 29**] as a far-reaching topological generalization of compact non-singular toric varieties. Consequently, our extension problem may be interpreted as the topological version of Demazure's work. (From this point we assume our groups in this report are always compact.)

In the first three chapters, we will study extended G-actions. In particular, in Chapter 1 and the next Chapter 2, 3, we will characterize extended G-actions which have codimension 0 (i.e., G acts transitively) and 1 principal orbits up to *essential isomorphism* (i.e., the induced effective actions are same: see Section 2.4), respectively. We often call a torus manifold on which G acts transitively a *homogeneous torus manifold*. For technical reasons, in this chapter, we do not assume that torus manifolds are omnioriented.

Let $\mathbb{Z}_2$ be defined as $\{I_{2m_j+1},\ -I_{2m_j+1}\} \subset O(2m_j+1)$, and let $PU(x)$ be a projective unitary group (see Example 2.2), $SO(x)$ a special orthogonal group, $O(x)$ an orthogonal group. Our main result (Theorem 3.4) is as follows:

THEOREM 1. Suppose a torus manifold $(M^{2n},\ T^n)$ extends to a transitive G-action, where G is a compact, connected Lie group whose maximal torus is T^n. Then $(M^{2n},\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\frac{\prod_{j=1}^{b}S^{2m_j}}{\mathcal{A}},\ \prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)\right),$$

where $\mathcal{A}$ can be any subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$, and $\prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)$ acts on $\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\prod_{j=1}^{b}S^{2m_j}/\mathcal{A}$ in the natural way, and $\sum_{i=1}^{a}l_i+\sum_{j=1}^{b}m_j=n$.

Furthermore, M^{2n} is orientable if and only if $\mathcal{A}\subset SO(2m_1+\cdots+2m_b+b)$.

We also have the following result (see Corollary 3.9):

COROLLARY 2. Suppose a compact, non-singular toric variety or a quasitoric manifold $(M^{2n},\ T^n)$ extends to a transitive G-action, where G is a compact, connected Lie group whose maximal torus is T^n. Then $(M^{2n},\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i),\ \prod_{i=1}^{a}PU(l_i+1)\right),$$

where $\sum_{i=1}^{a} l_i = n$.

Here, a T^n-manifold M^{2n} is called a *quasitoric manifold* over a simple polytope P^n if the following two conditions are satisfied (see [**7, 11**][1]):

(1) the T^n-action is *locally standard*, that is, locally modelled by the standard action on $\mathbb{C}^n$;
(2) there is the orbit projection map $\pi : M^{2n} \to M^{2n}/T^n = P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n, $k = 0,\ \cdots,\ n$.

Quasitoric manifolds were defined by Davis and Januszkiewicz as a topological counterpart of projective, non-singular toric varieties in [**11**]. We note that every smooth projective toric variety is a quasitoric manifold (e.g. [**7**, Chapter 5]). From our main result Theorem 1, we may conclude that if a compact, non-singular toric variety has a transitive G-action, then this manifold is a product of $\mathbb{C}P(l)$'s (Corollary 2). Hence, such manifolds are projective toric varieties; thus, they are also quasitoric manifolds.

The organization of this chapter is as follows. In Section 2, we first set up some notation and basic facts. Next we prove our main result Theorem 3.4, that is, we characterize homogeneous torus manifolds G/H in Section 3. A key lemma for this characterization is Lemma 3.2, as well as the classification result of simply connected, simple Lie groups and their maximal rank maximal connected subgroups in classical Lie theory proved in [**4**]. Finally, we remark that our methods, in particular Lemma 3.2, do not work for other T-manifolds in Section 4.

2. Preliminaries to the characterization

In this section, we recall some fundamental results. We start with recalling some basic notation associated to a torus manifold and an extended G-action.

2.1. Basic notations and examples. A *torus manifold* is a $2n$-dimensional, closed, connected, smooth manifold $M^{2n}(= M)$ with smooth, finite kernel action of an n-dimensional torus $T^n = (S^1)^n(= T)$ such that $M^T \neq \emptyset$ (see Section 2.4 for the kernel of an action). Let M^T denote the T-fixed point set. Automatically, every fixed point is isolated, because $\dim M = 2\dim T$ and T acts on M with finite kernel.

REMARK 2.1. In the paper [**17**], the definition of torus manifolds involves the choice of orientations of manifold M and its *characteristic submanifolds* called an *omniorientation* on M. Because we will classify extended actions up to essential isomorphism in this report, we do not need to choose an omniorientation on M. Moreover, the T-action on M does not need to be effective.

Let $\varphi : T^n \times M^{2n} \to M^{2n}$ be a T^n-action on M^{2n}. Assume that a compact, connected Lie group G has T^n as its maximal torus subgroup. If there exists an action $\Phi : G \times M^{2n} \to M^{2n}$ such that the restricted T^n-action $\Phi|_{T^n\times M^{2n}}$ is the given φ, then we call Φ an *extended G-action* of $(M^{2n},\ T^n)$, and we also denote Φ as $(M^{2n},\ G)$. If a principal G-orbit is of codimension k, we call $(M^{2n},\ G)$ a *codimension k extended G-action* of $(M^{2n},\ T^n)$. Here, the integer k satisfies $0 \leq k \leq n$. Because M and G are compact, if $(M^{2n},\ G)$ is a codimension 0 extended G-action then the G-action on M^{2n} is transitive and M^{2n} is a homogeneous manifold. The following three examples are standard and important.

[1] Davis and Januszkiewicz use the term "toric manifold" in [**11**], but in this report we use the term "quasitoric manifold" in [**7**] because we would like to reserve the use of the term "toric manifold" to mean a "non-singular toric variety".

EXAMPLE 2.2. If the T^n-action $\varphi : T^n \times \mathbb{C}P(n) \to \mathbb{C}P(n)$ on $\mathbb{C}P(n)$ is defined by

$$\varphi((t_1,\ \cdots,\ t_n),\ [z_0;\ z_1; \cdots;\ z_n]) = [z_0;\ t_1z_1; \cdots;\ t_nz_n],$$

where $(t_1,\ \cdots,\ t_n) \in T^n$ and $[z_0;\ z_1; \cdots;\ z_n] \in \mathbb{C}P(n)$, then we can easily check that this is a torus manifold whose fixed points are $(n+1)$ points:

$$[1;\ 0;\ \cdots;\ 0],\ [0;\ 1;\ 0;\ \cdots;\ 0],\ \cdots,\ [0;\ \cdots;\ 0;\ 1].$$

Considering the above T^n as the diagonal subgroup of $U(n+1)$ with a unit in the $(1,1)$-entry, this action extends to the transitive $U(n+1)/Z(U(n+1))$-action, where $Z(U(n+1))$ is the center of $U(n+1)$. Therefore $(\mathbb{C}P(n),\ U(n+1)/Z(U(n+1))) = (\mathbb{C}P(n),\ PU(n+1))$ is a codimension 0 extended action of the torus manifold $(\mathbb{C}P(n),\ T^n)$. Remark that $PU(n+1) = U(n+1)/Z(U(n+1)) \simeq SU(n+1)/\mathbb{Z}_{n+1} \approx SU(n+1)$ has T^n as its maximal torus subgroup, where $G \simeq G'$ means G and G' are isomorphic, $G \approx G'$ means G and G' have a same Lie algebra, and $\mathbb{Z}_{n+1}$ is the center of $SU(n+1)$.

EXAMPLE 2.3. Assume the T^n-action $\varphi : T^n \times S^{2n} \to S^{2n}$ on $S^{2n} \subset \mathbb{R}^{2n} \oplus \mathbb{R} = \mathbb{R}^{2n+1}$ is defined as follows: first we identify T^n with $SO(2)^n \subset SO(2n)$; and next T^n acts on $\mathbb{R}^{2n}$ by the restriction of the natural $SO(2n)$-action on $\mathbb{R}^{2n}$. Then we can easily check that this is a torus manifold whose fixed points are 2 points: the North pole $(0,\ \cdots,\ 0,\ 1)$ and the South pole $(0,\ \cdots,\ 0,\ -1)$ of S^{2n}. Moreover this action extends to the $SO(2n)$-action whose orbits are principal orbits S^{2n-1} (codimension 1 orbits) and two singular orbits which are the 2 fixed points of the T^n-action. Therefore $(S^{2n},\ SO(2n))$ is a codimension 1 extended action of the torus manifold $(S^{2n},\ T^n)$. Remark that $(S^{2n},\ T^n)$ also extends to a codimension 0 extended action $(S^{2n},\ SO(2n+1))$.

EXAMPLE 2.4. In the above Example 2.3, $S^{2n} \subset \mathbb{R}^{2n+1}$ has a free involution by $-I_{2n+1} \in O(2n+1)$, where I_{2n+1} is the identity element in the orthogonal group $O(2n+1)$. Now we define the manifold $\mathbb{R}P(2n)$ by $S^{2n}/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{I_{2n+1},\ -I_{2n+1}\}$. Because the T^n-action on S^{2n} commutes with the $\mathbb{Z}_2$-action, we can define a T^n-action on $\mathbb{R}P(2n)$ induced by the T^n-action on S^{2n}. Moreover, the two T^n-fixed points on S^{2n}, the North and South poles, go to the same point in $\mathbb{R}P(2n)$ under the $\mathbb{Z}_2$-quotient, and this point is the unique fixed point of this T^n-action on $\mathbb{R}P(2n)$. We can easily check $(\mathbb{R}P(2n),\ T^n)$ is effective, because $(S^{2n},\ T^n)$ is effective and $T^n \cap \mathbb{Z}_2 = \{I_{2n+1}\}$. Therefore, $(\mathbb{R}P(2n),\ T^n)$ is a torus manifold; however, we remark that $\mathbb{R}P(2n)$ is a non-orientable manifold. Furthermore, we have that $(\mathbb{R}P(2n),\ T^n)$ has a codimension 1 extended action $(\mathbb{R}P(2n),\ SO(2n))$ and a codimension 0 extended action $(\mathbb{R}P(2n),\ SO(2n+1))$, because the $SO(2n)$-action and the $SO(2n+1)$-action on S^{2n} in Example 2.3 commute with this $\mathbb{Z}_2$-action. Remark that the orbits of the codimension 1 extended action $(\mathbb{R}P(2n),\ SO(2n))$ consist of: principal orbits S^{2n-1}, one singular orbit which coincides with the unique T^n-fixed point of $(\mathbb{R}P(2n),\ T^n)$, and one exceptional orbit $\mathbb{R}P(2n-1)$.

In order to characterize torus manifolds which have codimension 0 extended actions, i.e., homogeneous torus manifolds, we will also need the following basic facts, summarized in Section 2.2 to 2.4.

2.2. Homogeneous space G/H with finite T-fixed points. First we discuss homogeneous spaces with T-actions. Let T be a maximal torus in a compact Lie group G, and H a closed subgroup in G. Suppose that $(G/H,\ T)$ is a torus manifold, that is, it satisfies the following three properties:

(1) the T-action on G/H has finite kernel;
(2) $\dim G/H = 2\dim T$;
(3) the T-fixed point set $(G/H)^T \neq \emptyset$.

Because of the third property, there is an element $gH \in G/H$ such that $TgH = gH$. It follows that $T \subset gHg^{-1}$ for some $g \in G$. Hence we can take a subgroup H as follows:

$$T \subset H^o \subset H \subset G,$$

where H^o is the identity component of H. Since T is a maximal torus in G, we have

$$\text{rank } G = \text{rank } H^o = \dim T = n,$$

where the *rank* of a compact connected Lie group is the dimension of a maximal torus subgroup. Consequently, we need to consider maximal rank subgroups of G.

2.3. Facts from classical Lie theory. In order to consider maximal rank subgroups, we recall some classical Lie theory (see [**38**, Chapter V]).

For any compact, connected Lie group G, there is a finite covering map:

$$p : \widetilde{G} = G_1 \times \cdots \times G_k \to G, \tag{2.1}$$

where G_i $(i = 1, \cdots, k)$ is a compact, simply connected, simple Lie group, or a compact, connected, commutative Lie group, i.e., a torus. Let the kernel of p be denoted by N. Then we have

$$G \simeq (G_1 \times \cdots \times G_k)/N,$$

where N is some finite central normal subgroup in $G_1 \times \cdots \times G_k$.

Now we have the following lemma for a product of Lie groups (see [**38**, Theorem 7.2]).

LEMMA 2.5. *Let G_i $(i = 1, \cdots, k)$ be compact, connected Lie groups and let G be their product. Assume H^o is the identity connected maximal rank subgroup in G. Then $H^o = H_1 \times \cdots \times H_k$, where H_i is a maximal rank subgroup in G_i.*

2.4. Essential isomorphism and a remark for the characterization. In this subsection, we define an essential isomorphism.

We first need some notations. The *kernel* of $(M,\ G)$ is defined as the intersection of all isotropy subgroups $\cap_{x\in M} G_x$. If the kernel N is the identity element, then this action is an *effective action*. The induced action $(M,\ G/N)$ is always effective, and we call it the *induced effective action*. We may now define an essential isomorphism.

DEFINITION 2.6. Let N be the kernel of $(M,\ G)$ and N' the kernel of $(M',\ G')$. We say that $(M,\ G)$ and $(M',\ G')$ are *essentially isomorphic* if their induced effective actions $(M,\ G/N)$ and $(M',\ G'/N')$ are weak equivariantly diffeomorphic, that is, there are an isomorphism $\rho : G/N \to G'/N'$ and a diffeomorphism $f : M \to M'$ such that $f(\varphi(g,x)) = \psi(\rho(g), f(x))$ for $(g,x) \in G/N \times M$, where $\varphi : G/N \times M \to M$ and $\psi : G'/N' \times M' \to M'$ are two induced effective actions.

EXAMPLE 2.7. In Example 2.2, $(\mathbb{C}P(n),\ PU(n+1))$ is essentially isomorphic to the natural transitive action $(\mathbb{C}P(n),\ SU(n+1))$.

EXAMPLE 2.8. Let $Spin(m)$ be the universal (double) covering of $SO(m)$ $(m \geq 3)$. This group $Spin(m)$ acts on a sphere and a real projective space through the projection to $SO(m)$. In Example 2.3 (resp. Example 2.4), the codimension 1 extended action $(S^{2n},\ SO(2n))$ (resp. $(\mathbb{R}P(2n),\ SO(2n))$) is essentially isomorphic to $(S^{2n},\ Spin(2n))$ (resp. $(\mathbb{R}P(2n),\ Spin(2n))$) for $n \geq 2$, and the codimension 0 extended action $(S^{2n},\ SO(2n{+}1))$ (resp. $(\mathbb{R}P(2n),\ SO(2n{+}1))$) is essentially isomorphic to the natural transitive action $(S^{2n},\ Spin(2n{+}1))$ (resp. $(\mathbb{R}P(2n),\ Spin(2n+1))$) for $n \geq 1$.

Let $\widetilde{G} = G_1 \times \cdots \times G_k$ be a covering of G defined in (2.1) such that each G_i $(i = 1, \cdots, k)$ is a compact, simply connected, simple Lie group, or a torus group. Then $(M,\ G)$ is essentially isomorphic to

$$(M,\ \widetilde{G}) = (M,\ G_1 \times \cdots \times G_k). \tag{2.2}$$

Therefore, we only need to consider products of simply connected, simple Lie groups and tori as the transformation groups on a homogeneous torus manifold. In the next section we characterize homogeneous torus manifolds.

3. Characterization of homogeneous torus manifolds

Assume $(M,\ G)$ is a codimension 0 extended G-action of a torus manifold $(M^{2n},\ T^n)$. In this section, we will classify such $(M,\ G)$ up to essential isomorphism.

3.1. Structure of torus manifolds. Now we can put $M = G/H$ and T is a maximal torus subgroup of H and G by the argument in Section 2.2. Moreover, a T-action of $(G/H,\ T)$ is defined by a natural inclusion of T to G. By (2.2), $(G/H,\ G)$ is essentially isomorphic to

$$(G/H,\ \widetilde{G}) = (G/H,\ G_1 \times \cdots \times G_k).$$

Let $p : \widetilde{G} \to G$ be the projection of (2.1). Then we have

$$G/H \cong \widetilde{G}/p^{-1}(H), \tag{3.1}$$

where $X \cong Y$ means X and Y are diffeomorphic. Therefore it is sufficient to classify $\widetilde{G}$ and its subgroup $p^{-1}(H)$. To classify such $\widetilde{G}$ and $p^{-1}(H)$, we first consider the identity component of $p^{-1}(H)$.

Let $\widetilde{H}$ (resp. $\widetilde{T}$) be the identity component of $p^{-1}(H)$ (resp. $p^{-1}(T)$)[2]. Because of Lemma 2.5, $\widetilde{G}/\widetilde{H}$ is decomposed into a product as follows:

$$\widetilde{G}/\widetilde{H} = G_1/H_1 \times \cdots \times G_k/H_k,$$

where $H_i \subset G_i$ is a maximal rank, connected subgroup for all $i = 1,\ \cdots,\ k$. Because T is a maximal torus subgroup of G and H, we see that $\widetilde{T}$ is also a maximal torus subgroup of $\widetilde{G}$ and $\widetilde{H}$ such that $p(\widetilde{T}) = T$. Moreover, we have the following lemma.

LEMMA 3.1. *If $(G/H,\ T)$ is a torus manifold, then $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is also a torus manifold, and each G_i is a compact, simply connected, simple Lie group.*

PROOF. We prove $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ satisfies the three properties in Section 2.2. Because $\widetilde{T}$ is a maximal torus subgroup of $\widetilde{G}$ and $\widetilde{H}$, we can easily check $(\widetilde{G}/\widetilde{H})^{\widetilde{T}} \neq \emptyset$, i.e., property (3) holds. Because $\dim G/H = 2\dim T$, we have $\dim \widetilde{G}/\widetilde{H} = 2\dim \widetilde{T}$, i.e., property (2) holds. Since $(G/H,\ T)$ is almost effective, $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is also almost effective, i.e., property (1) holds. Moreover, we have each G_i is not a torus by property (1), i.e., each G_i is a compact, simply connected, simple Lie group. $\square$

Taking a maximal torus subgroup T_i in H_i, the maximal torus subgroup $\widetilde{T}$ is decomposed into

$$\widetilde{T} = T_1 \times \cdots \times T_k \subset H_1 \times \cdots \times H_k \subset G_1 \times \cdots \times G_k.$$

The following structure lemma holds.

LEMMA 3.2. *Suppose that the torus manifold $(M,\ T)$ has a codimension 0 extended G-action and $M = G/H$ such that $T \subset H \subset G$. Put $\widetilde{G}/\widetilde{H} = G_1/H_1 \times \cdots \times G_k/H_k$, $\widetilde{T} = T_1 \times \cdots \times T_k$, and $G_i/H_i = M_i$ for all $i = 1,\ \cdots,\ k$, where $\widetilde{G}$ is a universal covering of G, its subgroup $\widetilde{H}$ (resp. $\widetilde{T}$) is the identity component of $p^{-1}(H)$ (resp. $p^{-1}(T)$), and T_i is a maximal torus subgroup in G_i and H_i. Then each factor $(M_i,\ T_i) = (G_i/H_i,\ T_i)$ is a torus manifold.*

PROOF. By Lemma 3.1, $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is a torus manifold and each G_i is a simply connected, simple Lie group. Because $(\widetilde{G}/\widetilde{H},\ \widetilde{T})$ is almost effective, we also have that $(G_i/H_i,\ T_i)$ is almost effective for all $i = 1,\ \cdots,\ k$. Since T_i is a maximal torus subgroup in G_i and H_i, we have $(G_i/H_i)^{T_i} \neq \emptyset$ for all $i = 1,\ \cdots,\ k$. Therefore, we have

$$2\dim T_i \leq \dim(G_i/H_i)$$

for all $i = 1,\ \cdots,\ k$. Hence, the following equation holds:

$$2\dim \widetilde{T} = \sum_{i=1}^{k} 2\dim T_i \leq \sum_{i=1}^{k} \dim(G_i/H_i) = \dim \widetilde{G}/\widetilde{H}.$$

On the other hand $2\dim \widetilde{T} = \dim \widetilde{G}/\widetilde{H}$. Consequently, we have $2\dim T_i = \dim G_i/H_i$ for all $i = 1,\ \cdots,\ k$, and each factor $(M_i,\ T_i) = (G_i/H_i,\ T_i)$ is a torus manifold. $\square$

[2] We remark that $p^{-1}(T) = \widetilde{T}$ by [**38**, Theorem 4.9 in Chapter V].

From Lemma 3.2, in order to classify $\widetilde{G}/\widetilde{H}$, we need to consider each factor G_i/H_i constructed by a compact, simply connected, simple Lie group G_i and its maximal rank connected subgroup H_i, such that

$$\dim G_i/H_i = \dim G_i - \dim H_i = 2\dim T_i = 2 \text{ rank } G_i = 2 \text{ rank } H_i.$$

In the next subsection we classify all codimension 0 extended G-actions of torus manifolds $(M,\ T^n)$ up to essential isomorphism.

3.2. Characterization of homogeneous torus manifolds. Let S be a compact, connected, simple Lie group, and S' a compact, connected, maximal rank, maximal subgroup of S. Here, a *maximal subgroup* means that if S'' is another compact, connected, maximal rank subgroup in S and there is an element $g \in S$ such that $S' \subset gS''g^{-1}$ then $S' = gS''g^{-1}$. For such S and S', the classification of these types, i.e., the Lie algebras of S and S', is known by classical Lie theory: see the Table 1 (see [**4**] or [**38**, Chapter V][3]).

TABLE 1. maximal rank subgroups

S	$A_l \quad (l \geq 1)$		$B_l \quad (l \geq 2)$
S'	$A_{i-1} \times A_{l-i} \times T^1 \quad (1 < i < l)$	$A_{l-1} \times T^1 \quad (i = 1,\ l)$	$B_{l-1} \times T^1 \quad (i = 1)$
S	$B_l \quad (l \geq 2)$		$C_l \quad (l \geq 3)$
S'	$D_i \times B_{l-i} \quad (1 < i < l)$	$D_l \quad (i = l)$	$C_i \times C_{l-i} \quad (1 \leq i < l)$
S	$C_l \quad (l \geq 3)$	$D_l \quad (l \geq 4)$	
S'	$A_{l-1} \times T^1 \quad (i = l)$	$D_{l-1} \times T^1 \quad (i = 1)$	$D_i \times D_{l-i} \quad (1 < i < l-1)$
S	$D_l \quad (l \geq 4)$	E_6	
S'	$A_{l-1} \times T^1 \quad (i = l-1,\ l)$	$D_5 \times T^1$	$A_1 \times A_5$
S	E_6	E_7	
S'	$A_2 \times A_2 \times A_2$	$D_6 \times A_1$	A_7
S	E_7		E_8
S'	$A_2 \times A_5$	$E_6 \times T^1$	D_8
S	E_8		
S'	A_8	$A_4 \times A_4$	$E_6 \times A_2$
S	E_8	F_4	
S'	$E_7 \times A_1$	$C_3 \times A_1$	$A_2 \times A_2$
S	F_4	G_2	
S'	B_4	A_2	$A_1 \times A_1$

Here, in the Table 1, $A_l \approx SU(l+1)$, $B_l \approx SO(2l+1)$, $C_l \approx Sp(l)$, $D_l \approx SO(2l)$ are the classical Lie groups and E_6, E_7, E_8, F_4, G_2 are the exceptional Lie groups (indices indicate their rank), and $S \approx X$ means S and X have the same Lie algebra. Note that each dimension of S is as follows:

$$\dim A_l = l^2 + 2l; \quad \dim B_l = \dim C_l = (2l+1)l; \quad \dim D_l = l(2l-1);$$
$$\dim E_6 = 78;\ \dim E_7 = 133;\ \dim E_8 = 248;\ \dim F_4 = 52;\ \dim G_2 = 14.$$

Therefore $\dim S/S'$ can be computed as in Table 2 below.

It follows easily from Table 2 that the following two cases are the only possible cases of $\dim S/S' = 2\,\text{rank}S$:

$$A_l/(A_{l-1} \times T^1) \quad \text{and} \quad B_l/D_l.$$

[3]The Table 1 is based on the list in [**38**]. In the list in [**4, 38**], the cases $i = 1,\ l$ in A_l do not appear; however, we can easily check these cases should be included in their list by making use of [**38**, Theorem 7.16 in Chapter V]. Moreover, for the cases that i is not fixed in B_l, C_l and D_l in [**38**], we can easily check that indices of S' (G' in [**38**]) should be the indices in the above list by making use of [**38**, Theorem 7.16 in Chapter V].

TABLE 2. dimension of S/S'

S/S'	$A_l/(A_{i-1}\times A_{l-i}\times T^1)$	$A_l/(A_{l-1}\times T^1)$	$B_l/(B_{l-1}\times T^1)$
dim	$2i(l-i+1);\quad 1<i<l$	$2l;\quad 1\leq l$	$2(2l-1);\quad 2\leq l$
S/S'	$B_l/(D_i\times B_{l-i})$	B_l/D_l	$C_l/(C_i\times C_{l-i})$
dim	$2i(2l-2i+1);\quad 1<i<l$	$2l;\quad 2\leq l$	$4i(l-i);\quad 1\leq i<l,\ 3\leq l$
S/S'	$C_l/(A_{l-1}\times T^1)$	$D_l/(D_{l-1}\times T^1)$	$D_l/(D_i\times D_{l-i})$
dim	$l(l+1);\quad 3\leq l$	$4(l-1);\quad 4\leq l$	$4i(l-i);\quad 1<i<l-1,\ 4\leq l$
S/S'	$D_l/(A_{l-1}\times T^1)$	$E_6/(D_5\times T^1)$	$E_6/(A_1\times A_5)$
dim	$l(l-1);\quad 4\leq l$	32	40
S/S'	$E_6/(A_2\times A_2\times A_2)$	$E_7/(D_6\times A_1)$	E_7/A_7
dim	54	64	70
S/S'	$E_7/(A_2\times A_5)$	$E_7/(E_6\times T^1)$	E_8/D_8
dim	90	54	128
S/S'	E_8/A_8	$E_8/(A_4\times A_4)$	$E_8/(E_6\times A_2)$
dim	168	200	162
S/S'	$E_8/(E_7\times A_1)$	$F_4/(C_3\times A_1)$	$F_4/(A_2\times A_2)$
dim	112	28	36
S/S'	F_4/B_4	G_2/A_2	$G_2/(A_1\times A_1)$
dim	16	6	8

If S'' is not a maximal subgroup but is a maximal rank compact connected subgroup, then S'' is a subgroup of a conjugation of one of the maximal subgroups S' in Table 1. Hence we have $\dim S/S'' > \dim S/S'$. By Table 2, we have $\dim S/S'' > \dim S/S' \geq 2\ \mathrm{rank} S$, hence such S'' does not occur. Moreover, if S_1' and S_2' are compact, connected, maximal rank, maximal compact subgroups in the compact, connected, simple Lie group S with same Lie algebra type, i.e., $S_1'\approx S_2'$, then S_1' and S_2' are unique up to conjugation in S (see [**38**, Chapter V]). Therefore we have the following Lemma 3.3.

LEMMA 3.3. *Let $(G_i/H_i,\ T_i)$ be a pair such that $G_i\supset H_i\supset T_i$ and $\dim G_i/H_i = 2\dim T_i = 2l$. Assume G_i is a connected, simple Lie group, H_i is a connected, closed subgroup, and* $\mathrm{rank}\ G_i = \mathrm{rank}\ H_i = \dim T_i$. *Then there are just the following two cases:*

$$G_i/H_i = SU(l+1)/S(U(l)\times U(1))\cong \mathbb{C}P(l);$$
$$G_i/H_i = Spin(2l+1)/Spin(2l)\cong SO(2l+1)/SO(2l)\cong S^{2l}.$$

From Lemma 3.3, we have $\widetilde{G}=\prod_{i=1}^{a} SU(l_i+1)\times\prod_{j=1}^{b} Spin(2m_j+1)$. Because an $Spin(m)$-action can be identified with an $SO(m)$-action up to essential isomorphism (see Section 2.3 and Example 2.8), we can assume that

$$\widetilde{G}=\prod_{i=1}^{a} SU(l_i+1)\times\prod_{j=1}^{b} SO(2m_j+1),$$

where $a+b=k$.

Because $\widetilde{H} = H_1 \times \cdots \times H_k$ is the identity component of $p^{-1}(H)$, we have the following relation:

$$\begin{aligned}
(3.2) \qquad & \widetilde{H} \subset p^{-1}(H) \subset N(\widetilde{H}) \subset \widetilde{G}; \\
& N(\widetilde{H}) = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \prod_{j=1}^{b} S(O(2m_j) \times O(1)),
\end{aligned}$$

where $N(\widetilde{H})$ is a normalizer group of $\widetilde{H}$ in $\widetilde{G}$. Hence, we have the following fibration for the torus manifold $G/H \cong \widetilde{G}/p^{-1}(H)$:

$$\mathcal{A} = p^{-1}(H)/\widetilde{H} \longrightarrow \widetilde{G}/\widetilde{H} \longrightarrow \widetilde{G}/p^{-1}(H) \cong G/H, \tag{3.3}$$

where $p^{-1}(H)/\widetilde{H} = \mathcal{A}$ is a subgroup in $N(\widetilde{H})/\widetilde{H} \simeq \prod_{j=1}^{b} \mathbb{Z}_2$, because of $S(O(2m_j)\times O(1))/SO(2m_j) \simeq \mathbb{Z}_2$ and the above (3.2). Note that we can regard $\mathbb{Z}_2$ as the group generated by the antipodal involution on S^{2m_j}, i.e., $\mathbb{Z}_2 \simeq O(2m_j+1)/SO(2m_j+1) = \{I_{2m_j+1},\ -I_{2m_j+1}\}$. Therefore, $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ freely. Moreover, we have that this $\mathcal{A}$-action on $\prod_{j=1}^{b} S^{2m_j}$ is orientation preserving if and only if $\mathcal{A} \subset SO(2m_1 + \cdots + 2m_b + b)$. Consequently, we have the following theorem.

THEOREM 3.4. *Suppose a torus manifold* $(M^{2n},\ T^n)$ *extends to a codimension* 0 *extended* G*-action, where* G *is a compact, connected Lie group whose maximal torus is* T^n*. Then* $(M^{2n},\ G)$ *is essentially isomorphic to*

$$\left(\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \frac{\prod_{j=1}^{b} S^{2m_j}}{\mathcal{A}},\ \prod_{i=1}^{a} PU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)\right),$$

where $\mathcal{A}$ *can be any subgroup of* $\prod_{j=1}^{b} \mathbb{Z}_2$ *whose factor* $\mathbb{Z}_2 = \{I_{2m_j+1},\ -I_{2m_j+1}\}$ *for* $j = 1, \cdots, b$*, and* $\prod_{i=1}^{a} PU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)$ *acts on* $\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j}/\mathcal{A}$ *in the natural way, and* $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n$*.*

Furthermore, M^{2n} *is orientable if and only if* $\mathcal{A} \subset SO(2m_1 + \cdots + 2m_b + b)$*.*

Here, we give two examples.

EXAMPLE 3.5. Let $I = I_{2m+b}$ be the identity element in $O(2m+b)$. We have $\{I,\ -I\} \subset \prod_{j=1}^{b} \mathbb{Z}_2 \subset O(2m+b)$, and the following manifold is one of the homogeneous torus manifolds in Theorem 3.4:

$$\mathbb{R}P_b(2m) = (\prod_{j=1}^{b} S^{2m_j})/\{I,\ -I\},$$

where $\{I,\ -I\}$ acts on $\prod_{j=1}^{b} S^{2m_j} \subset \prod_{j=1}^{b} \mathbb{R}^{2m_j+1} = \mathbb{R}^{2m+b}$ canonically. Note that if $b = 1$, this manifold $\mathbb{R}P_1(2m)$ is an even dimensional real projective space $\mathbb{R}P(2m)$. If b is even (resp. odd), then $\{I,\ -I\} \subset SO(2m+b)$ (resp. $\{I,\ -I\} \not\subset SO(2m+b)$). Therefore the following two conditions are equivalent by Theorem 3.4:

(1) $\mathbb{R}P_b(2m)$ is orientable;
(2) b is even.

EXAMPLE 3.6. Let us consider the product of three spheres $S^{2m_1} \times S^{2m_2} \times S^{2m_3} \subset \mathbb{R}^{2m_1+1} \times \mathbb{R}^{2m_2+1} \times \mathbb{R}^{2m_3+1}$. We define the group $\mathcal{A}$ as follows:

$$\begin{aligned}
\mathcal{A} &= \langle(-I_{2m_1+1},\ -I_{2m_2+1},\ I_{2m_3+1}),\ (I_{2m_1+1},\ -I_{2m_2+1},\ -I_{2m_3+1})\rangle \\
&= \{(I_{2m_1+1},\ I_{2m_2+1},\ I_{2m_3+1}),\ (-I_{2m_1+1},\ -I_{2m_2+1},\ I_{2m_3+1}), \\
&\qquad (I_{2m_1+1},\ -I_{2m_2+1},\ -I_{2m_3+1}),\ (-I_{2m_1+1},\ I_{2m_2+1},\ -I_{2m_3+1})\} \\
&\simeq \mathbb{Z}_2 \times \mathbb{Z}_2,
\end{aligned}$$

where $I_{2m_j+1} \in O(2m_j+1)$ is the identity element. Because $\mathcal{A} \subset SO(2m_1+2m_2+2m_3+3)$, we see that $(S^{2m_1} \times S^{2m_2} \times S^{2m_3})/\mathcal{A}$ is a homogeneous torus manifold and orientable by Theorem 3.4.

We can easily show the following corollaries.

COROLLARY 3.7. If a simply connected torus manifold has a codimension 0 extended G-action, $(M^{2n},\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\prod_{j=1}^{b}S^{2m_j},\ \prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)\right),$$

where $\sum_{i=1}^{a} l_i+\sum_{j=1}^{b} m_j=n$.

REMARK 3.8. We can assume $m_j\geq 2$ in the above Corollary 3.7, because $(S^2,\ SO(3))$ and $(\mathbb{C}P(1),\ PU(2))$ are essentially isomorphic. Therefore, diffeomorphism types of simply connected, homogeneous torus manifolds can be completely determined by sequences $(l_1,\ \cdots,\ l_a)$ and $(m_1,\ \cdots,\ m_a)$ such that $0<l_1\leq\cdots\leq l_a$, $2\leq m_1\leq\cdots\leq m_b$, $\sum_{i=1}^{a} l_i+\sum_{j=1}^{b} m_j=n$. We also remark that these sequences do not determine the omniorientation on M.

The set of torus manifolds is a topological generalization of *compact non-singular toric varieties*, and this set also contains all *quasitoric manifolds*. As is well known, every quasitoric manifold is simply connected and their cohomology rings are generated by second degree cohomology classes (see [**14**] for the case of toric varieties, and [**7**, Theorem 5.18], or [**11**, Theorem 4.14] for the case of quasitoric manifolds). Consequently, we have the following corollary by Corollary 3.7.

COROLLARY 3.9. If a compact non-singular toric variety or a quasitoric manifold $(M^{2n},\ T^n)$ has a codimension 0 extended G-action, then $(M^{2n},\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i),\ \prod_{i=1}^{a}PU(l_i+1)\right),$$

where $\sum_{i=1}^{a} l_i=n$.

REMARK 3.10. If a compact algebraic variety has a codimension 0 extended compact G-action, then this variety is non-singular. Hence, in this case we may omit the assumption of non-singularity in the above corollary.

4. On other T-manifolds

Finally, in this section, we give an application of the above argument for other T-manifolds $(M,\ T)$.

Suppose a T^m-manifold $(M^{2n},\ T^m)$ extends to a transitive G-action, where G has T^m as its maximal torus. Assume that this T^m-action is almost effective and has finitely many fixed points. Then we have $n\geq m$ (if $n=m$ then M is a torus manifold). From the same argument as in Section 2.2, we also have $M\cong G/H$ such that $T^m\subset H^o\subset G$ and rank $G=$ rank $H^o=m$. Therefore we can apply the same argument as for torus manifolds, and we obtain the diffeomorphism type of M^{2n}.

For example, applying the above argument for $(M^{4n},\ T^{n+1})$, we have the following proposition, where a *decomposable manifold* M means that the manifold M is diffeomorphic to $M_1\times M_2$ such that $\dim M_1,\ \dim M_2\neq 0$.

PROPOSITION 4.1. Assume that $(M^{4n},\ T^{n+1})$ has finitely many fixed points, M is simply connected, and that the T^{n+1}-action is almost effective. If $(M^{4n},\ T^{n+1})$ has a codimension 0 extended G-action (where *rank* $G=n+1$) and M is not a decomposable manifold, then M^{4n} is diffeomorphic to one of the followings:

$$\begin{aligned} G_2(\mathbb{C}^{n+2}) &= SU(n+2)/S(U(n)\times U(2));\\ \mathbb{H}P(n) &= Sp(n+1)/(Sp(n)\times Sp(1));\\ Q_{2n} &= SO(2n+2)/(SO(2n)\times SO(2)),\end{aligned}$$

where $G_2(\mathbb{C}^{n+2})$ is the complex Grassmannian of 2-planes in $\mathbb{C}^{n+2}$, $\mathbb{H}P(n)$ is the quaternionic projective space, and Q_{2n} is the complex quadric.

PROOF. Because M is not decomposable, we can assume that G is a simply connected, compact, simple Lie group. First we assume $M = G/H$. With a method similar to that demonstrated in Section 2.2, we can easily show that rank G = rank H. Because M is simply connected, we have $H = H^o$. Assume H is a maximal compact subgroup. In this case, we will see from Table 2 that the pair $(G,\ H)$ such that rank G = rank $H = n+1$ and $\dim G = \dim H = 4n$ is one of the three pairs in the statement of this proposition (remark that $(B_2,\ D_2) \approx (C_2,\ C_1 \times C_1)$ and $(D_4,\ A_3 \times T^1) \approx (D_4,\ D_3 \times T^1)$).

Next we assume $M = G/K$ and $K = K^o$ is not maximal. Then we have $\dim G/K > \dim G/H$ where H is maximal and rank G = rank H = rank $K = n+1$. If G is one of the next Lie groups: $B_{n+1},\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$, then we always have $\dim G/H > 4n$ from the list in Table 2. Therefore we can assume that G is one of the next three Lie groups: $A_{n+1},\ C_{n+1},\ D_{n+1}$.

If G is A_{n+1}, then an inequality $\dim G/H < 4n$ holds only when $A_{n+1}/(A_n \times T^1)$ from the list in Table 2. Thus, we have $G \supset H \supset K$ where $H \approx A_n \times T^1$. If K is maximal in such subgroup, we also have $K \approx (A_{j-1} \times A_{n-j} \times T^1) \times T^1$ for $1 \leq j \leq n$. Then we can easily check that $\dim G/K > 4n$. Hence this case does not occur.

If G is C_{n+1} or D_{n+1}, then we always have $\dim G/H \geq 4n$ from the list in Table 2. Hence only the maximal case occurs. $\square$

REMARK 4.2. In general $(M^{4n},\ T^{n+1})$ can be decomposed into a product $(M_1 \times \cdots \times M_k,\ T_1 \times \cdots \times T_k)$, but we can easily prove that the type decomposition of Lemma 3.2 does not hold (except $k = 1$). For example $(S^6 \times S^2,\ T^2 \times T^1)$ is one of the elements in the class $(M^{4n},\ T^{n+1})$ for $n = 2$, and it has the codimension 0 extended $G_2 \times SO(3)$-action, where $S^6 = G_2/SU(3)$ and $S^2 = SO(3)/SO(2)$. However, the two factors $(S^6,\ T^2)$ and $(S^2,\ T^1)$ are not in the class $(M^{4n},\ T^{n+1})$.

CHAPTER 2

Classification of quasitoric manifolds with codimension one extended actions

1. Introduction of Chapter 2

One of the essential problems in geometry is to find the most natural group action on the given space. A *torus manifold* is an even dimensional oriented manifold M^{2n} acted on by a half-dimensional torus T^n with non-empty fixed point set. Since the previous Chapter 1, we have studied the extended G-actions $(M^{2n},\ G)$ on torus manifolds $(M^{2n},\ T^n)$, where G is a compact, connected Lie group whose maximal torus is T^n. To study extended actions of torus manifolds is to find the natural group action on M which induces the given T^n-action on M.

In the previous Chapter 1, we classified homogeneous torus manifolds, i.e., torus manifolds with transitive extended G-actions (also see Theorem 2.3 in this chapter). If G acts on M transitively, then its principal orbit G/K is M itself, i.e., $M = G/K$. In other words, the codimension of the principal orbit in the transitive action is zero, i.e., $\dim M - \dim G/K = 0$. Therefore, we can regard the classification in Chapter 1 as the classification of codimension 0 extended actions. So we are naturally led to study codimension 1 extended actions, i.e., extended G-actions on torus manifolds with codimension 1 principal orbits.

A torus manifold is called a *quasitoric manifold* over a simple polytope P^n if the following two conditions are satisfied (see [**7, 11**]):

(1) the T^n-action is *locally standard*, that is, locally modelled by the standard T^n-action on $\mathbb{C}^n$;
(2) there is a projection map $\pi : M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n, $k = 0, \cdots, n$,

where an n-dimensional convex polytope P^n is *simple* if precisely n codimension-one faces (*facets*) meet at each vertex. The goal of the present chapter is to classify codimension 1 extended actions of T^n-actions on quasitoric manifolds. In order to classify such actions, we classify more general classes as in the following main theorem (Theorem 8.4, 8.6 and 8.8 for detail):

THEOREM 3. Suppose a simply connected torus manifold M has a codimension one extended G-action. If two singular orbits are also simply connected torus manifolds, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{j=1}^{b} S^{2m_j} \times N,\quad \prod_{j=1}^{b} SO(2m_j+1) \times H\right),$$

where $(N,\ H)$ is one of the followings:

N	H
$\left(\prod_{i=1}^{a} S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k)$
$\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$	$\prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$
$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$

where $\prod SO(2m_j+1)$ and $\prod SU(l_i+1)$ act naturally on $\prod S^{2m_j}$ and $\prod S^{2l_i+1}$ (or $\prod \mathbb{C}P(l_i)$) respectively; and $U(k)$, $S(U(k_1) \times U(k_2))$ and $SO(2k)$ also act naturally on $\mathbb{C}^k_{\mathfrak{a}}$, $\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}$ and $\mathbb{R}^{2k}$ respectively.

Using Theorem 3 and basic results for quasitoric manifolds, we reach our goal as the following corollary:

COROLLARY 4. If a quasitoric manifold M has a codimension one extended G-action, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}),\quad \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))\right).$$

For the classification of codimension 0 extended actions on torus manifolds, we only used classical Lie theory, in the previous Chapter 1. However, for the classification of codimension 1 extended actions on torus manifolds, we need to use not only classical Lie theory but also transformation group theory. In history of studying transformation group theory, there are so many classification results of actions with codimension 1 principal orbits, (e.g., [**2, 15, 22, 24, 50, 52**]). In particular, the Uchida's method in [**50**], developed the Wang's method in [**52**], is very powerful method in the case that we classify vast classes for compact transformation groups on compact manifolds with codimension 1 principal orbits. So we apply the Uchida's method to classify our case.

When we use the Uchida's method, we often need to divide our proof into two cases (see [**24, 50**]); specifically, in our case, the case that two singular orbits of G-actions are torus manifolds, and the other case that one of two singular orbits of G-actions is not a torus manifold (see Lemma 3.2). Proofs of these two cases are quite different; thus, we divide our chapter into two chapters: this chapter and the next Chapter 3. In this chapter, we focus on the first case, i.e., we classify the class which consists of simply connected torus manifolds $(M^{2n},\ T^n)$ with codimension 1 extended G-actions such that two singular orbits of extended G-actions are also simply connected torus manifolds (see Theorem 3). In particular, quasitoric manifolds with codimension 1 extended actions are contained in this class by Lemma 3.3 (also see Corollary 4).

Due to Corollary 2 in Chapter 1, a homogeneous quasitoric manifold is a product of $\mathbb{C}P(l)$'s only. On the other hand, by making use of the above Corollary 4, a quasitoric manifold which has codimension 1 extended actions is a complex projective bundle over a homogeneous quasitoric manifold. As we mentioned in the previous Chapter 1, our extension problem is reminiscent of the study of automorphism groups of *toric varieties* by Demazure in [**12**], where here a toric variety is a normal algebraic variety V on which an algebraic torus $(\mathbb{C}^*)^n$ acts with a dense orbit (see [**14**]). In fact, quasitoric manifolds were defined by Davis and Januszkiewicz as a topological counterpart of projective, non-singular toric varieties in [**11**]. In this chapter, the term "dimension" means a real dimension. Therefore, to classify codimension 1 extended G-actions belongs to purely topological problems. However, the resulting manifolds in Corollary 4 are projective, non-singular toric varieties, that is, these objects also belong to algebraic geometry. Remark that $\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$ is equivariantly diffeomorphic to $\prod_{i=1}^{a-1} \mathbb{C}_0^{l_i+1} \times_{(\mathbb{C}^*)^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$, where $\mathbb{C}_0^{l_i+1} = \mathbb{C}^{l_i+1} - \{0\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Finally, we note that the classification results for low dimensional torus manifolds are known, e.g., [**45**] for simply connected, 4-dimensional torus manifolds or [**36**] for simply connected, spin, 6-dimensional torus manifolds; however, in [**36**], there is a mistake (see Chapter 7 or [**25**]).

The organization of this chapter is as follows. In Section 2, we prepare some fundamental results from toric topology, classical Lie theory, the previous Chapter 1, and transformation group theory. In Section 3, we show that a G-orbit through T-fixed points is a singular orbit, and give some difference between quasitoric manifolds and torus manifolds with codimension 1 extended G-actions. From Section 4, we assume that our torus manifold is simply connected, our extended G-action of torus manifold has two singular orbits and they are simply connected torus manifolds. Moreover, from this section, we start the Uchida's method; thus, we first show a topological type of two singular orbits. In Section 5, we next classify two tubular neighborhoods of two singular orbits by computing slice representations. In Section 6, we get possibilities for G and two singular isotropy subgroups by comparing two slice representations. In Section 7, we study more precise structure of slice representations and get possibilities for two tubular neighborhoods in $(M,\ G)$. The key lemmas of this classification are Lemma 7.1, 7.2 and 7.3, that is, two slice representations of two singular isotropy groups must coincide up to sign. We also have principal isotropy subgroups in this section. Finally, in Section 8, we compute the attaching maps between boundaries of two tubular neighborhoods. Then we can construct our manifolds explicitly as in Theorem 3. In final Section 9, as an application, we show the following fact: codimension 1 (resp. 0) extended actions of quasitoric manifolds can be lifted to actions on moment-angle manifolds with codimension 1 (resp. 0) principal orbits. In other wards, codimension 1 (resp. 0) extended actions of quasitoric manifolds are always induced from actions on moment-angle manifolds with codimension 1 (resp. 0) principal orbits.

2. Preliminary

In this first section, we recall some basic notations and facts from toric topology [**17, 29**], classical Lie theory [**38**], the previous Chapter 1, and transformation group theory [**6, 21**]. Throughout of this chapter, T^n is always an n-dimensional, compact, abelian group, i.e., T^n is a product of n circles $(S^1)^n$, we often call it an n-dimensional torus or just a torus.

2.1. Torus manifold. A torus manifold introduced in [**17, 29**] is the main research object in this chapter. First we recall this definition. A *torus manifold* is a pair $(M^{2n},\ T^n)$ of a smooth, $2n$-dimensional, compact manifold M^{2n} and an n-dimensional (half dimensional) torus T^n which satisfies the following two properties:

(1) M^{2n} is an oriented manifold equipped with an almost effective T^n-action (also see Remark 2.1);
(2) its fixed point set is non-empty, i.e., $M^T \neq \emptyset$,

and it is often denoted by $(M,\ T)$ or M simply. Automatically, the fixed point set M^T is finite and the principal orbit (defined in the next Section 2.2) is T^n itself.

A compact, connected, codimension two submanifold of M without boundary is called *characteristic* if it is a connected component of the set fixed pointwise by a certain circle subgroup of T and contains at least one T-fixed point. There are only finitely many characteristic submanifolds and they are orientable, because M is compact and orientable.

REMARK 2.1. For technical reasons, we assume a T-action on M is an *almost effective*, that is, the intersection of all isotropy subgroups $\cap_{x\in M}T_x$ is finite set (if this set only consists of the identity element, then this action is *effective*).

2.2. Review of classical Lie theory and the previous chapter. In order to study extended actions of torus actions on torus manifolds, we next introduce some terminology and fact from classical Lie theory, transformation group theory and the previous Chapter 1. We start with recalling general terminology.

Let $(M,\ G)$ be a pair of a manifold M and its smooth G-action. A *principal orbit* in $(M,\ G)$ is a maximal orbit in $(M,\ G)$. A *singular orbit* in $(M,\ G)$ is an orbit whose dimension is strictly less than the dimension of principal orbits. An *exceptional orbit* in $(M,\ G)$ is an orbit which is not maximal but whose dimension is the same as that of a principal orbit (see Example 3.6). Let G_x be an isotropy subgroup of $x \in M$, i.e., $G_x = \{g \in G \mid gx = x\}$.

In this and the next chapters, we will classify $(M,\ G)$ up to essential isomorphism. So we next define an essential isomorphism. Let N be the intersection of all isotropy subgroups $\cap_{x\in M}G_x$ of $(M,\ G)$. Then the induced action $(M,\ G/N)$ is always effective, and we call it the *induced effective action*. If two induced effective actions of $(M,\ G)$ and $(M',\ G')$ are weak equivariantly diffeomorphic, that is, there are an isomorphism $\rho : G/N \to G'/N'$ and a diffeomorphism $f : M \to M'$ such that $f(\varphi(g,x)) = \psi(\rho(g), f(x))$ for $(g,x) \in G/N \times M$ (where $\varphi : G/N \times M \to M$ and $\psi : G'/N' \times M' \to M'$ are two induced effective actions), then we call that $(M,\ G)$ and $(M',\ G')$ are *essentially isomorphic*. Now we may recall the following facts (see [**38**, Section 5]). For any compact, connected Lie group G, there is a finite covering, homomorphic map:

$$p : \widetilde{G} = G_1 \times \cdots \times G_k \to G, \tag{2.1}$$

where G_i is a compact, (simply) connected, simple Lie group, or a torus for $i = 1, \cdots, k$. Let the kernel of p be denoted as N. Because p is a surjective homomorphism, we have

$$G \simeq (G_1 \times \cdots \times G_k)/N,$$

where N is some finite, central, normal subgroup in $G_1 \times \cdots \times G_k$. By the definition of p, we have that $(M,\ G)$ can be lifted to $(M,\ \widetilde{G})$ and they are essentially isomorphic.

A *rank* of G is the dimension of a maximal torus subgroup of G. The following lemma is known for a maximal rank subgroup H^o of G (see [**38**, Theorem 7.2]).

LEMMA 2.2. *Let G_i $(i = 1, \cdots, k)$ be compact, connected Lie groups and let G be their product. Assume H^o is a compact, connected, maximal rank subgroup in G. Then $H^o = H_1 \times \cdots \times H_k$, where H_i is a maximal rank subgroup in G_i.*

We next recall the results of the previous Chapter 1. Before we state the results in Chapter 1, we need to introduce some actions. Let $\mathbb{Z}_2$ be defined as $\{I_{2m_j+1},\ -I_{2m_j+1}\} \subset O(2m_j+1)$, where $O(2m_j+1)$ is the orthogonal group. We remark that $\mathbb{Z}_2$ acts on the $2m_j$-dimensional sphere $S^{2m_j} \subset \mathbb{R}^{2m_j+1}$ canonically (we call this action the *antipodal action* on sphere). Let $\mathcal{A}$ be a subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$. Then $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ through the canonical $\prod_{j=1}^{b}\mathbb{Z}_2$-action on $\prod_{j=1}^{b} S^{2m_j}$ (the product of antipodal actions). For codimension 0 extended G-actions, we proved the following two classification results (see Chapter 1; Theorem 1, Corollary 3.7 and Remark 3.8):

THEOREM 2.3. *Let $(M^{2n},\ T^n)$ be a $2n$-dimensional, compact, connected manifold M equipped with an almost effective, half dimensional torus T-action with fixed points. Suppose $(M^{2n},\ T^n)$ extends to a codimension 0 extended G-action, where G is a compact, connected Lie group whose maximal torus is T^n. Then $(M^{2n},\ G)$ is essentially isomorphic to*

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i) \times \frac{\prod_{j=1}^{b} S^{2m_j}}{\mathcal{A}},\ \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)\right),$$

where the above group acts on M^{2n} in the natural way, and $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n$.

COROLLARY 2.4. If a simply connected torus manifold has a codimension 0 extended G-action, $(M^{2n},\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a}\mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j},\ \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1)\right),$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n$ and $m_j \geq 2$ for all $j = 1, \cdots, b$.

REMARK 2.5. In Chapter 1 (Theorem 1 and Corollary 3.7), we did not use $SU(l+1)$. Instead, we used $PU(l+1)$ as the transformation group, where $PU(l+1)$ is defined as the quotient of $SU(l+1)$ by its center $Z(SU(l+1))$. However, $(\prod \mathbb{C}P(l),\ \prod SU(l+1))$ is essentially isomorphic to $(\prod \mathbb{C}P(l),\ \prod PU(l+1))$ (see Example 2.7 in Chapter 1). So we may change $PU(l+1)$'s into $SU(l+1)$'s of Chapter 1 (Theorem 1 and Corollary 3.7). For technical reasons, in this chapter, we use $SU(l+1)$.

In order to classify codimension 1 extended G-actions, we need to use the above results.

2.3. Review of transformation group theory. Next we recall some fact from transformation group theory. For the general codimension 1 actions, the following theorem is known (see [**6**, 8.2 Theorem in Chapter IV] or [**50**, Lemma 1.2.1]):

THEOREM 2.6. *Let G be a compact, connected Lie group and M a compact, connected manifold without boundary such that $H^1(M;\ \mathbb{Z}_2) = 0$. Assume G acts smoothly on M with codimension one orbits G/K. Then all codimension one orbits G/K are principal orbits, and there are just two singular orbits G/K_1 and G/K_2. Furthermore, there exists a closed, invariant tubular neighborhood X_s (of G/K_s for $s = 1,\ 2$) such that*

$$M = X_1 \cup X_2 \text{ and } X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

The Figure 1 shows the image of $(M,\ G)$ with codimension 1 principal orbits.

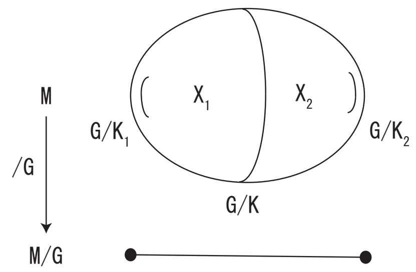

FIGURE 1. The orbit structure of $(M,\ G)$ with codimension 1 orbits.

Once we know the singular orbits G/K_1 and G/K_2, then their tubular neighborhoods X_1 and X_2 are also known by the following *differentiable slice theorem*, the *slice theorem* for short (see, e.g., [**6, 21**]).

THEOREM 2.7 (differentiable slice theorem). *Let G be a compact Lie group and M a smooth G-manifold, and G_x an isotropy subgroup of $x \in M$. Then, for all $x \in M$, there is a closed G-invariant neighborhood X (of the orbit $G(x) \cong G/G_x$) such that $X \cong G \times_{G_x} D_x$ as a G-diffeomorphism, where $D_x \subset \mathbb{R}^N$ is some closed disk, $G \times_{G_x} D_x$ is the quotient space $(G \times D_x)/G_x$ induced by the G_x-action on $G \times D_x$. Here, this G_x-action is defined as follows: G_x canonically acts on G as a subgroup of G; and on a closed disk D_x through an orthogonal representation $\sigma : G_x \to O(D_x)$, where $O(D_x)$ is an orthogonal group of $D_x \subset \mathbb{R}^N$ $(N = \dim D_x = \dim M - \dim G(x))$.*

In Theorem 2.7, σ is called a *slice representation* of G_x, and D_x a *slice* of X on x. We identify a tubular neighborhood X of $G(x)$ with $G \times_{G_x} D_x$. The slice theorem will be often used throughout this report.

Suppose a compact Lie group G acts on a manifold M smoothly and it has a fixed point $p \in M^G$. Using the slice theorem, the tangent space $T_p(M)$ of $p \in M^G$ is an orthogonal G-representation space. We call it a *tangential representation space*, or simply a *tangential representation* on p. Let α_i be a representation from T to $S^1 \simeq SO(2)$, i.e., $\alpha_i : T \to S^1 \simeq SO(2)(\in Hom(T,\ S^1) \simeq \mathbb{Z}^n)$, and let $V(\alpha_i) \simeq \mathbb{R}^2$ be a representation space of α_i. For tangential representation spaces of torus manifolds, the following proposition holds.

PROPOSITION 2.8. Let $(M,\ T)$ be a torus manifold and $p \in M$ a fixed point. Then there is the following decomposition for the tangential representation space on p:

$$T_p(M) \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_n)$$

such that $\{\alpha_1,\ \cdots,\ \alpha_n\}$ is linearly independent in $Hom(T,\ S^1) \otimes \mathbb{R} \simeq \mathbb{R}^n$.

PROOF. Because $p \in M^T$, we may regard $T_p(M)$ as an orthogonal T-representation space by the slice theorem. From the definition of the torus manifold, the T-action on M is an almost effective. It follows that there is a non-degenerate representation ρ from T^n to the orthogonal group $O(T_p(M)) \simeq O(2n)$, i.e., the image of ρ is also an n-dimensional torus. Moreover, the image of ρ is in the special orthogonal group $SO(2n)$ because T^n is connected. Therefore, the image of ρ and the diagonal maximal torus $SO(2) \times \cdots \times SO(2) \subset SO(2n)$ are conjugate in $SO(2n)$. This gives an equivalence between ρ and $\alpha_1 \oplus \cdots \oplus \alpha_n$ for some $\alpha_i = k_i \circ \pi_i$, where π_i is the i-th projection $\pi_i : SO(2) \times \cdots \times SO(2) \to SO(2)$ and k_i is a non-degenerate representation $k_i : SO(2) \to SO(2)$, i.e., $k_i \neq 1 : SO(2) \to \{I_2\} \subset SO(2)$ $(i = 1,\ \cdots,\ n)$. Moreover, we can easily see that $\{\alpha_1,\ \cdots,\ \alpha_n\}$ is linearly independent in $Hom(T,\ S^1) \otimes \mathbb{R} \simeq \mathbb{R}^n$ by their definitions. □

3. Orbits in extended actions

In this section, we study orbits in extended actions $(M,\ G)$ of torus manifolds $(M,\ T)$.

3.1. G-orbits on T-fixed points. By the definition of the torus manifold $(M,\ T)$, there are non-empty isolated fixed points M^T. We first consider a G-orbit on a fixed point $p \in M^T$.

If p is in the T-fixed point set M^T, then we have

$$T \subset G_p \subset G, \tag{3.1}$$

where G_p is the isotropy subgroup of p. We have the following lemma.

LEMMA 3.1. *Suppose that a torus manifold $(M^{2n},\ T^n)$ has a codimension 1 extended G-action. Then a G-orbit G/K_1 of a T-fixed point is a singular orbit in $(M,\ G)$, i.e., $\dim G/K_1 < 2n-1$. In other words, there is at least one singular orbit in $(M,\ G)$.*

PROOF. Let p be a fixed point ($p \in M^T$). We shall prove that the orbit $G(p)$ is a singular orbit, i.e., $\dim G(p) < 2n-1$. By the above relation (3.1), rank G_p^o coincides with rank $G = \dim T$. It follows that the Euler character of G/G_p^o is not zero (e.g. [**16**, Theorem 1.1 (2), (3)]), and thus $\dim G/G_p = \dim G/G_p^o$ is even. Because $(M^{2n},\ G)$ is a codimension 1 extended action, the dimension of a principal orbit is $2n-1$. Therefore, we have $\dim G(p) = \dim G/G_p = \dim G/G_p^o < 2n-1$. □

Moreover, we see that a G-orbit on a fixed point $p \in M^T$ is a torus manifold by the following lemma.

LEMMA 3.2. *Let G/K_1 be a singular orbit of $(M,\ G)$ which contains a fixed point of $(M,\ T)$, i.e., $G/K_1 \cap M^T \neq \emptyset$. Then there is some subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is a torus manifold.*

PROOF. Let $p : \widetilde{G} \to G$ be a covering of G in (2.1), and let $\widetilde{K_1}$ (resp. $\widetilde{T}$) be a connected component of $p^{-1}(K_1)$ (resp. $p^{-1}(T)$). By (3.1), we have $\widetilde{T}$ is a maximal torus subgroup of $\widetilde{G}$ and $\widetilde{K_1}$. By the argument of Section 2.2 and Lemma 2.2, there is the following decomposition:

$$\widetilde{G} = G_1' \times G_1'',\ \widetilde{K_1} = K_1' \times G_1'',\ \widetilde{T} = T_1' \times T_1'',$$

where G_1' and G_1'' are products of compact, simply connected, simple Lie groups and tori, and G_1'' is the same factor in $\widetilde{G}$ and $\widetilde{K_1}$, that is, the connected component of the intersection of all isotropy subgroup $\cap_{x \in G/K_1} \widetilde{G}_x$ (i.e., the *kernel* of the $\widetilde{G}$ action on G/K_1). Here, G_1' and G_1'' satisfy that rank $G_1' =$ rank $K_1' = \dim T_1'$ and rank $G_1'' = \dim T_1''$.

Because $\widetilde{K_1} \subset p^{-1}(K_1) \subset \widetilde{G}$, we also have the following decomposition:

$$p^{-1}(K_1) = H_1' \times G_1'',$$

where $K_1' \subset H_1' \subset G_1'$ and the connected component of H_1' is K_1'. Then the projection p induces the diffeomorphism between G_1'/H_1' and G_1/K_1. We remark that T_1'' is a connected component of the kernel of $\widetilde{T}$-action on G_1'/H_1'. Now we may prove that $(G_1'/H_1',\ T_1')$ is an *unoriented* torus manifold, i.e., a torus manifold which does not assume orientability. Because T_1' is a maximal torus of G_1' and $K_1' = (H_1')^o$, we have that this T_1'-action on G_1'/H_1' is almost effective and there are fixed points. Moreover, we have the following decomposition on the fixed point $q \in G/K_1 \cap M^T$:

$$T_qM = T_qG/K_1 \oplus N_qG/K_1.$$

Now T_1' acts on T_qG/K_1 and T_1'' acts on N_qG/K_1 through the finite covering p. Moreover, using a T-action on T_qM, we have an irreducible decomposition of $T_qM = V(\alpha_1) \oplus \cdots \oplus V(\alpha_n)$ (by Proposition 2.8). Therefore, we have $\dim G/K_1 = 2n - 2k_1$ and $2k_1 = \dim N_qG/K_1 = 2\dim T_1''$. Hence, $(G_1'/H_1',\ T_1')$ is an unoriented torus manifold.

We define T' as $p(T_1')$ and T'' as $p(T_1'')$. Because $(G_1'/H_1',\ T_1')$ is an unoriented torus manifold, $(G_1/K_1,\ T')$ is also an unoriented torus manifold. Furthermore, we see that G_1/K_1 is one of the fixed pointwise connected component of $M^{T''}$. Therefore, G_1/K_1 can be written as some connected component of an intersection of characteristic manifolds. Because all characteristic manifolds are orientable, G_1/K_1 is an oriented manifold. Hence, $(G_1/K_1,\ T')$ is a torus manifold. □

In the next subsection, we recall quasitoric manifolds briefly, and give some differences between quasitoric manifolds and torus manifolds with codimension 1 extended G-actions.

3.2. Differences between quasitoric manifolds and torus manifolds. The concept of a quasitoric manifold introduced in [**11**] is a topological counterpart of the projective toric variety. Let P^n be a simple convex polytope, i.e., a convex hull of some vertices such that neighborhoods of all vertices look like simplexes. If the torus manifold $(M^{2n},\ T^n)$ satisfies the following two properties:

1. T^n-action is *locally standard*, that is, locally looks like the standard torus representation in $\mathbb{C}^n$;
2. there is a projection map $\pi : M^{2n} \to P^n$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of k-dimensional face of P^n, $k = 0,\ \cdots,\ n$,

then we call $(M^{2n},\ T^n)$ a *quasitoric manifold.* For quasitoric manifolds with codimension 1 extended G-actions, we have the following lemma.

LEMMA 3.3. *Suppose that the quasitoric manifold $(M^{2n},\ T^n)$ has a codimension 1 extended G-action. Then there are just two singular orbits in $(M,\ G)$, and these two singular orbits are quasitoric manifolds.*

PROOF. Let $p \in M^T$. We first prove that an orbit $G(p) = G/K_1$ is a quasitoric manifolds. By Lemma 3.2, there is a subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is a torus manifold. Because $(M,\ T)$ satisfies the locally standard property, $(G/K_1,\ T')$ is also locally standard.

Moreover, G/K_1 is a fixed pointwise connected component of $M^{T''}$ (see the proof of Lemma 3.2). In other wards, G/K_1 can be constructed by an intersection of some characteristic submanifolds M_i, i.e., $G/K_1 = \cap M_i$. Because $(M,\ T)$ is a quasitoric manifold, an orbit space M_i/T of a characteristic submanifold M_i corresponds with a *facet* in M/T, i.e., a codimension 1 subface in a simple convex polytope M/T. Hence, $(G/K_1)/T$ is a subface of M/T, and it becomes a simple convex polytope. Thus, G/K_1 is a quasitoric manifold.

If all $p \in M^T$ are included in G/K_1, then the convex hull of M^T and $(G/K_1)^T$ must be same, that is, $M/T = (G/K_1)/T$. This gives a contradiction to dimensions of M/T and $(G/K_1)/T$. Hence, both two singular orbits G/K_1 and G/K_2 contain fixed points of $(M,\ T)$, and we have that G/K_2 is also a quasitoric manifold by the same argument for G/K_1. $\square$

This Lemma 3.3 does not hold for all torus manifolds, the following two examples give a simple difference between quasitoric manifolds and torus manifolds.

EXAMPLE 3.4. Let $(M,\ T) = (\mathbb{C}P(2),\ T^2)$ be the torus manifold defined by the standard multiplication of T^2 on the last two coordinates in $[z_0 : z_1 : z_2] \in \mathbb{C}P(2)$ (also see Example 2.2 in Chapter 1). This torus manifold has a codimension 1 extended $G = PU(2) \times T^1$-action: for $[z_0 : z_1 : z_2] \in \mathbb{C}P(2)$, $PU(2) = U(2)/Z(U(2))$ acts on the first two coordinates $(z_0,\ z_1)$ by the canonical multiplication where $Z(U(2))$ is the center of $U(2)$; and T^1 also acts on the third coordinate z_2 by the canonical multiplication. Now we can easily check $(M,\ T)$ is a quasitoric manifold (also see the left "triangle" in Figure 2), and $(M,\ G)$ has codimension 1 orbits $G([1 : 0 : 1]) \cong \mathbb{C}P(1) \times S^1$ and two singular orbits $G([1 : 0 : 0]) \cong \mathbb{C}P(1)$ and $G([0 : 0 : 1]) \cong \{*\}$ (one point). We can also check both singular orbits are quasitoric manifolds (also see Lemma 3.3).

EXAMPLE 3.5. Let $(M,\ T) = (S^4,\ T^2)$ be the torus manifold defined by the standard multiplication of $T^2 = SO(2) \times SO(2)$ on $S^4 \cap \mathbb{R}^4$, where $S^4 \subset \mathbb{R} \oplus \mathbb{R}^4$ (also see Example 2.3 in Chapter 1). This torus manifold has a codimension 1 extended $G = SO(3) \times T^1$-action: for $(x, y) \in S^4 \subset \mathbb{R}^3 \oplus \mathbb{R}^2$, $SO(3)$ canonically acts on $x \in \mathbb{R}^3$; and $T^1 \simeq SO(2)$ also acts on $y \in \mathbb{R}^2$ canonically. Now we can check $(M,\ T)$ is not a quasitoric manifold because its orbit space is not a convex polytope (see the right "half-moon" in Figure 2, this half-moon is not a convex polytope), and $(M,\ G)$ has codimension 1 orbits $G(e_1, f_1) \cong S^2 \times S^1$ and two singular orbits $G(e_1,\ 0) \cong S^2$ and $G(0,\ f_1) \cong S^1$, where $e_1 = (1,\ 0,\ 0) \in \mathbb{R}^3$ and $f_1 = (1,\ 0) \in \mathbb{R}^2$. We can easily check S^1 is not a torus manifold because its dimension is not even. Hence, Lemma 3.3 does not hold in this case.

The Figure 2 shows the difference of the quasitoric manifold and the torus manifold in the above Examples 3.4 and 3.5.

We also remark the following example:

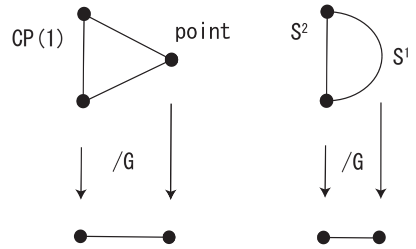

FIGURE 2. This is an image of a reason why Lemma 3.3 does not hold for all torus manifolds, and which submanifolds correspond to singular orbits. The left triangle shows the orbit space $\mathbb{C}P(2)/T^2$, the right half-moon also shows the orbit space S^4/T^2, and interval shows the orbit space of $\mathbb{C}P(2)/(PU(2) \times T^1)$ and $S^4/(SO(3) \times T^1)$.

EXAMPLE 3.6. Let $(S^4,\ T^2)$ be a torus manifold defined in Example 3.5. Then we may naturally define the product of two copies $(S^4 \times S^4,\ T^2 \times T^2)$, and this is a torus manifold with 4 fixed points. If N and S denote the 2 fixed points in $(S^4,\ T^2)$, then the 4 fixed points in $(S^4 \times S^4,\ T^2 \times T^2)$ can be denoted as $(N,\ N)$, $(N,\ S)$, $(S,\ N)$ and $(S,\ S)$.

Let $\mathbb{Z}_2$ be the group generated by $(-I_5,\ -I_5)$, where $-I_5$ is the antipodal involution on $S^4 \subset \mathbb{R}^5$ and I_5 is the identity map on $\mathbb{R}^5$. We remark that $-I_5$ does not preserve an orientation on S^4; however, $(-I_5,\ -I_5)$ preserves an orientation on $S^4 \times S^4$. Now we may consider the following manifold

$$(S^4 \times S^4)/\mathbb{Z}_2 = S^4 \times_{\mathbb{Z}_2} S^4.$$

Since $(-I_5,\ -I_5)$ preserves an orientation of $S^4 \times S^4$ and $(-I_5,\ -I_5)$ commutes with $T^2 \times T^2$-action on $S^4 \times S^4$, we have that $S^4 \times_{\mathbb{Z}_2} S^4$ is an oriented manifold equipped with $T^2 \times T^2$-action induced from $(S^4 \times S^4,\ T^2 \times T^2)$. Moreover, there are 2 fixed points denoted by $[N:\ N] = [S:\ S]$ and $[N:\ S] = [S:\ N]$. Therefore, $(S^4 \times_{\mathbb{Z}_2} S^4,\ T^2 \times T^2)$ is a torus manifold (also see Theorem 2.3).

This action extends to the canonical $G = SO(5) \times SO(4)$-action on $S^4 \times_{\mathbb{Z}_2} S^4$. Then we have the following three orbit types: $G([e_1:\ e_1]) = (SO(5) \times SO(4))/(SO(4) \times SO(4)) = S^4$; $G([e_1:\ e_2]) \cong (SO(5) \times SO(4))/(SO(4) \times SO(3) \times \mathbb{Z}_2) \cong S^4 \times_{\mathbb{Z}_2} S^3$; and $G([e_1:\ e_1 + e_2]) = (SO(5) \times SO(4))/(SO(4) \times SO(3)) = S^4 \times S^3$. Here, $e_1,\ \cdots,\ e_5$ are the canonical basis of $\mathbb{R}^5$. Therefore, in this case there are one singular orbit S^4, principal orbits $S^4 \times S^3$, and the exceptional orbit $S^4 \times_{\mathbb{Z}_2} S^3$.

From the above Example 3.6, we know that there is a case which has an exceptional orbit in $(M,\ G)$. However, in this chapter, we do not need to deal with such kind of actions (we will discuss with such actions in the next Chapter 3) by the assumption that our manifold M is simply connected and Theorem 2.6 (see Section 4).

3.3. Singular orbit G/K_1. Next we study more precise structures of a singular orbit by making use of Theorem 2.3, i.e., the classification result for homogeneous torus manifolds. Let $(M,\ G)$ be a codimension 1 extended G-action of a torus manifold $(M,\ T)$, and let $(G/K_1,\ T)$ be one of the singular orbits through T-fixed points on M. By (3.1), we see that G and K_1 have the same maximal torus T. Moreover, by Lemma 3.2, there is some subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is a torus manifold.

By the argument of Section 2.2, $(M,\ G)$ is essentially isomorphic to $(M,\ \widetilde{G})$, where $\widetilde{G}$ is a product of connected, compact, simple Lie groups. Let $p:\widetilde{G} \to G$ be a finite projection in (2.1), and let $\widetilde{K_1}$ (resp. $\widetilde{T}$) be the identity component of $p^{-1}(K_1)$ (resp. $p^{-1}(T)$). We note that $p(\widetilde{K_1}) = K_1^o$, $p(\widetilde{T}) = T$ and $\widetilde{G}/\widetilde{K_1} \cong G/K_1^o$ induced by p, where K_1^o is the identity component of K_1. Let G_1'' be the same factors in $\widetilde{G}$ and $\widetilde{K_1}$. By the proof of Lemma 3.2, there are the following decompositions:

$$\widetilde{T} = T_1' \times T_1'' \subset \widetilde{K_1} = K_1' \times G_1'' \subset p^{-1}(K_1) = H_1' \times G_1'' \subset \widetilde{G} = G_1' \times G_1''$$

such that the identity component of H_1' is K_1'.

Now $(G_1'/H_1',\ T_1')$ is a torus manifold. Therefore, we have from Theorem 2.3,

$$G_1' = \prod_{i=1}^a SU(l_i+1) \times \prod_{j=1}^b SO(2m_j+1) \text{ and } H_1' = \prod_{i=1}^a S(U(1)\times U(l_i)) \times \mathcal{S},$$

where we may take $\mathcal{S}$ as the following subgroup:

$$\prod_{j=1}^b SO(2m_j) \subset \mathcal{S} \subset \Big(\prod_{j=1}^b S(O(1)\times O(2m_j))\Big),$$

such that the manifold $\prod_{j=1}^b SO(2m_j+1)/\mathcal{S}$ has an orientation.

In summary, we have the following proposition by Theorem 2.3.

Proposition 3.7. Suppose that a torus manifold $(M,\ T)$ extends to a codimension 1 extended action. Then this codimension 1 extended action is essentially isomorphic to $(M,\ G)$, such that its singular orbit G/K_1 which contains one of the fixed points M^T can be denoted as follows:

$$G = G_1' \times G_1'', \quad K_1^o = K_1' \times G_1'' \quad \text{and} \quad K_1 = H_1' \times G_1'',$$

where G_1'' is a product of compact, connected, simple Lie groups or tori, and $(G_1',\ K_1',\ H_1')$ is

$$G_1' = \prod_{i=1}^a SU(l_i+1) \times \prod_{j=1}^b SO(2m_j+1),$$

$$K_1' = \prod_{i=1}^a S(U(l_i)\times U(1)) \times \prod_{j=1}^b SO(2m_j),$$

$$H_1' = \prod_{i=1}^a S(U(l_i)\times U(1)) \times \mathcal{S},$$

where $\sum_{i=1}^a l_i + \sum_{j=1}^b m_j = n - \operatorname{rank} G_1''$.

4. Assumptions and structures of two singular orbits

In the remainder of this chapter, we assume $(M,\ T)$ is a torus manifold, where M is simply connected. Because M is simply connected, we have $H^1(M;\ \mathbb{Z}_2) = 0$. Hence, by Theorem 2.6, a codimension 1 extended action has just two singular orbits. Moreover, by the argument of Section 2.2, we may only consider that a codimension 1 extended action $(M,\ G)$ such that G is a product of compact, connected, simple Lie groups, up to essential isomorphism.

Because of Lemma 3.1, one of the singular orbit has T-fixed points. Thus, we need to discuss the following two cases:

- both of two singular orbits contain T^n-fixed points (see e.g. Example 3.4);
- one singular orbit contains all T^n-fixed points, but the other singular orbit has no T^n-fixed points (see e.g. Example 3.5).

In this chapter, we only discuss the first case, i.e., both singular orbits contain T^n-fixed points. Moreover, we assume two singular orbits are simply connected. In order to classify quasitoric manifolds with codimension 1 extended G-actions, it is enough to classify such torus manifolds, by using Lemma 3.3 in this chapter and [**11**, Corollary 3.9], i.e., the fact that quasitoric manifolds are simply connected.

In this case, Proposition 3.7 holds for two singular orbits. Because G/K_1 and G/K_2 are simply connected, we also have $K_1 = K_1^o$ and $K_2 = K_2^o$. Since $SU(2)/S(U(1)\times U(1)) \cong SO(3)/SO(2) \cong S^2$, we can regard $S^2 = SU(2)/S(U(1)\times U(1))$ if there is an S^2 factor in G/K_s. It follows that we can assume $m_j \geq 2$ for all $j = 1,\ \cdots,\ b$ in Proposition 3.7. In summary, we have the following proposition.

Proposition 4.1. If two singular orbits G/K_1 and G/K_2 contain T^n-fixed points and they are simply connected, then there are the following decompositions: for $s = 1,\ 2$,

$$G = G_s' \times G_s'', \quad K_s = K_s^o = K_s' \times G_s'',$$

where G_s'' is a product of simply connected, compact, simple Lie groups or tori, and

$$G_1' = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1), \quad K_1' = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \prod_{j=1}^{b} SO(2m_j);$$

$$G_2' = \prod_{i=1}^{c} SU(l_i' + 1) \times \prod_{j=1}^{d} SO(2m_j' + 1), \quad K_2' = \prod_{i=1}^{c} S(U(l_i') \times U(1)) \times \prod_{j=1}^{d} SO(2m_j'),$$

where $\sum_{i=1}^{a} l_i + \sum_{j=1}^{b} m_j = n - \text{rank}\, G_1''$, and $\sum_{i=1}^{c} l_i' + \sum_{j=1}^{d} m_j' = n - \text{rank}\, G_2''$, and $m_j,\ m_j' \geq 2$. Furthermore, for the maximal torus subgroup $T_s' \subset T$ of G_s' and K_s', $(G_s'/K_s',\ T_s')$ is a simply connected torus manifold.

From the next section, we start with classifying codimension 1 extended G-actions of torus manifolds which satisfy the above assumptions.

5. Tubular neighborhoods of singular orbits

From the previous section, we assume that M is a simply connected torus manifold and two singular orbits G/K_1 and G/K_2 are also simply connected torus manifolds: we have already known structures of G/K_s in Proposition 4.1 ($s = 1,\ 2$). As the next step of the classification, we will investigate tubular neighborhoods of singular orbits by computing slice representations (see Theorem 2.7). In order to compute slice representations, we first prepare some fact.

5.1. Preparation. Because a singular orbit G/K_s is a torus manifold, we can define its dimension as $2k_s = \dim M - \dim G/K_s$, i.e.,

$$\dim G/K_s = 2n - 2k_s$$

for $s = 1,\ 2$ and $k_s \geq 1$.

By the slice theorem, we also know that a tubular neighborhood X_s of G/K_s is as follows:

$$X_s \cong G \times_{K_s} D^{2k_s}$$

such that K_s acts on D^{2k_s} by the slice representation

$$\sigma_s : K_s \to O(2k_s).$$

Since $(M,\ G)$ has a codimension 1 principal orbits, we also have that K_s acts transitively on a sphere $S^{2k_s - 1} \subset D^{2k_s}$ through this slice representation $\sigma_s : K_s \to O(2k_s)$, and a principal isotropy subgroup of K_s-action on D^{2k_s} is isomorphic to a principal isotropy subgroup of $(M,\ G)$. Hence, the slice representation can be computed using the classification of transitive actions on spheres.

5.2. Transitive actions on sphere. The transitive actions on spheres were studied by Borel, Montgomery and Samelson, and Poncet, and the following results are proved in a series of the papers [**3, 39, 46**].

THEOREM 5.1. *Let G be a compact, connected Lie group acting effectively and transitively on a homotopy sphere Σ^k and H be the isotropy subgroup, namely, $G/H \cong \Sigma^k$. Then there always exists a simple normal subgroup G_1 of G that is already transitive on Σ^k, i.e., $G_1/(G_1 \cap H) = \Sigma^k$.*

THEOREM 5.2. *Let G_1 be a one of the simple groups in G such that $G_1/H_1 \cong \Sigma^k$. Then we have that:*

- *if k is even, $G_1 = SO(k+1)$ or the exceptional Lie group G_2 in the case $k = 6$;*
- *if $k = 2l - 1$ and l odd, $G_1 = SO(k+1)$ or $SU(l)$;*
- *if $k = 2l - 1$ and l even, $G_1 = SO(k+1)$, $SU(l)$, $Sp(l/2)$, $Spin(9)$ (in the case $k = 15$, $l = 8$), or $Spin(7)$ (in the case $k = 7$, $l = 4$).*

As we may check easily in each of the above cases, we have a unique embedding of G_1 into $SO(k+1)$ such that $G_1 \cap SO(k) = H_1$. Hence, Σ^k is diffeomorphic to the standard sphere S^k.

THEOREM 5.3. *Let $G_1(\subset G)$ be the simple subgroup in Theorem 5.2. Then $G_1 \subset G \subset N(G_1)^o \subset SO(k+1)$, where $N(G_1)^o$ is the identity component of the normalizer of G_1 in $SO(k+1)$, and the following holds.*

- *In cases $G_1 = SO(k+1)$, G_2 $(k=6)$, $Spin(7)$ $(k=7,\ l=4)$, or $Spin(9)$ $(k=15,\ l=8)$, we have that $N(G_1)^o = G_1$, hence $G_1 = G$.*
- *In the case $G_1 = SU(l)$, we see that $N(G_1)^o = U(l)$, hence we may have either $G = G_1$ or $U(l)$.*
- *In the case $G_1 = Sp(\frac{l}{2})$, $N(G_1)^o$ is the subgroup of $SO(k+1)$ generated by $Sp(\frac{l}{2})$, and the S^3-subgroup realized as right multiplications of unit quaternions, As a group, $N(Sp(\frac{l}{2}))^o$ is isomorphic to $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^3$, where $\mathbb{Z}_2$ is the subgroup generated by $(-Id,\ -1)$. Hence, G is either $Sp(\frac{l}{2})$ or $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^1$ or $Sp(\frac{l}{2}) \times_{\mathbb{Z}_2} S^3$.*

The above results are also referred in the paper [**18**].

In particular, we see the following results from the above theorems.

COROLLARY 5.4. Assume the connected subgroup H in $O(2l)$ acts on S^{2l-1} transitively and its rank is l, i.e., rank $H = l$. Then $H \simeq U(l)$ or $SO(2l)$ in $O(2l)$.

In the next Section 5.3, we will discuss about slice representations.

5.3. Slice representations (rough structures). By Lemma 3.2 and the assumptions in Section 4, two singular isotropy subgroups can be denoted as $K_s = G_p$ $(s = 1,\ 2)$ for some fixed point $p \in M^T$. From the argument in Section 3.3, there are the following decompositions for this given $p \in M^T$:

$$G = G'_s \times G''_s \supset K_s = K^o_s = K'_s \times G''_s \supset T = T'_s \times T''_s, \tag{5.1}$$

where G'_s and G''_s are products of compact, connected, simple Lie groups and tori, and T'_s and T''_s are their maximal tori. Then a slice representation is as follows:

$$\sigma_s : K_s = K'_s \times G''_s \to SO(2k_s) \subset O(2k_s).$$

Remark that we can take the target of the slice representation σ_s as $SO(2k_s)$ because K_s is connected.

By the decomposition (5.1), we have that T''_s is in the kernel of the T-action on $G/K_s \cong G'/K'_s$. It follows that $T''_s \subset G''_s$ acts almost effectively on the normal space of $p \in G/K_s \cap M^T$. Moreover by Lemma 3.2 and Section 5.1, we have $\dim T''_s = k_s$. Therefore, we have $\sigma_s(T''_s) = T^{k_s} \subset SO(2k_s)$, i.e., a maximal torus of $SO(2k_s)$.

Because of the decomposition of $K_s = K'_s \times G''_s$, we know that K'_s commutes with G''_s in K_s. In particular, K'_s also commutes with $T''_s(\subset G''_s)$. It follows that the image $\sigma_s(K'_s)$ of K'_s is in the group $\{g \in SO(2k_s) \mid gt = tg \text{ for all } t \in T^{k_s}\}$: we remark that this group (the centralizer of T^{k_s} in $SO(2k_s)$) is T^{k_s} itself. Therefore, we have $\sigma_s(K'_s) \subset T^{k_s}$. Hence, G''_s acts transitively on S^{2k_s-1} through the slice representation $\sigma_s : K'_s \times G''_s \to SO(2k_s)$. Because rank $\sigma_s(G''_s) = k_s = \dim T''_s =$ rank G''_s, we have from Corollary 5.4:

$$\begin{aligned} G''_s \quad \approx \quad & \sigma_s(G''_s) \simeq U(k_s) \quad \text{or} \\ & SO(2k_s), \end{aligned}$$

where $X \approx Y$ means that they have the same Lie algebra. Now we may consider that if $k_s = 1$ then $SO(2) = U(1)$; therefore, we may assume

$$\begin{aligned} G''_s \quad = \quad & SU(k_s) \times T^1 \quad \text{or} \\ & SO(2k_s) \text{ and } k_s \geq 2. \end{aligned}$$

Because $\sigma_s(K'_s)$ is also in the centralizer of $\sigma_s(G''_s)$, we see

$$\begin{aligned} &\sigma_s(K'_s) \subset Z(U(k_s)) \quad (\text{if } \sigma_s(G''_s) \simeq U(k_s)) \quad \text{or} \\ &\sigma_s(K'_s) \simeq \{e\} \quad (\text{if } \sigma_s(G''_s) \simeq SO(2k_s),\ k_s \geq 2), \end{aligned}$$

where $Z(U(k_s)) \simeq T^1$ is the center of $U(k_s)$, i.e., the diagonal subgroup of $U(k_s)$.

In summary, we have the following two lemmas by the above arguments and Proposition 4.1.

LEMMA 5.5. *Two singular orbits G/K_s $(s=1,\ 2)$ satisfy the followings:*

- $(G,\ K_s)=(G'_s\times G''_s,\ K'_s\times G''_s)$;
- $G'_1=\prod_{i=1}^{a}SU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)$ *and* $G'_2=\prod_{i=1}^{c}SU(l'_i+1)\times\prod_{j=1}^{d}SO(2m'_j+1)$;
- $K'_1=\prod_{i=1}^{a}S(U(l_i)\times U(1))\times\prod_{j=1}^{b}SO(2m_j)$ *and* $K'_2=\prod_{i=1}^{c}S(U(l'_i)\times U(1))\times\prod_{j=1}^{d}SO(2m'_j)$;
- $G''_s\approx SU(k_s)\times T^1$ *or* $SO(2k_s)$ *and* $k_s\geq 2$.

Furthermore, $m_j,\ m'_j\geq 2$.

LEMMA 5.6. *Two slice representations $\sigma_s: K_s=K'_s\times G''_s\to SO(2k_s)$ $(s=1,\ 2)$ satisfy the following list:*

	G''_s	$\sigma_s(G''_s)$	$\sigma_s(K'_s)$
(1)	$SU(k_s)\times T^1$	$U(k_s)\subset SO(2k_s)$	$Z(U(k_s))\simeq T^1\subset U(k_s)\subset SO(2k_s)$
(2)	$SO(2k_s)$	$SO(2k_s)$	$\{e\}\subset SO(2k_s)$

where the right list means $\sigma_s(K'_s)\subset Z(U(k_s))$ for (1) and $\sigma_s(K'_s)=\{e\}$ (identity) for (2).

6. Possibility for G, K_1, K_2

In this section we shall give a possibility for a transformation Lie group G and two singular isotropy subgroups K_1, K_2.

6.1. Decomposition of G. We first remark that the decompositions of G in Lemma 5.5 depends on $s=1,\ 2$, i.e., two decompositions $G'_1\times G''_1$ and $G'_2\times G''_2$ may not be the same decomposition. However, for all compact Lie groups G, there is an unique covering $\widetilde{G}$ consists of the product of simply connected, simple Lie group and tori. Therefore, the coverings of two G's for $s=1,\ 2$ are the same coverings. Using Lemma 5.5 and the relation $\widetilde{G}=\widetilde{G'_1}\times\widetilde{G''_1}=\widetilde{G'_2}\times\widetilde{G''_2}$, we know that $\widetilde{G}$ satisfies

$$\prod_{i=1}^{a}SU(l_i+1)\times\prod_{j=1}^{b}Spin(2m_j+1)\times X=\prod_{i=1}^{c}SU(l'_i+1)\times\prod_{j=1}^{d}Spin(2m'_j+1)\times Y,$$

where $(X,\ Y)$ is one of the followings:

(1) $(X,\ Y)=(SU(k_1)\times T^1,\ SU(k_2)\times T^1)$;
(2) $(X,\ Y)=(Spin(2k_1),\ Spin(2k_2))$, and $k_1,\ k_2\geq 2$;
(3) $(X,\ Y)=(SU(k_1)\times T^1,\ Spin(2k_2))$, and $k_2\geq 2$;
(4) $(X,\ Y)=(Spin(2k_1),\ SU(k_2)\times T^1)$, and $k_1\geq 2$.

Note that $m_j,\ m'_j\geq 2$.

Comparing left and right sides of the above equations from (1) to (4), we will find a possibility for G.

6.2. The cases (3) and (4). In this subsection, we shall prove that the above two cases (3) and (4) do not occur.

In the case (4), if we change the role of k_1 and k_2 then it is the same as the case (3). Hence, we can regard the last two cases (3) and (4) as the same cases.

First, we remark the following well-known lemma.

LEMMA 6.1. *The following three statements hold:*

- *if* $Spin(2m)\simeq SU(m+1)$, *then* $m=3$ *and* $Spin(6)\simeq SU(4)$;
- *if* $Spin(2m+1)\simeq SU(m+1)$, *then* $m=1$ *and* $Spin(3)\simeq SU(2)$;
- $Spin(4)\simeq SU(2)\times SU(2)$.

For the other m in Lemma 6.1, there are no isomorphisms between $Spin(2m)$ (or $Spin(2m+1)$) and $SU(m+1)$, because their dimensions are different. Therefore, for each case from (1) to (4), we can regard

$$b=d\quad\text{and}\quad Spin(2m_j+1)=Spin(2m'_j+1),$$

because $m_j,\ m'_j\geq 2$. In particular, we can regard $SO(2m_j+1)=SO(2m'_j+1)$ for $j=1,\ \cdots,\ b$ in Lemma 5.5.

Hence, for the equation (3), we have

$$\prod_{i=1}^{a} SU(l_i+1)\times SU(k_1)\times T^1=\prod_{i=1}^{c} SU(l_i+1)\times Spin(2k_2).$$

It follows that $Spin(2k_2)=T^1$, i.e., $k_2=1$. However, we can assume $k_s\geq 2$ for $Spin(2k_s)$. Hence, this case does not occur, and the equation (4) also does not occur because we can regard (3) and (4) are the same cases.

6.3. The case (2). Suppose the equation (2). If $Spin(2k_1)=Spin(2k_2)$, then we have that $k_1=k_2$, $a=c$ and $l_i=l_i'$ for all $i=1,\ \cdots,\ a$ (we call this case the case $(2)-(a)$).

If $Spin(2k_1)\neq Spin(2k_2)$, then there are the following four cases using Lemma 6.1:

- **(b):** if $Spin(2k_1)=SU(l_c'+1)$, $(k_1,\ l_c')=(3,\ 3)$ and $Spin(2k_2)=SU(l_a+1)$, $(k_2,\ l_a)=(3,\ 3)$, then we have that $a=c$ and $l_i=l_i'$ for all $i=1,\ \cdots,\ a-1$ (the case $(2)-(b)$);
- **(c):** if $Spin(2k_1)=SU(l_c'+1)$, $(k_1,\ l_c')=(3,\ 3)$ and $Spin(2k_2)=SU(l_{a-1}+1)\times SU(l_a+1)$, $(k_2,\ l_{a-1},\ l_a)=(2,\ 1,\ 1)$, then we have that $a=c+1$ and $l_i=l_i'$ for all $i=1,\ \cdots,\ a-2(=c-1)$ (the case $(2)-(c)$);
- **(d):** if $Spin(2k_1)=SU(l_{c-1}'+1)\times SU(l_c'+1)$, $(k_1,\ l_{c-1}',\ l_c')=(2,\ 1,\ 1)$ and $Spin(2k_2)=SU(l_{a-1}+1)\times SU(l_a+1)$, $(k_2,\ l_{a-1},\ l_a)=(2,\ 1,\ 1)$, then we have that $a=c$ and $l_i=l_i'$ for all $i=1,\ \cdots,\ a-2$ (the case $(2)-(d)$);
- **(e):** if $Spin(2k_1)\cap Spin(2k_2)\neq\{e\}$, that is, $Spin(2k_1)=SU(l_c'+1)\times SU(2)$, $(k_1,\ l_c')=(2,\ 1)$ and $Spin(2k_2)=SU(l_a+1)\times SU(2)$, $(k_2,\ l_a)=(2,\ 1)$ $(Spin(2k_1)\cap Spin(2k_2)=SU(2))$, then we have that $a=c$ and $l_i=l_i'$ for all $i=1,\ \cdots,\ a-1$ also hold (the case $(2)-(e)$).

In this subsection, we shall prove that these cases (from $(2)-(b)$ to $(2)-(e)$) do not occur. Before we prove this fact, we give the following remark.

Remark 6.2. Because G_s'' acts transitively on the boundary of the slice, we can take the principal isotropy subgroup as $K=\sigma_s^{-1}(SO(2k_s-1))\subset K_1\cap K_2$. Hence, we can regard the subgroups $SO(2m_j)$, $S(U(l_i)\times U(1))\subset K_1,\ K_2$ in Lemma 5.5 as the same subgroup in G by conjugating K_1 and K_2 in G, that is, they are in $K_1\cap K_2$.

Suppose that the case $(2)-(b)$ occurs. Using Lemma 5.5, the above arguments and Remark 6.2, we have that

- $G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a-1}SU(l_i+1)\times SU(4)\times SU(4)$,
- $K_1=\prod_{j=1}^{b}SO(2m_j)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times SU(4)\times S(U(3)\times U(1))$,
- $K_2=\prod_{j=1}^{b}SO(2m_j)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times S(U(3)\times U(1))\times SU(4)$,

where $k_s=3$, $\dim G/K_s=2n-2k_s=2n-6$ and we identify $Spin(2k_s)$ as $SU(4)$. By Lemma 5.6, for the slice representation $\sigma_s:K_s\to SO(2k_s)=SO(6)$, we have that

$$\sigma_s(Spin(2k_s))=\sigma_s(SU(4))=SO(6).$$

Because $\sigma_s(K_s)=SO(6)$ acts transitively on the sphere $S^5\cong SO(6)/SO(5)\cong K_s/K$, the restricted representation $\sigma_s|_{Spin(2k_s)}$ is the double covering $Spin(6)$ to $SO(6)$ by Theorem 5.2. Hence, we have that the following conjugation in G by $K\equiv\sigma_s^{-1}(SO(5))$:

$$\sigma_1^{-1}(SO(5))=\prod_{j=1}^{b}SO(2m_j)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times Spin(5)\times S(U(3)\times U(1))$$

$$\equiv\sigma_2^{-1}(SO(5))=\prod_{j=1}^{b}SO(2m_j)\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times S(U(3)\times U(1))\times Spin(5).$$

Moreover, we see that $Spin(5)\times S(U(3)\times U(1))\subset\sigma_1^{-1}(SO(5))$ and $S(U(3)\times U(1))\times Spin(5)\subset\sigma_2^{-1}(SO(5))$ are conjugate in $SU(4)\times SU(4)\subset G$. In particular, $Spin(5)\times\{I_4\}$ and $S(U(3)\times U(1))\times\{I_4\}$ are conjugate in $SU(4)\times\{I_4\}$. This gives a contradiction, because $\dim Spin(5)=10\neq\dim S(U(3)\times U(1))=9$. Hence, the case $(2)-(b)$ does not occur.

For the other cases $(2)-(c)$ to $(2)-(e)$, we can similarly prove that these cases do not occur. Hence, the case $(2)-(a)$ only occurs for the equation (2).

6.4. The case (1) and the summary. Suppose the equation (1). If $SU(k_1) = SU(k_2)$, then we have that $k_1 = k_2$, $a = c$ and $l_i = l_i'$ for all $i = 1, \cdots, a$ (we call this case the case $(1)-(a)$).

If $SU(k_1) \neq SU(k_2)$, then we can put $SU(l_a+1) = SU(k_2)$, $SU(l_a'+1) = SU(k_1)$, $a = c$ and $l_i = l_i'$ for all $i = 1, \cdots, a-1$ (we call this case the case $(1)-(b)$). Therefore, there are two cases $(1)-(a)$ and $(1)-(b)$ for the equation (1).

Hence, we have the following lemma by the above arguments, Lemma 5.5 and Remark 6.2.

LEMMA 6.3. *If (M, G) is a codimension one extended action of a simply connected torus manifold and its two singular orbits are simply connected torus manifolds, then the compact connected Lie group $G = \prod_{j=1}^{b} SO(2m_j+1) \times \widehat{G}$, and two singular isotropy subgroups $K_1 = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K_1}$ and $K_2 = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K_2}$, where $\widehat{G}$, $\widehat{K_1}$ and $\widehat{K_2}$ are one of the followings:*

(1) (a) $\widehat{G} = \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$,
$\widehat{K_1} = \widehat{K_2} = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k) \times T^1$,
where $\dim G/K_s = 2n - 2k$ *and* $k \geq 1$*;*

(b) $\widehat{G} = \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_1) \times SU(k_2) \times T^1$,
$\widehat{K_1} = \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times SU(k_1) \times S(U(k_2-1) \times U(1)) \times T^1$,
$\widehat{K_2} = \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times S(U(k_1-1) \times U(1)) \times SU(k_2) \times T^1$,
where $\dim G/K_s = 2n - 2k_s$ *and* $k_s \geq 1$*;*

(2) (a) $\widehat{G} = \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$,
$\widehat{K_1} = \widehat{K_2} = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,
where $\dim G/K_s = 2n - 2k$ *and* $k \geq 2$,

where $SO(2m_j)$, $S(U(l_i) \times U(1)) \subset K_1 \cap K_2$.

In order to give a complete classification of G, K_1 and K_2, we study precise structures of slice representations for all cases $(1)-(a)$, $(1)-(b)$ and $(2)-(a)$ in Lemma 6.3, in the next Section 7 and 8.

7. Possibility for K and slice representations

In this section, we shall compute the slice representations for each cases in Lemma 6.3.

First we prepare the following notation:

$$(t_1, \cdots, t_a) = \left(\begin{pmatrix} A_1 & 0 \\ 0 & t_1 \end{pmatrix}, \cdots, \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix} \right) \in \prod_{i=1}^{a} S(U(l_i) \times U(1)),$$

where $A_i \in U(l_i)$ and $\det A_i^{-1} = t_i$.

7.1. Slice representations and K for the case $(1)-(a)$. In this subsection, we study the slice representations for the case $(1)-(a)$ in Lemma 6.3, that is,

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$,
- $K_1 = K_2 = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k) \times T^1$,

where $\dim G/K_s = 2n - 2k$ and $k \geq 1$. By Lemma 5.6, for the slice representation $\sigma_s : K_s \to U(k) \subset SO(2k)$, we have that

$$\sigma_s(S(U(l_i) \times U(1))) = \{I_{2k}\} \text{ or } Z(U(k)) \subset U(k) \text{ for all } i = 1, \cdots, a,$$
$$\sigma_s(SU(k) \times T^1) = U(k).$$

Because $\sigma_s(S(U(l_i) \times U(1))) \subset Z(U(k))$ for all $i = 1, \cdots, a$ and the diagonal subgroup $Z(U(k)) \subset U(k)$ is the abelian group, we have that

$$\sigma_s \begin{pmatrix} A_i & 0 \\ 0 & t_i \end{pmatrix} = t_i^{\omega} I_k \in Z(U(k)) \subset U(k),$$

where $A_i \in U(l_i)$, $t_i = \det A_i^{-1} \in U(1)$, $\omega \in \mathbb{Z}$ and $I_k \in U(k)$ is the identity matrix. Hence, the image of $\prod_{i=1}^a S(U(l_i) \times U(1)) \times SU(k) \times T^1$ is as follows:

$$\sigma_1((t_1, \cdots, t_a), A, t) = t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^\alpha A \in U(k), \tag{7.1}$$

$$\sigma_2((t_1, \cdots, t_a), A, t) = t_1^{\beta_1} \cdots t_a^{\beta_a} t^\beta A \in U(k), \tag{7.2}$$

where $(A, t) \in SU(k) \times T^1$ and $(\alpha_1, \cdots, \alpha_a, \alpha), (\beta_1, \cdots, \beta_a, \beta) \in \mathbb{Z}^{a+1}$.

We next show a principal isotropy subgroup K. Because $\sigma_s(K_s) = U(k)$ acts on the sphere $S^{2k-1} \cong U(k)/U(k-1) \cong K_s/K$ transitively, we have $\sigma_s^{-1}(U(k-1)) \equiv K$ (conjugation in G) for $s = 1, 2$. Hence, the following conjugation holds:

$$\begin{aligned} \sigma_1^{-1}(U(k-1)) &= \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \,\middle|\, x^{-1} = \det X = t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^\alpha \right\} \\ \equiv \quad \sigma_2^{-1}(U(k-1)) &= \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix}, t \right) \,\middle|\, y^{-1} = \det Y = t_1^{\beta_1} \cdots t_a^{\beta_a} t^\beta \right\}, \end{aligned}$$

where $X, Y \in U(k-1)$ and we omit the subgroup $\prod_{j=1}^b SO(2m_j) \subset \ker \sigma_s \subset K$.

In order to analyse relations between $(\alpha_1, \cdots, \alpha_a, \alpha)$ and $(\beta_1, \cdots, \beta_a, \beta)$, we first consider the following subgroups:

$$\sigma_1^{-1}(U(k-1)) \cap T^1 = \{(e, t) \mid t \in T^1, t^\alpha = 1\} \simeq \mathbb{Z}_\alpha \text{ and}$$
$$\sigma_2^{-1}(U(k-1)) \cap T^1 = \{(e, t) \mid t \in T^1, t^\beta = 1\} \simeq \mathbb{Z}_\beta,$$

where $e \in \prod_{j=1}^b SO(2m_j) \times \prod_{i=1}^a S(U(l_i) \times U(1)) \times SU(k)$ is the identity element and $\mathbb{Z}_\omega = \{t \in T^1 \mid t^\omega = 1\}$ $(\omega \in \mathbb{Z})$. Since $\sigma_1^{-1}(U(k-1)) \equiv \sigma_2^{-1}(U(k-1))$ in G, we have $\sigma_1^{-1}(U(k-1)) \cap T^1 = \sigma_2^{-1}(U(k-1)) \cap T^1$. Therefore, we have $\mathbb{Z}_\alpha \simeq \mathbb{Z}_\beta$, i.e., $|\alpha| = |\beta|$. Because the slice representation $\sigma_s : K_s \to U(k) \subset O(2k)$ is equivalent up to conjugate in the target group $O(2k)$, we can put $\alpha \geq 0$ and $\beta \geq 0$. Hence, we have $\alpha = \beta$. Because $\sigma_s(SU(k) \times T^1) = U(k)$, we also have $\alpha = \beta \neq 0$, i.e., $\alpha = \beta \in \mathbb{N}$ (i.e., natural number).

Next we consider the following subgroups:

$$\begin{aligned} &\sigma_1^{-1}(U(k-1)) \cap (S(U(l_a) \times U(1)) \times T^1) \\ = \;& \left\{ \left(e, \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix}, t \right) \,\middle|\, \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix} \in S(U(l_a) \times U(1)),\ t \in T^1,\ t_a^{\alpha_a} t^\alpha = 1 \right\} \end{aligned}$$

and

$$\begin{aligned} &\sigma_2^{-1}(U(k-1)) \cap (S(U(l_a) \times U(1)) \times T^1) \\ = \;& \left\{ \left(e, \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix}, t \right) \,\middle|\, \begin{pmatrix} A_a & 0 \\ 0 & t_a \end{pmatrix} \in S(U(l_a) \times U(1)),\ t \in T^1,\ t_a^{\beta_a} t^\beta = 1 \right\}, \end{aligned}$$

where $e \in \prod_{j=1}^b SO(2m_j) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times SU(k)$ is the identity element. Therefore, by using $\alpha = \beta$ and the above two subgroups (they are conjugate), we have that $t_a^{\alpha_a - \beta_a} = 1$ for all $t_a \in U(1)$. Hence, we have $\alpha_a = \beta_a$. Similarly we can prove the equation $\alpha_i = \beta_i$ for all $i = 1, \cdots, a$. Thus, we have the following lemma.

LEMMA 7.1. *Let $\sigma_s : K_s \to SO(2k) \subset O(2k)$ be a slice representation of the case $(1) - (a)$. Then $\sigma_1 = \sigma_2$, and they are defined by $\sigma_s(SO(2m_j)) = \{e\}$ for all $j = 1, \cdots, b$ and*

$$\sigma_s((t_1, \cdots, t_a), A, t) = t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^\alpha A \in U(k) \subset SO(2k),$$

where $((t_1, \cdots, t_a), A, t) \in \prod_{i=1}^a S(U(l_i) \times U(1)) \times SU(k) \times T^1$, $(\alpha_1, \cdots, \alpha_a) \in \mathbb{Z}^a$, $\alpha \in \mathbb{N}$ and $s = 1, 2$.

Furthermore, we have that the principal isotropy group is $K = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K}$ in the case $(1)-(a)$, where $\widehat{K} \subset \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(U(k-1) \times U(1)) \times T^1$ is as follows:

$$\left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \middle| \ x^{-1} = \det X = t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^{\alpha} \right\}.$$

7.2. Slice representations and K for the case $(1)-(b)$. In this subsection, we study the slice representations for the case $(1)-(b)$ in Lemma 6.3, that is,

- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_1) \times SU(k_2) \times T^1$,
- $K_s = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times SU(k_s) \times S(U(k_r-1) \times U(1)) \times T^1$,

where $s + r = 3$ $(s,\ r \geq 1)$, $\dim G/K_s = 2n - 2k_s$ and $k_s \geq 1$. By Lemma 5.6, we have for the slice representation $\sigma_s : K_s \to U(k_s) \subset SO(2k_s)$:

$$\begin{aligned}
&\sigma_s(S(U(l_i) \times U(1))) = \{I_{2k_s}\} \text{ or } Z(U(k_s)) \subset U(k_s) \text{ for all } i = 1, \cdots, a-1;\\
&\sigma_s(S(U(k_r-1) \times U(1))) = \{I_{2k_s}\} \text{ or } Z(U(k_s)) \subset U(k_s);\\
&\sigma_s(SU(k_s) \times T^1) = U(k_s).
\end{aligned}$$

Because of a similar reason for (7.1) and (7.2) in Section 7.1, the images of $\prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times SU(k_s) \times S(U(k_r-1) \times U(1)) \times T^1$ of slice representations are as follows:

$$\begin{aligned}
&\sigma_1 \left((t_1, \cdots, t_{a-1}),\ A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x^{\alpha_a} t^{\alpha} A \in U(k_1);\\
&\sigma_2 \left((t_1, \cdots, t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ B,\ t \right) = t_1^{\beta_1} \cdots t_{a-1}^{\beta_{a-1}} y^{\beta_a} t^{\beta} B \in U(k_2),
\end{aligned}$$

where $(A,\ t) \in SU(k_1) \times T^1$, $X \in U(k_2-1)$, $\det X^{-1} = x \in U(1)$ and $(B,\ t) \in SU(k_2) \times T^1$, $Y \in U(k_1-1)$, $\det Y^{-1} = y \in U(1)$.

We next show a principal isotropy subgroup K. Because $\sigma_s(K_s) = U(k_s)$ acts on the sphere $S^{2k_s-1} \cong U(k_s)/U(k_s-1) \cong K_s/K$ transitively, we have $\sigma_s^{-1}(U(k_s-1)) \equiv K$ (conjugation in G) for $s = 1,\ 2$. Hence, the following conjugation holds:

$$\begin{aligned}
&\sigma_1^{-1}(U(k_1-1))\\
= &\left\{ \left((t_1, \cdots, t_{a-1}), \begin{pmatrix} C & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \middle| \ c^{-1} = \det C = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x^{\alpha_a} t^{\alpha} \right\}\\
\equiv\ &\sigma_2^{-1}(U(k_2-1))\\
= &\left\{ \left((t_1, \cdots, t_{a-1}), \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} D & 0 \\ 0 & d \end{pmatrix}, t \right) \middle| \ d^{-1} = \det D = t_1^{\beta_1} \cdots t_{a-1}^{\beta_{a-1}} y^{\beta_a} t^{\beta} \right\},
\end{aligned}$$

where $C \in U(k_1-1),\ D \in U(k_2-1)$ and we omit the subgroup $\prod_{j=1}^{b} SO(2m_j) \subset \ker \sigma_s \subset K$. Therefore, we may assume the following equations:

$$c = y;\ d = x, \tag{7.3}$$

because the subgroup $S(U(k_1-1) \times U(1)) \times S(U(k_2-1) \times U(1)) \subset \sigma_1^{-1}(U(k_1-1))$ coincides with the subgroup $S(U(k_1-1) \times U(1)) \times S(U(k_2-1) \times U(1)) \subset \sigma_2^{-1}(U(k_2-1))$ up to conjugate in $SU(k_1) \times SU(k_2) \subset G$.

We can similarly prove $\alpha = \beta > 0$ and $\alpha_i = \beta_i$ for all $i = 1, \cdots, a-1$ like the case $(1)-(a)$ in Section 7.1, by analysing $\sigma_s^{-1}(U(k_s-1)) \cap T^1$ and $\sigma_s^{-1}(U(k_s-1)) \cap (S(U(l_i) \times U(1)) \times T^1)$.

Moreover, we have the following equation using the above $\sigma_1^{-1}(U(k_1-1)) \equiv \sigma_2^{-1}(U(k_2-1))$:

$$t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x^{\alpha_a} t^{\alpha} c = t_1^{\beta_1} \cdots t_{a-1}^{\beta_{a-1}} y^{\beta_a} t^{\beta} d.$$

Therefore, we have the equation $x^{\alpha_a - 1} = y^{\beta_a - 1}$ by using $\alpha_i = \beta_i$ $(i = 1, \cdots, a-1)$, $\alpha = \beta$ and Eq. (7.3). This equation holds for all $x,\ y \in U(1)$; therefore, we have $\alpha_a = \beta_a = 1$. Hence, we have the following lemma.

LEMMA 7.2. *Let $\sigma_s : K_s \to SO(2k_s) \subset O(2k_s)$ be a slice representation of the case $(1)-(b)$. Then σ_s is defined by $\sigma_s(SO(2m_j)) = \{e\}$ for all $j = 1, \cdots, b$ and*

$$\sigma_s\left((t_1, \cdots, t_{a-1}),\ A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t\right) = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x t^{\alpha} A \in U(k_s) \subset SO(2k_s),$$

where $(t_1, \cdots, t_{a-1}) \in \prod_{i=1}^{a-1} S(U(l_i) \times U(1))$, $A \in SU(k_s)$, $t \in T^1$, $X \in U(k_r - 1)$, $x = \det X^{-1}$, $(\alpha_1, \cdots, \alpha_{a-1}) \in \mathbb{Z}^{a-1}$, $\alpha \in \mathbb{N}$ and $s + r = 3$ $(s, r \geq 1)$.

Furthermore, we have that the principal isotropy group is $K = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K}$ in the case $(1)-(b)$, where $\widehat{K} \subset \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times S(U(k_1 - 1) \times U(1)) \times S(U(k_2 - 1) \times U(1)) \times T^1$ is as follows:

$$\left\{ \left((t_1, \cdots, t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t\right) \ \middle|\ t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x y t^{\alpha} = 1 \right\}.$$

7.3. Slice representations and K for the case $(2)-(a)$. In this subsection, we study the slice representations and a principal isotropy subgroup K for the case $(2)-(a)$ in Lemma 6.3, that is,

- $G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times SO(2k)$,
- $K_1 = K_2 = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,

where $\dim G/K_s = 2n - 2k$ and $k \geq 2$. By Lemma 5.6, we have that the slice representation $\sigma_s : K_s \to SO(2k)$ satisfies

$$\sigma_s(SO(2k)) = SO(2k)$$

and other factors are trivial. Therefore we have the following lemma.

LEMMA 7.3. *Let $\sigma_s : K_s \to SO(2k) \subset O(2k)$ be a slice representation of the case $(2)-(a)$. Then $\sigma_1 = \sigma_2$, and σ_s are defined by*

$$\sigma_s(SO(2m_j)) = \sigma_s(S(U(l_i) \times U(1))) = \{I_{2k}\} \text{ for all } j \text{ and } i,$$
$$\sigma_s|_{SO(2k)} = id_{SO(2k)},$$

where $\sigma_s|_{SO(2k)}$ is a restricted representation of σ_s to $SO(2k)$, and $id_{SO(2k)}$ is the identity representation, where $k \geq 2$.

Furthermore, we have that the principal isotropy group in the case $(2)-(a)$ is as follows:

$$K = \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k-1).$$

PROOF. Because $\sigma_s(K_s) = SO(2k)$ acts transitively on the sphere $S^{2k-1} \cong SO(2k)/SO(2k-1) \cong K_s/K$, we see that the restricted representation $\sigma_s|_{SO(2k)}$ is the identity. Because $SO(2m_j)$ and $S(U(l_i) \times U(1))$ (for all $j = 1, \cdots, b$ and $i = 1, \cdots, a$) commute with $SO(2k)$ in K_s and $k \geq 2$, we also have $\sigma_s(SO(2m_j)) = \sigma_s(S(U(l_i) \times U(1))) = \{I_{2k}\}$. Hence, the slice representation σ_s of $(2)-(a)$ is unique for $s = 1, 2$. It follows the first statement of this lemma.

Because $K \equiv \sigma_s^{-1}(SO(2k-1))$, $S^{2k-1} \cong SO(2k)/SO(2k-1) \cong K_s/K$ and $\sigma_s|_{SO(2k)}$ is the identity, we also have the second statement of this lemma. $\square$

8. Attaching maps from ∂X_1 to ∂X_2 and constructions of G-manifolds

In this final section for our classification, we devote to study attaching maps from ∂X_1 to ∂X_2, and construct the G-manifolds for each cases $(1)-(a)$, $(1)-(b)$ and $(2)-(a)$ in Lemma 6.3.

8.1. Preparation. Because of Theorem 2.6, we see $\partial X_1 = \partial X_2 = G/K$, that is, ∂X_s is a codimension 1 principal orbit. Hence, the attaching map f can be taken from the G-equivariant automorphism group $Aut_G(G/K)$ on G/K. As is well known, there is the following isomorphism:

$$Aut_G(G/K) \simeq N(K;G)/K, \tag{8.1}$$

where $N(K;G)$ is the normalizer of K in G (see [**21**]).

Attaching two boundaries ∂X_1 and ∂X_2 by $f \in N(K;G)/K$, we can construct a G-manifold, and such manifold is denoted by

$$M(f) = X_1 \cup_f X_2. \tag{8.2}$$

In order to check that $M(f)$ and $M(f')$ are equivariantly diffeomorphic or not for two attaching maps f and f', the following lemma is useful (see [**50**, Lemma 5.3.1]).

LEMMA 8.1. *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms, where ∂X_i means a boundary of X_i. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions are satisfied:*

(1) *f is G-diffeotopic to f';*
(2) *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1;*
(3) *$f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.*

As in [**15**], we call this lemma the *Uchida's criterion*.

Because of the Uchida's criterion (1), we do not need to compute (8.1). Instead, for attaching maps, we may compute

$$N(K;G)/N(K;G)^o, \tag{8.3}$$

where $N(K;G)^o$ is a connected component of $N(K;G)$.

After the criterion, we construct the G-manifolds explicitly. The following lemma gives one of the construction methods of such G-manifolds.

LEMMA 8.2. *Suppose that a compact, connected Lie group H acts on a compact manifold N with codimension one orbits H/K and two singular isotropy subgroups K_1 and K_2. Then, for a compact, connected Lie group G such that $G \supset H$, a compact, connected manifold*

$$M = G \times_H N$$

has the natural G-action on the first factor G by the left multiplication, and this action has codimension one orbits G/K and two singular isotropy subgroups K_1 and K_2.

From the next subsection 8.2, we analyse the attaching maps and construct G-manifolds, for each cases $(1)-(a)$, $(1)-(b)$ and $(2)-(a)$ in Lemma 6.3.

8.2. The case $(1)-(a)$**.** First we compute the set of attaching maps $N(K;G)/N(K;G)^o$ for the case $(1)-(a)$ (see (8.3)), and prove this case has only one attaching map up to G-diffeomorphism. In this case, $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$, and $K = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K}$ where $\widehat{K}$ is as follows by Lemma 7.1:

$$\left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; x^{-1} = \det X = t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^{\alpha} \right\}.$$

Hence, we have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a} W_{l_i+1}$$

because of $\alpha \in \mathbb{N}$, where $\prod_{j=1}^{b} S(O(2m_j) \times O(1))/SO(2m_j) \simeq \prod_{j=1}^{b} \mathbb{Z}_2$ and

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0. \end{cases}$$

Remark the above $S_2 (\simeq \mathbb{Z}_2)$ is the Weyl group of $SU(2)$.

In order to check the Uchida's criterion (2) or (3), we need to study a tubular neighborhood. By the slice theorem (Theorem 2.7), the tubular neighborhood X_s and its boundary ∂X_s in this case is as follows:

$$
\begin{aligned}
(8.4) \qquad & X_s = G \times_{K_s} D^{2k} \\
&\cong \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} SU(l_i+1) \times_{\prod_{i=1}^{a} S(U(l_i)\times U(1))} D(\mathbb{C}^k) \right); \\
(8.5) \qquad & \partial X_s \cong G \times_{K_s} (K_s/K) \\
&\cong \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} SU(l_i+1) \times_{\prod_{i=1}^{a} S(U(l_i)\times U(1))} S(\mathbb{C}^k) \right),
\end{aligned}
$$

where $D(\mathbb{C}^k)(\cong D^{2k})$ is a disk and $S(\mathbb{C}^k)(\cong S^{2k-1})$ is a sphere in $\mathbb{C}^k$, and $\prod_{i=1}^{a} S(U(l_i)\times U(1))$ acts on $\mathbb{C}^k$ by the scalar multiplication defined by $\sigma'_s(t_1, \cdots, t_a) = t_1^{\alpha_1}\cdots t_a^{\alpha_a}$. Note that the above manifold has the following G-action:

(1) $\prod_{j=1}^{b} S^{2m_j}$ has the canonical transitive $\prod_{j=1}^{b} SO(2m_j+1)$-action;
(2) $\prod_{i=1}^{a} SU(l_i+1)$ has also the canonical transitive $\prod_{i=1}^{a} SU(l_i+1)$-action on itself;
(3) $\mathbb{C}^k$ has an action of $SU(k)\times T^1$ by $\sigma''_s(A,\ t) = At^{\alpha} \in U(k)$ for $(A,\ t)\in SU(k)\times T^1$.

Now we may check the Uchida's criterion (2) or (3).

First, we take the attaching map $f_j(\neq I_{2m_j+1})$ in $\mathbb{Z}_2 \simeq S(O(2m_j)\times O(1))/SO(2m_j) \subset \prod_{j=1}^{b}\mathbb{Z}_2$. Then this attaching map $f_j : \partial X_1 = G/K \to G/K = \partial X_2$ can be regarded as $f_j(gK) = gf_jK$, i.e., the non-trivial $SO(2m_j+1)$-equivariant diffeomorphism on the S^{2m_j} factor in $\partial X_s = G/K$ for $s = 1,\ 2$. Let $I : \partial X_2 = G/K \to G/K = \partial X_1$ be the identity attaching map. We shall prove $I\circ f_j : \partial X_1 = G/K \to G/K = \partial X_1$ is extendable to the equivariant diffeomorphism $X_1 \to X_1$. Remark that we can identify $I\circ f_j : G/K \to G/K$ as $f_j : G/K \to G/K$. Because $f_j = I\circ f_j$ induces an identity map on the $S(\mathbb{C}^k)$ factor in $\partial X_1 = G/K$ (see (8.5)), we have the following commutative diagram:

$$
\begin{array}{ccccc}
\partial X_1 & = & G\times_{K_1} K_1/K & \xrightarrow{\pi} & G/K \\
 & & R_{f_j}\times id \downarrow & & \downarrow f_j \\
\partial X_1 & = & G\times_{K_1} K_1/K & \xrightarrow{\pi} & G/K
\end{array}
$$

where $\pi([g,\ kK]) = gkK$ (equivariantly diffeomorphic), $id : K_1/K = S(\mathbb{C}^k) \to K_1/K = S(\mathbb{C}^k)$ is the identity, R_{f_j} is the product of the map $S^{2m_j}\to S^{2m_j}$ (involution) and identities for the other factors. Now $id : K_1/K \cong S(\mathbb{C}^k) \to S(\mathbb{C}^k) \cong K_1/K$ is extendable to $id : D(\mathbb{C}^k)\to D(\mathbb{C}^k)$. Hence, $R_{f_j}\times id$ is extendable to the equivariant diffeomorphism $G\times_{K_1} D(\mathbb{C}^k) \to G\times_{K_1} D(\mathbb{C}^k)$, i.e., $I\circ f_j : \partial X_1 \to \partial X_1$ is extendable to $X_1\to X_1$. Hence, we have that $M(f_j)\cong M(I)$ for all $j = 1, \cdots, b$ by the Uchida's criterion (Lemma 8.1 (2)).

Next, we take the attaching map $f_i(\neq I_{l_i+1})$ in W_{l_i+1} (when $l_i = 1$ and $\alpha_i = 0$). Because $\alpha_i = 0$, the $S(U(l_i)\times U(1))$ factor acts trivially on $S(\mathbb{C}^k)$ in ∂X_s. Hence, there is a trivial product factor $SU(l_i+1)/S(U(l_i)\times U(1)) \cong \mathbb{C}P(1)\cong S^2$ in ∂X_s ($l_i = 1$), that is, $\partial X_s \cong \mathbb{C}P(1)\times N$ for some manifold N, and the composition of two attaching maps $I\circ f_i : \partial X_s(\cong \mathbb{C}P(1)\times N) \to \partial X_s(\cong \mathbb{C}P(1)\times N)$ can be regarded as an $SU(l_i+1)$-equivariant diffeomorphism on this $\mathbb{C}P(1)\cong S^2$ factor in ∂X_s and trivially on the other factors N. Therefore, we can similarly show that $I\circ f_i$ is extendable to the G-equivariant diffeomorphism map $X_s\to X_s$, and $M(f_i)\cong M(I)$ by the Uchida's criterion (2) or (3).

Hence, the following proposition holds by the above argument and Lemma 7.1.

LEMMA 8.3. *For all the attaching map $f : \partial X_1 \to \partial X_2$ in the case $(1)-(a)$, we have $M(f)\cong M(I)$ where $I : \partial X_2 \to \partial X_1$ is the identity attaching map. In particular, for fixed integers $\alpha_1, \cdots, \alpha_a$ and a natural number α, a G-manifold M which satisfies the case $(1)-(a)$ is unique up to essential isomorphism.*

Hence, the case $(1)-(a)$ is only determined by the integers $\alpha_1, \cdots, \alpha_a$ and the natural number α which appear in the slice representation, up to essential isomorphism. $M(\mathfrak{a}) = M(\alpha_1, \cdots, \alpha_a, \alpha)$ denotes such manifold.

Finally we construct G-manifolds of this case $(1)-(a)$. Using the above X_s in (8.4) and Lemma 8.2 and 8.3, we can easily show the following manifold corresponds with $M(\mathfrak{a})$ (by considering orbits of the G-action on $M(\mathfrak{a})$):

$$\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} SU(l_i+1) \times_{\prod_{i=1}^{a} S(U(l_i)\times U(1))} S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \right),$$

where $S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \simeq S^{2k}$ is a sphere in $\mathbb{C}^k \oplus \mathbb{R}$, $\mathbb{C}_{\mathfrak{a}}^k$ has an $\prod_{i=1}^{a} S(U(l_i) \times U(1))$-action by the representation $\mathfrak{a}(t_1, \cdots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a}$, and the G-action on this manifold is induced from the G-action on X_s (see the lists (1), (2) and (3) below just (8.4)).

Because we classify $(M(\mathfrak{a}), G)$ up to essential isomorphism, we can regard G as $G/\ker$ where ker is a kernel of $(M(\mathfrak{a}), G)$. Hence, we can put $\alpha = 1$. Moreover, we can regard $SU(k)\times T^1$ as $U(k)$ through the finite covering $SU(k) \times T^1 \to SU(k) \times_{\mathbb{Z}_k} T^1 \simeq U(k)$, where $\mathbb{Z}_k$ is the center of $SU(k)$. Hence, we can put our transformation group G as $\prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k)$.

Now we can easily show this manifold is equivariantly diffeomorphic to

$$\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \right),$$

where T^a acts on $\prod_{i=1}^{a} S^{2l_i+1} (\subset \prod_{i=1}^{a} \mathbb{C}^{l_i+1})$ canonically and on $S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \cap \mathbb{C}_{\mathfrak{a}}^k$ through the representation $\mathfrak{a} : T^a \to S^1$ such that $\mathfrak{a}(t_1, \cdots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a}$. Hence, we have the following theorem.

THEOREM 8.4. *If the torus manifold (M, T) has codimension one extended G-action of the case $(1)-(a)$, then (M, G) is essential isomorphic to as follows:*

- $M(\mathfrak{a}) = \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \right)$;
- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k)$,

where $(t_1, \cdots, t_a) \in T^a$ acts on $\mathbb{C}_{\mathfrak{a}}^k$ by the following scalar multiplication:

$$\mathfrak{a}(t_1, \cdots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a},$$

and G acts on $M(\mathfrak{a})$ canonically as follows:

(1) *$\prod_{j=1}^{b} SO(2m_j+1)$ acts on the $\prod_{j=1}^{b} S^{2m_j}$ factor in $M(\mathfrak{a})$;*
(2) *$\prod_{i=1}^{a} SU(l_i+1)$ acts on the $\prod_{i=1}^{a} S^{2l_i}$ factor in $M(\mathfrak{a})$;*
(3) *$U(k)$ acts on the $S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \cap \mathbb{C}^k$ factor in $M(\mathfrak{a})$.*

8.3. The case $(1)-(b)$. First we compute the set of attaching maps $N(K;G)/N(K;G)^o$ for the case $(1)-(b)$ (see (8.3)). In this case, $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_1) \times SU(k_2) \times T^1$ and $K = \prod_{j=1}^{b} SO(2m_j) \times \widehat{K}$ where $\widehat{K}$ is as follows by Lemma 7.2:

$$\left\{ \left((t_1, \cdots, t_{a-1}), \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \middle| \; t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} xyt^{\alpha} = 1 \right\}.$$

Because $\alpha \in \mathbb{N}$, we also have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a-1} W_{l_i+1},$$

where

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0 \end{cases}$$

By a similar arguments in Section 8.2 for the case $(1)-(a)$, we have the following lemma by Lemma 7.2.

LEMMA 8.5. *For all the attaching map $f : \partial X_s \to \partial X_s$ in the case $(1)-(b)$, we have $M(f) \cong M(I)$ where $I : \partial X_s \to \partial X_s$ is the identity attaching map. In particular, for fixed integers $\alpha_1, \cdots, \alpha_{a-1}$ and a natural number α, a G-manifold M which satisfies the case $(1)-(b)$ is unique up to essential isomorphism.*

Hence, the case $(1)-(b)$ is also determined by the integers $\alpha_1, \cdots, \alpha_{a-1}$ and the natural number α which appear in the slice representation, up to essential isomorphism. Let $M(\mathfrak{b})$ denote such manifold.

Finally, we construct the G-manifolds $M(\mathfrak{b})$ of this case $(1)-(b)$. Remark, in this case, tubular neighborhoods X_s are G-diffeomorphic to the following:

$$\prod_{j=1}^{b} S^{2m_j} \times \left\{ \left(\prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_r) \right) \times_{\prod_{i=1}^{a-1} S(U(l_i)\times U(1))\times S(U(k_r-1)\times U(1))} D(\mathbb{C}_{\mathfrak{b}_s}^{k_s}) \right\},$$

where $s+r=3$ $(s,\ r \geq 1)$, $\prod_{i=1}^{a-1} S(U(l_i)\times U(1)) \times S(U(k_r-1)\times U(1))$ acts on $D(\mathbb{C}_{\mathfrak{b}_s}^{k_s})$ by the representation $\sigma_s'(t_1, \cdots t_{a-1},\ x) = t_1^{\alpha_a} \cdots t_{a-1}^{\alpha_{a-1}} x$. The G-action on X_s is as follows: $\prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a-1} SU(l_i+1) \times SU(k_r)$ acts canonically; and $SU(k_s)\times T^1$ acts on $D(\mathbb{C}_{\mathfrak{b}_s}^{k_s})$ by $\sigma_s''(A,\ t) = At^{\alpha}$. By using Lemma 8.2, 8.5 and the above X_s, the following manifold corresponds with $M(\mathfrak{b})$:

$$\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a-1} SU(l_i+1) \times_{\prod_{i=1}^{a-1} S(U(l_i)\times U(1))} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})) \right),$$

where $P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})) \cong \mathbb{C}P(k_1+k_2-1)$ is a complex projective space, $\mathbb{C}_{\mathfrak{b}}^{k_1}$ has an $\prod_{i=1}^{a-1} S(U(l_i)\times U(1))$-action by the representation $\mathfrak{b}(t_1, \cdots, t_{a-1}) = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}}$, and the G-action on this manifold is induced from the G-action on the above X_s.

By the same reasons in the case $(1)-(a)$, we can put $\alpha = 1$. Moreover, we can regard $SU(k_1)\times SU(k_2)\times T^1$ as $S(U(k_1)\times U(k_2))$ through the following finite covering $SU(k_1)\times SU(k_2)\times T^1 \to S(U(k_1)\times U(k_2))$:

$$(A,\ B,\ t) \mapsto \begin{pmatrix} At^{k_2} & 0 \\ 0 & Bt^{-k_1} \end{pmatrix}.$$

By this covering, we also have $(SU(k_1)\times SU(k_2)) \times_{\mathbb{Z}_{k_1k_2}} T^1 \simeq S(U(k_1)\times U(k_2))$. Hence, we can regard G as $\prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1)\times U(k_2))$.

Now we can easily show this manifold is equivariantly diffeomorphic to

$$\prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}) \right),$$

where T^{a-1} acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ canonically and $P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$ through the representation $\mathfrak{b}$. Hence, we have the following theorem.

THEOREM 8.6. *If the torus manifold $(M,\ T)$ has codimension one extended G-action of the case $(1)-(b)$, then $(M,\ G)$ is essential isomorphic to as follows:*

- $M(\mathfrak{b}) = \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})) \right)$;
- $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1)\times U(k_2))$,

where $(t_1, \cdots, t_{a-1}) \in T^{a-1}$ acts on $\mathbb{C}_{\mathfrak{b}}^{k_1}$ by

$$\mathfrak{b}(t_1, \cdots, t_{a-1}) = t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}},$$

and G-action on $M(\mathfrak{b})$ is as follows:

(1) $\prod_{j=1}^{b} SO(2m_j+1)$ *acts on the* $\prod_{j=1}^{b} S^{2m_j}$ *factor in* $M(\mathfrak{b})$*;*
(2) $\prod_{i=1}^{a-1} SU(l_i+1)$ *acts on the* $\prod_{i=1}^{a-1} S^{2l_i+1}$ *factor in* $M(\mathfrak{b})$*;*
(3) $S(U(k_1)\times U(k_2))$ *acts on the* $P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$ *factor in* $M(\mathfrak{b})$.

8.4. The case $(2)-(a)$**.** First we compute the set of attaching maps $N(K;G)/N(K;G)^o$ for the case $(2)-(a)$ (see (8.3)). Because $G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a}SU(l_i+1)\times SO(2k)$ and $K=\prod_{j=1}^{b}SO(2m_j)\times\prod_{i=1}^{a}S(U(l_i)\times U(1))\times SO(2k-1)$ by Lemma 7.3, we have

$$N(K;G)/N(K;G)^o\simeq\prod_{j=1}^{b}\mathbb{Z}_2\times\prod_{i=1}^{a}W_{l_i}\times W,$$

where $W=S(O(2k-1)\times O(1))/SO(2k-1)=\{I_{2k},\ -I_{2k}\}\simeq\mathbb{Z}_2$. Because of the similar reason of the case $(1)-(a)$ and $(1)-(b)$, we have that $M(f)\cong M(I)$ for all attaching maps $f\in\prod_{j=1}^{b}\mathbb{Z}_2\times\prod_{i=1}^{a}W_{l_i}$. Hence, we need to consider the attaching map $f=-I_{2k}$ in W. Remark this attaching map can be taken from the center of $SO(2k)\subset K_s$. Therefore, the following map is well-defined and commute:

$$\begin{array}{ccccc}\partial X_s & = & G\times_{K_s}K_s/K & \xrightarrow{\pi} & G/K\\ & & R_f\times id\downarrow & & \downarrow f\\ \partial X_s & = & G\times_{K_s}K_s/K & \xrightarrow{\pi} & G/K\end{array}$$

where $\pi([g,\ kK])=gkK$, $f(kK)=kfK$ and $(R_f\times id)([g,\ k_1K])=[gf,\ k_sK]$. Here, $id:K_s/K\cong S^{2k-1}\to S^{2k-1}\cong K_s/K$ is extendable to $id:D^{2k}\to D^{2k}$. Therefore, $R_f\times id$ is extendable to the equivariant diffeomorphism $G\times_{K_s}D^{2k}=X_s\to X_s=G\times_{K_s}D^{2k}$. Hence, $M(f)\cong M(I)$ by the Uchida's criterion. Thus, we have the following lemma by Lemma 7.3.

LEMMA 8.7. *For all the attaching map $f:\partial X_s\to\partial X_s$ in the case $(2)-(a)$, we have $M(f)\cong M(I)$ where I is the identity attaching map. In particular, a G-manifold M which satisfies the case $(2)-(a)$ is unique up to essential isomorphism.*

By Lemma 7.3, the tubular neighborhood X_s is as follows:

$$\prod_{j=1}^{b}S^{2m_j}\times\prod_{i=1}^{a}\mathbb{C}P(l_i)\times D(\mathbb{R}^{2k}).$$

Therefore, we have the following theorem by Lemma 8.2 and 8.7.

THEOREM 8.8. *If the torus manifold $(M,\ T)$ has codimension one extended G-action of the case $(2)-(a)$, then $(M,\ G)$ is essential isomorphic to as follows:*

- $M(\mathfrak{c})=\prod_{j=1}^{b}S^{2m_j}\times\prod_{i=1}^{a}\mathbb{C}P(l_i)\times S(\mathbb{R}^{2k}\oplus\mathbb{R})$;
- $G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a}SU(l_i+1)\times SO(2k)$,

where G acts on $M(\mathfrak{c})$ canonically.

As the result, we get the main result Theorem 3.

Because cohomology rings of quasitoric manifolds are generated by the degree 2 elements (see [**7, 11**]), we can easily show the following corollary:

COROLLARY 8.9. If a quasitoric manifold M has a codimension one extended G-action, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a-1}S^{2l_i+1}\times_{T^{a-1}}P(\mathbb{C}_b^{k_1}\oplus\mathbb{C}^{k_2}),\quad\prod_{i=1}^{a-1}SU(l_i+1)\times S(U(k_1)\times U(k_2))\right).$$

9. On moment-angle manifolds

Finally, we observe some relation between a *moment-angle manifold* and our classification results in Corollary 8.9. A moment-angle manifold is defined as follows (see [**5, 7, 11**]). Let P be a simple convex polytope with the set of facets $\mathcal{F}=\{F_1,\ \cdots,\ F_m\}$. For each facet $F_i\in\mathcal{F}$, the 1-dimensional coordinate subgroup of the m-torus $T^{\mathcal{F}}\simeq T^m$ corresponding to F_i is denoted by T_{F_i}. Then assign to every face L the coordinate subtorus

$$T_L=\prod_{F_i\supset L}T_{F_i}\subset T^{\mathcal{F}}.$$

For every point $q \in P$, $L(q)$ denotes the unique face containing q in its relative interior.

Then the *moment-angle manifold* $\mathcal{Z}_P$ is the identification space

$$\mathcal{Z}_P = (T^{\mathcal{F}} \times P)/\sim,$$

where $(t_1,\ p) \sim (t_2,\ q)$ if and only if $p = q$ and $t_1^{-1}t_2 \in T_{L(p)}$.

We remark that moment-angle manifolds $\mathcal{Z}_P$ have natural T^m-actions on their $T^{\mathcal{F}}$ factors. Moreover, there is the following relations between quasitoric manifolds M over P and the moment-angle manifold $\mathcal{Z}_P$ over P (see [**7**, Proposition 6.5]):

PROPOSITION 9.1. There is the subtorus $H \subset T^{\mathcal{F}}$ such that $H \simeq T^{m-n}$ and H acts freely on $\mathcal{Z}_P$, where m is a number of facets in P and $2n$ is a dimension of M. Furthermore, this freely action defies a principal T^{m-n}-bundle $\mathcal{Z}_P \to M$.

In our case, the orbit space of the quasitoric manifold $M(\mathfrak{b}) = \prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$ becomes a product of simplices $\prod_{i=1}^{a-1} \Delta^{l_i} \times \Delta^{k_1+k_2-1}$. For the moment-angle manifold $\mathcal{Z}_{P_1\times P_2}$ satisfies the following relation (see [**7**, Proposition 6.4]):

$$\mathcal{Z}_{P_1\times P_2} = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}.$$

Moreover, the moment-angle manifold over the simplex Δ^n becomes an odd dimensional sphere S^{2n+1} (see [**7**, Example 6.7]). Therefore, the moment angle manifold over $\prod_{i=1}^{a-1} \Delta^{l_i} \times \Delta^{k_1+k_2-1}$ is as follows:

$$\mathcal{Z}(\mathfrak{b}) = \prod_{i=1}^{a-1} S^{2l_i+1} \times S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}),$$

where $S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}) \cong S^{2k_1+2k_2-1}$.

Remark that the number of facets in $\prod_{i=1}^{a-1} \Delta^{l_i} \times \Delta^{k_1+k_2-1}$ and the half of dimension of $M(\mathfrak{b})$ are

$$m = \sum_{i=1}^{a-1}(l_i+1) + k_1 + k_2 \quad \text{and} \quad n = \sum_{i=1}^{a-1} l_i + k_1 + k_2 - 1.$$

Therefore, in our case, $H = T^{a-1} \times S^1$ (see Proposition 9.1). By the definition of $M(\mathfrak{b})$ and $\mathcal{Z}(\mathfrak{b})$, this group H acts on $\mathcal{Z}(\mathfrak{b})$ as follows:

(1) $T^{a-1} \subset H$ acts naturally on the $\prod_{i=1}^{a-1} S^{2l_i+1}$ factor, and acts on the $S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}) \cap \mathbb{C}^{k_1}_{\mathfrak{b}}$ factor through the representation $\mathfrak{b}$;
(2) $S^1 \subset H$ acts only on the $S(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}) = S^{2k_1+2k_2-1} \subset \mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}$ factor naturally.

Moreover, $\mathcal{Z}(\mathfrak{b})$ has the natural action of $G = \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$, with codimension one principal orbits $\prod_{i=1}^{a-1} S^{2l_i+1} \times S^{2k_1-1} \times S^{2k_2-1}$, and two singular orbits $\prod_{i=1}^{a-1} S^{2l_i+1} \times S^{2k_1-1}$ and $\prod_{i=1}^{a-1} S^{2l_i+1} \times S^{2k_2-1}$. Furthermore, we see that this G-action on $\mathcal{Z}(\mathfrak{b})$ is commute with the $H = T^{a-1} \times S^1$-action and it induces the codimension one action on $M(\mathfrak{b})$. We can also show the above fact for quasitoric manifolds with codimension 0 extended G-actions (such quasitoric manifolds are only products of complex projective spaces, see Corollary 2.4). Hence, we have the following theorem from our classification results.

THEOREM 9.2. *If a quasitoric manifold M^{2n} has a codimension 0 or 1 extended G-actions, then its orbit space of T^n-action becomes a product of simplices $\prod_{i=1}^{a} \Delta^{l_i}$, where $\sum_{i=1}^{a} l_i = n$, and there is the following principal T^a-bundle for such quasitoric manifolds:*

$$\mathcal{Z} = \prod_{i=1}^{a} S^{2l_i+1} \to M^{2n}.$$

Furthermore, a codimension 0 and 1 extended G-actions on M can be lifted to G-actions on $\mathcal{Z}$ with codimension 0 and 1 principal orbits respectively. In other wards, all of codimension 0 and 1 extended G-actions on M can be induced from G-actions on $\mathcal{Z}$ with codimension 0 and 1 principal orbits respectively.

Remark 9.3. We can easily see that two singular orbits of $(\mathcal{Z}(\mathfrak{b}),\ G)$ are moment-angle manifolds of two singular orbits of $(M,\ G)$ respectively.

Remark 9.4. According to our classification results in Theorem 3, for torus manifolds with some assumptions, we can easily show that there is a similar principal T^a-bundle like the above moment-angle manifold:

$$\mathcal{T}(\mathfrak{a}) = \prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a} S^{2l_i+1} \times S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \longrightarrow M(\mathfrak{a});$$
$$\mathcal{T}(\mathfrak{b}) = \prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a-1} S^{2l_i+1} \times S(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}) \longrightarrow M(\mathfrak{b});$$
$$\mathcal{T}(\mathfrak{c}) = \prod_{j=1}^{b} S^{2m_j} \times \prod_{i=1}^{a} S^{2l_i+1} \times S(\mathbb{R}^{2k} \oplus \mathbb{R}) \longrightarrow M(\mathfrak{c}),$$

where $S(\mathbb{C}_{\mathfrak{a}}^k \oplus \mathbb{R}) \cong S(\mathbb{R}^{2k} \oplus \mathbb{R}) \cong S^{2k}$. Moreover, all codimension 1 extended G-actions on $M(\mathfrak{a})$, $M(\mathfrak{b})$ and $M(\mathfrak{c})$ can be lifted to G-actions on $\mathcal{T}(\mathfrak{a})$, $\mathcal{T}(\mathfrak{b})$ and $\mathcal{T}(\mathfrak{c})$ (they are denoted by $\mathcal{T}$) with codimension 1 principal orbits respectively. In other wards, all of codimension 1 extended G-actions on M in Theorem **??** can be induced from G-actions on $\mathcal{T}$ with codimension 1 principal orbits.

CHAPTER 3

Classification of torus manifolds with codimension one extended actions

1. Introduction of Chapter 3

This chapter is a continuation of Chapter 1 and 2 devoted to the study of the extended G-actions on torus manifolds $(M^{2n},\ T^n)$. In the first Chapter 1, we classified the homogeneous (*unoriented*) torus manifolds and their transformation groups up to essential isomorphism, where here an unoriented torus manifold means a torus manifold which is not assumed omniorientations. By using classical Lie theory, we proved such torus manifolds are only products of even dimensional spheres and complex projective spaces divided by finite groups. In the second Chapter 2, we classified quasitoric manifolds with codimension 1 extended G-actions up to essential isomorphism and studied relations with moment-angle manifolds. In order to classify such quasitoric manifolds, we classified more general class which involve them, i.e., simply connected torus manifolds with codimension 1 extended G-actions whose two singular orbits are also simply connected torus manifolds. To classify such torus manifolds, we used the part of the Uchida's method in [**50**]. The Uchida's method is the strong method to classify codimension 1 compact Lie group actions up to essential isomorphism. In the case that we apply the Uchida's method to classify codimension 1 actions, we usually need to divide our proof into two cases (compare the method of [**24**, Section 7, 8] and that of [**24**, Section 10, 11]). In particular, for torus manifolds with codimension 1 extended G-actions, (as we mentioned in Chapter 2) we divide our proof into the following two cases: the case that two singular orbits of G-actions are torus manifolds; and the case that one of two singular orbits of G-actions is not a torus manifold. Remark that if a singular orbit is a torus manifold then this is a homogeneous torus manifold (see Lemma 2.1); therefore, we know such singular orbit type by using Chapter 1 (see Lemma 2.2). In the previous Chapter 2, we only studied the former case because of its purpose. However, in general, the latter case also occurs (see Example 3.5 in Chapter 2). The goal of this chapter is to classify all unoriented torus manifolds with codimension one extended G-actions up to essential isomorphism. In particular, in this chapter, we put a special emphasis on the proof of the latter case, because the proof of the former case is almost similar to that of Chapter 2 even if in the case of the classification of such unoriented torus manifolds.

Now we state the main result of this chapter. Put $m_j,\ l_i \in \mathbb{N}\cup\{0\}$ for $j=1,\ \cdots,\ b$ and $i=1,\ \cdots,\ a$, and $\mathcal{A}\subset\prod_{j=1}^{b}\mathbb{Z}_2$. We get the following three theorems (see Section 3.1, 5.1 and 8.1 for detail).

THEOREM 5 (Theorem 3.1). Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and both of them are torus manifolds, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{j=1}^{b} S^{2m_j}\times_{\mathcal{A}} N,\quad \prod_{j=1}^{b} SO(2m_j+1)\times H\right),$$

such that $(N,\ H)$ is one of the following three types:

N	H
$(\prod_{i=1}^{a} S^{2l_i+1}) \times_{T^a} S(\mathbb{C}_{\mathfrak{a}}^{k} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k)$
$\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2})$	$\prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$
$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$

where $k,\ k_1,\ k_2 \in \mathbb{N}$.

THEOREM 6 (Theorem 8.1). Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and one of them is not a torus manifold, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} N, \quad \prod_{j=1}^{b-1} SO(2m_j+1) \times H\right),$$

such that $(N,\ H)$ is one of the following two types:

N	H
$\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k_1) \times SO(2k_2-1)$
$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1) \times SO(2k_2-1)$

where $k_1 \in \mathbb{N}$, $k_2 \geq 2$.

THEOREM 7 (Theorem 5.1). Suppose a torus manifold M has a codimension one extended G-action. If there is an exceptional orbit, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} N, \quad \prod_{j=1}^{b} SO(2m_j+1) \times H\right),$$

such that $(N,\ H)$ is one of the following two types:

N	H
$(\prod_{i=1}^{a} S^{2l_i+1}) \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k_1)$
$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k_1} \oplus \mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1)$

where $k_1 \in \mathbb{N}$.

From the above theorems (see Section 3.1, 5.1 and 8.1 for detail), the finite group $\mathcal{A}$ or $\mathcal{A}\times\mathbb{Z}_2$ acts only on $\prod_{j=1}^{b} S^{2m_j}$ and on the fibre of N (*not* on $\prod_{i=1}^{a} S^{2l_i+1}$ and $\prod_{i=1}^{a} \mathbb{C}P(l_i)$). Therefore, we also have that if an unoriented torus manifold has a codimension one extended G-action, then this manifold is a fibre bundle over the homogeneous torus manifold $\prod_{j=1}^{b} S^{2m_j}/\mathcal{A}' \times \prod_{i=1}^{a} \mathbb{C}P(l_i)$ (see Chapter 1) whose fibre is a complex projective space $\mathbb{C}P(l)$, an even dimensional sphere S^{2m}, or an even dimensional real projective space $\mathbb{R}P(2m)$. Therefore, we can easily show the following corollary:

COROLLARY 8. If a non-singular toric variety or a quasitoric manifold M has a codimension one extended G-action, then $(M,\ G)$ is essentially isomorphic to

$$\left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}), \quad \prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))\right).$$

Remark that the manifold in Corollary 8 is equivariantly diffeomorphic to

$$\prod_{i=1}^{a-1} \mathbb{C}_o^{l_i+1} \times_{(\mathbb{C}^*)^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}),$$

where $\mathbb{C}_o^{l_i+1} = \mathbb{C}^{l_i+1} - \{o\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ (removed the origin).

The organization of this chapter is as follows. In Section 2, we first set up some notation and basic facts from Chapter 1 and 2. Then we know that, in order to classify codimension one extended actions, we need to consider the three cases: the cases (1), (2) and (3). In Section 3, we classify the case (1), i.e., two singular orbits are torus manifolds. The proof of this case is similar to the previous classification in Chapter 2. Hence, we can apply the arguments of the proofs in Chapter 2, and show Theorem 5. In Section 4, we give a preparation for the cases (2) and (3). In particular, we know the following key fact in this section: in order to classify the cases (2) and (3), we may only study the isotropy subgroup $K_2 \subset G$ and its slice representation σ_2 by Remark 4.2, where G/K_1 is the singular orbit of the codimension 1 extended action which is a torus manifold and G/K_2 is the other non-principal orbit. In Section 5, we state the main theorem and give the remark for the case (2), i.e., one of non-principal orbits is an exceptional orbit. In particular, we divide this case into (2)-(a) and (2)-(b). In Section 6 and 7, we study the cases (2)-(a) and (2)-(b) respectively, and prove Theorem 7. In Section 8, we state the main theorem of the case (3), i.e., one of the non-principal orbits is not a torus manifold but a singular orbit, and we divide this case into the cases (3)-(a) and (3)-(b). In Section 9 and 10, we classify the case (3)-(a) and the case (3)-(b) respectively, and prove Theorem 6.

2. Review of the previous chapters

In order to classify codimension 1 extended actions, in this section, we recall the basic facts (see [**6, 17, 50**] and Chapter 1 and 2 for detail).

2.1. Codimension 1 extended G-actions and their singular orbits. Let $(M,\ T)$ be a torus manifold, and let G be a compact, connected Lie group whose maximal torus is T. We next recall the basic facts for extended G-actions of $(M,\ T)$.

Suppose a T^n-action on M^{2n} extends to a G-action on M with codimension 1 principal orbits, i.e., $(2n-1)$-dimensional orbit. Then we call $(M,\ T)$ has a *codimension* 1 *extended G-action*, and such extended G-action on M is denoted by $(M,\ G)$. We will classify such $(M,\ G)$ up to *essential isomorphism* (see Chapter 1 and 2 for detail)

For codimension 1 extended actions of $(M,\ T)$, the following lemma holds (see Lemma 3.1, 3.2 in Chapter 2).

LEMMA 2.1. *Suppose that an (unoriented) torus manifold $(M^{2n},\ T^n)$ has a codimension* 1 *extended G-action. Then a G-orbit G/K_1 of T-fixed points is a singular orbit in $(M,\ G)$, i.e., $\dim G/K_1 < 2n-1$.*

Furthermore, there is some subtorus $T' \subset T$ such that $(G/K_1,\ T')$ is an (unoriented) torus manifold.

PROOF. In the previous Chapter 2, we assume the orientation of the torus manifold. However, we can also apply the proofs of Lemma 3.1, 3.2 in Chapter 2 to the case of unoriented torus manifolds. Hence, we can prove this lemma with a method similar to the proofs of Lemma 3.1, 3.2 in Chapter 2. ☐

In this chapter, we do not need to consider the orientation on the singular orbit G/K_1. Hence, we can directly apply the main result in Chapter 1 to a homogeneous torus manifold G/K_1. Moreover, we get the following lemma using the argument in Section 3.3 in Chapter 2.

LEMMA 2.2. *Suppose that a torus manifold $(M^{2n},\ T^n)$ extends to a codimension* 1 *extended action. Then this codimension* 1 *extended action is essentially isomorphic to $(M,\ G)$ which satisfies that, for its singular orbit G/K_1 with $M^T \cap G/K_1 \neq \emptyset$, the pair $(G,\ K_1)$ is as follows:*

$$\left(\prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b} SO(2m_j+1) \times G_1'',\ \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times G_1''\right),$$

where $\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S}_1 \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1))$ and G_1'' is a product of connected, simple Lie groups and tori.

2.2. Slice representations of G/K_1. In this subsection, we study the slice representation of G/K_1 for the case of unoriented torus manifolds.

Due to Lemma 2.1, G/K_1 is a torus manifold. Therefore, we can put $\dim G/K_1 = 2n-2k_1$ for $k_1 \geq 1$. Let $K_1' = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1$ (see Lemma 2.2), and let $Z(X)$ be the centralizer of a subgroup X in $O(2k_1)$, i.e., $Z(X) = \{g \in O(2k_1) \mid gx = xg \text{ for all } x \in X\}$.

Using a method similar to the proof of Lemma 5.6 in Chapter 2, we have the following lemma for $K_1 = K_1' \times G_1''$ and the slice representation $\sigma_1 : K_1 \to O(2k_1)$.

LEMMA 2.3. *For G_1'' in $K_1 = K_1' \times G_1''$, there are the following two cases:*

$$G_1'' = SU(k_1) \times T^1, \quad \text{and} \quad G_1'' = SO(2k_1).$$

For these two cases, the slice representation $\sigma_1 : K_1 = K_1' \times G_1'' \to O(2k_1)$ satisfies the following list:

	G_1''	$\sigma_1(G_1'')$	$\sigma_1(K_1')$
(1)	$SU(k_1) \times T^1$	$U(k_1) \subset O(2k_1)$	$Z(U(k_1)) \simeq T^1 \subset U(k_1) \subset O(2k_1)$
(2)	$SO(2k_1)$	$SO(2k_1)$	$Z(SO(2k_1)) = \{\pm I_{2k_1}\} \subset O(2k_1)$

where the right list means $\sigma_1(K_1') \subset Z(U(k_1))$ for (1) and $\sigma_1(K_1') \subset Z(SO(2k_1))$ for (2).

Furthermore, the image of σ_1 is in $SO(2k_1)$, i.e., $\sigma_1 : K_1 \to SO(2k_1)$.

REMARK 2.4. We give the following four remarks.

(1) If $G_1'' = SO(2k_1)$ then we can regard $k_1 \geq 2$, because $SO(2) \simeq T^1 = SU(1) \times T^1$ for $k_1 = 1$.
(2) In the previous Chapter 2, $\sigma_1(K_1')$ is always connected, because K_1 is connected. However, in this chapter, K_1 is not always connected (because G_1'' is connected but K_1' might not be connected for $K_1 = K_1' \times G_1''$).
(3) Moreover, in Chapter 2, we can assume $m_j \geq 2$ (where m_j is defined in Lemma 2.2). However, in this chapter, we assume $m_j \geq 1$.
(4) $K_1/K \cong K_1^o/K^o \cong S^{2k_1-1}$ because $\dim G/K_1 = 2n - 2k_1$ and the tubular neighborhood of G/K_1 is $G \times_{K_1} D^{2k_1}$ such that K_1 acts transitively on $\partial D^{2k_1} (\cong K_1/K)$ through σ_1.

In order to apply the same arguments in Chapter 2 to the case $m_j \geq 1$, we need to study the case $m_j = 1$. Let $r_j : \prod_{j=1}^b SO(2m_j + 1) \to SO(2m_j + 1)$ be the natural projection to the j-th factor. Note that we have for $\mathcal{S}_1$ in Lemma 2.2

$$r_j(\mathcal{S}_1) = SO(2m_j) \quad \text{or} \quad S(O(2m_j) \times O(1)) \subset SO(2m_j + 1).$$

The following lemma holds.

LEMMA 2.5. *Assume $m_j = 1$. Then there are the following two cases:*

(1) *if $r_j(\mathcal{S}_1) = SO(2m_j) (= SO(2))$, then we can regard $SO(2m_j + 1) = SO(3)$ as $SU(2) = SU(l_{a+1} + 1)$ up to essential isomorphism;*
(2) *if $r_j(\mathcal{S}_1) = S(O(2m_j) \times O(1)) (= S(O(2) \times O(1)))$, then there is some inclusion $\iota : S(O(2) \times O(1)) \to \mathcal{S}_1$ such that* $\mathrm{Im}\ \iota \cap SO(2m_j) = SO(2m_j)$*, and for the slice representation σ_1 we have*

$$\sigma_1 \circ \iota(S(O(2) \times O(1))) \subset \{\pm I_{2k_1}\} \subset SO(2k_1).$$

PROOF. The first statement can be easily proved using the fact that $SO(3) \approx SU(2)$, i.e., these Lie algebras are same (also see Section 2.2 in Chapter 2). We may only prove the second statement.

Suppose $r_j(\mathcal{S}_1) = S(O(2m_j) \times O(1)) (= S(O(2) \times O(1)))$. Because $\prod_{j=1}^b SO(2m_j) \subset \mathcal{S}_1 \subset \prod_{j=1}^b S(O(2m_j) \times O(1))$, we can easily see that there is some inclusion $\iota : S(O(2) \times O(1)) \to \mathcal{S}_1$ such that $\mathrm{Im}\ \iota \cap SO(2m_j) = SO(2m_j)$. We may only prove that this inclusion ι satisfies $\sigma_1 \circ \iota(S(O(2) \times O(1))) \subset \{\pm I_{2k_1}\} \subset SO(2k_1)$. Let

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(2).$$

Then $J^2 = I_2$. Therefore, by the right list in Lemma 2.3, we have that

$$\sigma_1 \circ \iota \left(\begin{pmatrix} J & 0 \\ 0 & -1 \end{pmatrix} \right) \in \{\pm I_{2k_1}\} \subset U(2k_1) \subset SO(2k_1). \tag{2.1}$$

If $G_1'' = SO(2k_1)$, then the statement $\sigma_1 \circ \iota(S(O(2)\times O(1))) \subset \{\pm I_{2k_1}\}$ is straightforward because of Lemma 2.3. Assume $G_1'' = SU(k_1)\times T^1$. Now we can put

$$\sigma_1 \circ \iota\left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right) = A^{\gamma} \in SO(2) \subset SO(2k_1)$$

for $A \in SO(2)$, where $\gamma \in \mathbb{Z}$ and $A^{\gamma} \in SO(2) \simeq T^1 \subset U(k_1) \subset SO(2k_1)$ for the diagonal subgroup $T^1 \subset U(k_1)$, because $SO(2)$ is the abelian group. Hence, by Eq. (2.1), we have that

$$\begin{aligned} A^{\gamma} &= \sigma_1 \circ \iota\left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma_1 \circ \iota\left(\begin{pmatrix} J & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= \sigma_1 \circ \iota\left(\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= A^{-\gamma}. \end{aligned}$$

It follows that $\gamma = 0$; hence, we have the second statement. □

If $m_j \geq 2$, then we can easily show that $\sigma_1 \circ \iota \circ r_j(\mathcal{S}_1) \subset \{\pm I_{2k_1}\}$. Hence, by the above Lemma 2.5, we can regard $\sigma_1(\mathcal{S}_1) \subset \{\pm I_{2k_1}\}$. This corresponds with the previous fact that $\sigma_s(\prod_{j=1}^{b} SO(2m_j)) = \{e\}$ in Lemma 7.1, 7.2, 7.3 in Chapter 2. Because $\{\pm I_{2k_1}\}$ is the center of $SO(2k_1)$, we can use the same argument in the previous Section 7 in Chapter 2 for analyzing the slice representation σ_1.

2.3. Global structures of codimension 1 extended actions. We next study the global structure of $(M,\ G)$. Even if M is non-oriented, the following structure theorem holds (see [**6**, 8.2 Theorem in Chapter IV]):

THEOREM 2.6. *Let $(M,\ G)$ be a G-manifold M with codimension 1 orbits. If every orbit is principal, then M is a G/K-bundle over S^1. Otherwise, there are two non-principal orbits G/K_1 and G/K_2 such that $K_1 \cap K_2 \supset K$ (K is the principal isotropy subgroup). Furthermore, there exists a closed, invariant tubular neighborhood X_s of G/K_s for $s = 1,\ 2$ such that*

$$M = X_1 \cup X_2 \quad \textit{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2 = G/K,$$

where ∂X_s means the boundary of X_s.

Because of Lemma 2.1, we can assume G/K_1 is a singular orbit and a torus manifold throughout this chapter; moreover, because of Theorem 2.6, there are the following three cases:

(1) G/K_2 is a torus manifold (automatically G/K_2 is a singular orbit);
(2) G/K_2 is an exceptional orbit, namely, $\dim G/K_2 = \dim G/K = 2n-1$;
(3) G/K_2 is not a torus manifold but a singular orbit.

We call the above cases *the case* (1), (2) and (3), respectively. Remark that if M is simply connected, then we do not need to consider the case (2) (see Theorem 2.6 in Chapter 2 or [**50**, Lemma 1.2.1]). From the next section, we start to classify for each above case.

Before we go to the next section, we introduce the following Lemma 2.7 for the attaching map of $f : \partial X_1 \to \partial X_2$. In the final part of the classification, we compute the attaching maps f from ∂X_1 to ∂X_2, and construct the manifold $M(f) = X_1 \cup_f X_2$ by using f. For two attaching maps f and f', we know whether $M(f)$ and $M(f')$ are diffeomorphic or not, by making use of the following Uchida's criterion (see [**50**, Lemma 5.3.1]).

LEMMA 2.7 (Uchida's criterion). *Let $f, f' : \partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms. Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions are satisfied:*

(1) *f is G-diffeotopic to f';*
(2) *$f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1;*
(3) *$f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.*

We remark that this criterion also holds for non-orientable cases.

Using the above criterion (1), we can take the attaching map f from the group $N(K;G)/N(K;G)^o$, where $N(K;G)$ is the normalizer of K in G and $N(K;G)^o$ is its identity component (also see Section 8.1 in Chapter 2).

3. The case (1): two singular orbits are torus manifolds

The goal of this section is to classify the case (1). So, in this section, we assume that the other singular orbit G/K_2 is also a torus manifold. Then we can put $\dim G/K_s = 2n - 2k_s$ for $k_s \geq 1$ and $s = 1,\ 2$. Note that the proof of this case is similarly to that of the previous classification Chapter 2.

3.1. Notations and main theorem. First, we state the main theorem of this section. In order to state it, we prepare some notations. A manifold $X \times_H Y$ denotes a quotient manifold of $X \times Y$ divided by a free H-action. The manifold $\left(\prod_{i=1}^a S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R})$ is the quotient manifold of $\left(\prod_{i=1}^a S^{2l_i+1}\right) \times S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R})$ divided by the following T^a-action: T^a acts on $\prod_{i=1}^a S^{2l_i+1}$ by the a-times product of the scaler S^1-action on $S^{2l_i+1} \subset \mathbb{C}^{l_i+1}$ for $i = 1,\ \cdots,\ a$ (in other words, T^a acts on $\prod_{i=1}^a S^{2l_i+1}$ naturally); and T^a acts on the $2k$-dimensional sphere $S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \subset \mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}$ through the representation $\mathfrak{a} : T^a \to S^1$ such that $\mathfrak{a}(t_1,\ \cdots,\ t_a) \mapsto t_1^{\alpha_1} \cdots t_a^{\alpha_a}$ for some $\alpha_1,\ \cdots,\ \alpha_a \in \mathbb{Z}$, that is, $\mathbb{C}^k_{\mathfrak{a}} \simeq \mathbb{C}^k$ (as a vector space) is the representation space of the representation $\mathfrak{a}$ (S^1 acts on this space by the scaler multiplication). The manifold $\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$ is the projectify of the complex vector bundle $\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} (\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$, where T^{a-1} acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ naturally; and T^{a-1} acts on the representation space $\mathbb{C}^{k_1}_{\mathfrak{b}} \simeq \mathbb{C}^{k_1}$ through the representation $\mathfrak{b} : T^{a-1} \to S^1$, and on $\mathbb{C}^{k_2}$ trivially. A group $\mathcal{A}$ is a subgroup of $\prod_{j=1}^b \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the antipodal involution on S^{2m_j} for $j = 1,\ \cdots,\ b$.

Now we may state the main theorem in this section.

Theorem 3.1. *Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and both of them are torus manifolds, then $(M,\ G)$ is essentially isomorphic to*

$$\left(\prod_{j=1}^b S^{2m_j} \times_{\mathcal{A}} N,\quad \prod_{j=1}^b SO(2m_j+1) \times H\right),$$

such that $(N,\ H)$ is one of the followings:

	N	H
(a)	$\left(\prod_{i=1}^a S^{2l_i+1}\right) \times_{T^a} S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R})$	$\prod_{i=1}^a SU(l_i+1) \times U(k)$
(b)	$\left(\prod_{i=1}^{a-1} S^{2l_i+1}\right) \times_{T^{a-1}} P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$	$\prod_{i=1}^{a-1} SU(l_i+1) \times S(U(k_1) \times U(k_2))$
(c)	$\prod_{i=1}^a \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R})$	$\prod_{i=1}^a SU(l_i+1) \times SO(2k)$

where $\mathcal{A}$ acts on $\prod_{j=1}^b S^{2m_j}$ as the subgroup $\prod_{j=1}^b \mathbb{Z}_2$ and on the fibre of N through the following representations:

- **(a):** $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\} \subset S^1$ *on* $S(\mathbb{C}^k_{\mathfrak{a}} \oplus \mathbb{R}) \cap \mathbb{C}^k_{\mathfrak{a}}$;
- **(b):** $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\} \subset S^1$ *on* $\mathbb{C}^{k_1}_{\mathfrak{b}}$*-factor in* $P(\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2})$;
- **(c):** $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k}\} \subset SO(2k)$ *on* $S(\mathbb{R}^{2k} \oplus \mathbb{R}) \cap \mathbb{R}^{2k}$;

respectively.

Here, G-actions on M are as follows: $\prod SO(2m_j+1)$ and $\prod SU(l_i+1)$ act naturally on $\prod S^{2m_j}$ and $\prod S^{2l_i+1}$, respectively; and $U(k)$, $S(U(k_1) \times U(k_2))$ and $SO(2k)$ act naturally on $\mathbb{C}^k_{\mathfrak{a}}$, $\mathbb{C}^{k_1}_{\mathfrak{b}} \oplus \mathbb{C}^{k_2}$ and $\mathbb{R}^{2k}$, respectively.

From the next subsection, we start to prove the above theorem.

3.2. Singular isotropy subgroups and images of their slice representations. Because G/K_2 is a torus manifold, the pair $(G,\ K_2)$ as well as $(G,\ K_1)$ which is described in Lemma 2.2 satisfies the following property:

$$(G,\ K_2)=\left(\prod_{i=1}^{c}SU(l_i'+1)\times\prod_{j=1}^{d}SO(2m_j'+1)\times G_2'',\ \prod_{i=1}^{c}S(U(l_i')\times U(1))\times\mathcal{S}_2\times G_2''\right),$$

where $\prod_{j=1}^{d}SO(2m_j')\subset\mathcal{S}_2\subset\prod_{j=1}^{d}S(O(1)\times O(2m_j'))$ and G_2'' is a product of connected, simple Lie groups and tori.

By the same argument of Section 6, 7 in Chapter 2 for G/K_1^o and G/K_2^o (where K_s^o is the identity component of K_s), we have

$$(G,\ K_1,\ K_2)=(\prod_{j=1}^{b}SO(2m_j+1)\times\widehat{G},\ \mathcal{S}_1\times\widehat{K_1},\ \mathcal{S}_2\times\widehat{K_2})$$

such that $(\widehat{G},\ \widehat{K_1},\ \widehat{K_2})$ is as follows:

(a): $\widehat{G}=\prod_{i=1}^{a}SU(l_i+1)\times SU(k)\times T^1$,
$\widehat{K_1}=\widehat{K_2}=\prod_{i=1}^{a}S(U(l_i)\times U(1))\times SU(k)\times T^1$,
where $\dim G/K_s=2n-2k$ and $k\geq 1$ ($k_1=k_2=k$);

(b): $\widehat{G}=\prod_{i=1}^{a-1}SU(l_i+1)\times SU(k_1)\times SU(k_2)\times T^1$,
$\widehat{K_1}=\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times SU(k_1)\times S(U(k_2-1)\times U(1))\times T^1$,
$\widehat{K_2}=\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times S(U(k_1-1)\times U(1))\times SU(k_2)\times T^1$,
where $\dim G/K_s=2n-2k_s$ and $k_s\geq 1$;

(c): $\widehat{G}=\prod_{i=1}^{a}SU(l_i+1)\times SO(2k)$,
$\widehat{K_1}=\widehat{K_2}=\prod_{i=1}^{a}S(U(l_i)\times U(1))\times SO(2k)$,
where $\dim G/K_s=2n-2k$ and $k\geq 2$ ($k_1=k_2=k$).

We call the above cases *the case* (1)-(a), (1)-(b) and (1)-(c), respectively.

Next, we consider the slice representation σ_s for $s=1,\ 2$. Because of Lemma 2.3 and 2.5, we have $\sigma_s:K_s=K_s'\times G_s''\to SO(2k_s)$. Moreover, there are the following two cases:

G_s''	$\sigma_s(G_s'')$	$\sigma_s(K_s')$
$SU(k_s)\times T^1$	$U(k_s)\subset O(2k_s)$	$Z(U(k_s))\simeq T^1\subset U(k_s)\subset SO(2k_s)$
$SO(2k_s)$	$SO(2k_s)$	$Z(SO(2k_s))=\{\pm I_{2k_s}\}\subset SO(2k_s)$

where the right list means $\sigma_s(K_s')\subset Z(U(k_s))$ or $Z(SO(2k_s))$.

In order to classify the above each case, we analyze the followings: principal isotropy subgroups $K=\sigma_1^{-1}(SO(2k_1-1))=\sigma_2^{-1}(SO(2k_2-1))$; attaching maps $f:G/K=\partial X_1\to\partial X_2=G/K\in N(K;G)/N(K;G)^o$ (see Section 2.3); and constructions of G-manifolds as $M(f)=X_1\cup_f X_2$, where X_s denotes a G-invariant tubular neighborhood of G/K_s for $s=1,\ 2$.

3.3. The case (1)-(a). In this subsection, we study the case (1)-(a), that is,

- $G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a}SU(l_i+1)\times SU(k)\times T^1$,
- $K_1=\mathcal{S}_1\times\prod_{i=1}^{a}S(U(l_i)\times U(1))\times SU(k)\times T^1$,
- $K_2=\mathcal{S}_2\times\prod_{i=1}^{a}S(U(l_i)\times U(1))\times SU(k)\times T^1$

where $\prod_{j=1}^{b}SO(2m_j)\subset\mathcal{S}_s\subset\prod_{j=1}^{b}S(O(2m_j)\times O(1))$, $\dim G/K_s=2n-2k$, and $k\geq 1$ for $s=1,\ 2$.

In order to know the precise structure of K, we analyze slice representations $\sigma_s:K_s\to SO(2k)$. By Section 3.2, the slice representation $\sigma_s:K_s\to U(k)\subset SO(2k)$ satisfies the followings:

$$\sigma_s(S(U(l_i)\times U(1)))=\{I_{2k}\}\text{ or }Z(U(k))=T^1\subset U(k)\text{ for all }i=1,\ \cdots,\ a;$$
$$\sigma_s(SU(k)\times T^1)=U(k),$$

where $Z(U(k))=T^1\subset U(k)$ is the center of $U(k)$, i.e., the diagonal subgroup. Because $\sigma_s(\mathcal{S}_s^o)=\sigma_s(\prod_{j=1}^{b}SO(2m_j))=\{1\}\subset T^1\subset U(k)$ (by making use of Lemma 2.5) and $\mathcal{S}_s/\prod_{j=1}^{b}SO(2m_j)\simeq$

$\mathcal{A}_s \subset \prod_{j=1}^{b} \mathbb{Z}_2$ (i.e., $\mathcal{A}_s$ is a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$ generated by antipodal involutions), we have that

$$\sigma_s(\mathcal{S}_s) = \sigma_{\mathbb{C}}(\mathcal{A}_s) \subset \{\pm 1\} \subset T^1 \subset U(k),$$

where $\sigma_{\mathbb{C}} : \mathcal{A}_s \to \{\pm 1\}$. If $\sigma_s(\mathcal{S}_s) = \{1\}$ then we can apply the same argument in Chapter 2. So we assume $\sigma_s(\mathcal{S}_s) = \{\pm 1\}$.

Now, the principal isotropy subgroup K is as follows ($X \equiv Y$ means that two groups X and Y are conjugate in G):

$$\begin{aligned} K &\equiv \sigma_1^{-1}(U(k-1)) = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_1(A) x t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^{\alpha} = 1 \right\} \\ &\equiv \sigma_2^{-1}(U(k-1)) = \left\{ \left(B,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ t \right) \ \middle|\ \sigma_2(B) y t_1^{\beta_1} \cdots t_a^{\beta_a} t^{\beta} = 1 \right\}, \end{aligned}$$

where $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$; $X,\ Y \in U(k-1)$, $x,\ y \in T^1$ such that $x \det X = y \det Y = 1$; and

$$(t_1,\ \cdots,\ t_a) = \left(\begin{pmatrix} U_1 & 0 \\ 0 & t_1 \end{pmatrix},\ \cdots,\ \begin{pmatrix} U_a & 0 \\ 0 & t_a \end{pmatrix} \right) \in \prod_{i=1}^{a} S(U(l_i) \times U(1)).$$

First we take $A = B = e \in \mathcal{S}_1 \cap \mathcal{S}_2$ (the identity elements in $\prod_{j=1}^{b} SO(2m_j+1)$). Then $\sigma_1(A) = 1 = \sigma_2(B)$. Hence, by using the the same argument of Section 7.1 in Chapter 2 for $A = B = e$, we have that

$$\alpha_i = \beta_i \in \mathbb{Z} \text{ for } i = 1,\ \cdots,\ a, \quad \text{and} \quad \alpha = \beta \in \mathbb{N}.$$

Moreover, if $k = 1$ then we can take $\alpha = \beta = 1$ (up to essential isomorphism). Next, we consider the following part in K (if $k \geq 2$):

$$\begin{aligned} \mathcal{R}_1 &= \left\{ \left(A,\ e,\ \begin{pmatrix} J & 0 \\ 0 & \sigma_1(A) \end{pmatrix},\ 1 \right) \ \middle|\ A \in \mathcal{S}_1 \right\} \\ &\equiv \left\{ \left(B,\ e,\ \begin{pmatrix} J & 0 \\ 0 & \sigma_2(B) \end{pmatrix},\ 1 \right) \ \middle|\ B \in \mathcal{S}_2 \right\} = \mathcal{R}_2, \end{aligned}$$

where $e \in \prod_{i=1}^{a} S(U(l_i) \times U(1))$ is the identity element and $J \in U(k-1)$ such that $J = I_{k-1}$ (if $\sigma_s(C_s) = 1$) or $\det J = -1$ with $J^2 = I_{k-1}$ (if $\sigma_s(C_s) = -1$) for $s = 1,\ 2$ and $C_1 = A,\ C_2 = B$. For $\mathcal{R}_1$ and $\mathcal{R}_2$, we can easily show the following isomorphisms (not identity in G):

$$\mathcal{S}_1 \simeq \mathcal{R}_1, \quad \mathcal{S}_2 \simeq \mathcal{R}_2.$$

Because $\mathcal{R}_1 \equiv \mathcal{R}_2$, we see that

$$\mathcal{S}_1 = \mathcal{S}_2 \subset K_1 \cap K_2$$

by the definition of $\mathcal{S}_s$, i.e., we can regard $K_1 = K_2$ in G. If $k = 1$, then we can apply the same above argument by taking the set $\{(A,\ e,\ \sigma_1(A))\} \equiv \{(B,\ e,\ \sigma_2(B))\}$, and we can regard $K_1 = K_2$. Therefore, by using the above argument, we have $\sigma_1 = \sigma_2$. Moreover, K is as follows:

$$K = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_1(A) x t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^{\alpha} = 1 \right\}$$

As the result of the above argument, the tubular neighborhood $X_s = G \times_{K_s} D^{2k}$ ($s = 1,\ 2$) is equivariantly diffeomorphic to the following manifold:

$$\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} D(\mathbb{C}^k_{\mathfrak{a}}) \right), \tag{3.1}$$

where T^a acts on $\prod_{i=1}^{a} S^{2l_i+1}$ naturally, and on $D(\mathbb{C}^k_{\mathfrak{a}})$ by $\mathfrak{a} : (t_1,\ \cdots,\ t_a) \mapsto t_1^{\alpha_1} \cdots t_a^{\alpha_a}$; and $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_2 (\simeq \mathcal{S}_1/\mathcal{S}_1^o = \mathcal{S}_2/\mathcal{S}_2^o)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$, and on $D(\mathbb{C}^k_{\mathfrak{a}})$ by $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$ (induced by σ_1). Here, $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k) \times T^1$

acts on this manifold as follows: $\prod_{j=1}^{b} SO(2m_j+1)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ naturally; $\prod_{i=1}^{a} SU(l_i+1)$ acts on $\prod_{i=1}^{a} S^{2l_i+1}$ naturally; and $SU(k)\times T^1$ acts on $D(\mathbb{C}^k_{\mathfrak{a}})$ by $(A,\ t)\mapsto At^{\alpha}\in U(k)$.

Next, we analyze attaching maps $f:\partial X_1\to\partial X_2$. Because of $\alpha\in\mathbb{N}$, we have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^{b}\mathbb{Z}_2\times\prod_{i=1}^{a}W_{l_i+1},$$

where $\prod_{j=1}^{b} S(O(2m_j)\times O(1))/SO(2m_j)\simeq\prod_{j=1}^{b}\mathbb{Z}_2$ and

$$W_{l_i+1}=\begin{cases}\{I_{l_i+1}\} & \text{if } l_i\geq 2 \text{ or } \alpha_i\neq 0\\ S_2 & \text{if } l_i=1 \text{ and } \alpha_i=0.\end{cases}$$

Here, the above $S_2(\simeq\mathbb{Z}_2)$ is the Weyl group of $SU(2)$. Therefore, by the same argument as Section 8.2 in Chapter 2, we can show that $I\circ f:\partial X_1\to\partial X_1$ is extendable to the equivariant map $X_1\to X_1$, where $I:G/K=\partial X_2\to\partial X_1=G/K$ is the identity attaching map. Hence, $M(f)\cong M(I)$ for all attaching maps f by the Uchida's criterion (2). As the result, we have that if the case (1) holds for $(M,\ G)$, then such $(M,\ G)$ is only determined by the representations $\mathfrak{a}:T^a\to T^1$ and $\sigma_{\mathbb{C}}:\mathcal{A}\to\{\pm1\}$ (up to essential isomorphism), i.e., if we fix the representations $\mathfrak{a}$ and $\sigma_{\mathbb{C}}$, then $(M,\ G)$ is unique up to essential isomorphism. Thus, we have that such manifold M and G are as follows (up to essential isomorphism):

$$M=\prod_{j=1}^{b}S^{2m_j}\times_{\mathcal{A}}\left(\prod_{i=1}^{a}S^{2l_i+1}\times_{T^a}S(\mathbb{C}^k_{\mathfrak{a}}\oplus\mathbb{R})\right),$$
$$G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a}SU(l_i+1)\times U(k),$$

by computing the orbits G/K, G/K_1 and G/K_2. This corresponds with the first case of Theorem 3.1.

3.4. The case (1)-(b). In this subsection, we study the case (1)-(b), that is,

- $G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a-1}SU(l_i+1)\times SU(k_1)\times SU(k_2)\times T^1$,
- $K_1=\mathcal{S}_1\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times SU(k_1)\times S(U(k_2-1)\times U(1))\times T^1$,
- $K_2=\mathcal{S}_2\times\prod_{i=1}^{a-1}S(U(l_i)\times U(1))\times S(U(k_1-1)\times U(1))\times SU(k_2)\times T^1$,

where $\prod_{j=1}^{b}SO(2m_j)\subset\mathcal{S}_s\subset\prod_{j=1}^{b}S(O(2m_j)\times O(1))$, $\dim G/K_s=2n-2k_s$ and $k_s\geq 1$ for $s=1,\ 2$.

In order to know the precise structure of K, we analyze the slice representation $\sigma_s:K_s\to SO(2k_s)$. The following property can be proved similarly to that of the case (1)-(a) (see Section 3.2, 3.3):

$$\begin{aligned}
&\sigma_s(S(U(l_i)\times U(1)))=\{I_{2k_s}\}\text{ or }Z(U(k_s))\subset U(k_s)\text{ for all } i=1,\ \cdots,\ a-1;\\
&\sigma_s(S(U(k_r-1)\times U(1)))=\{I_{2k_s}\}\text{ or }Z(U(k_s))\subset U(k_s);\\
&\sigma_s(SU(k_s)\times T^1)=U(k_s);\\
&\sigma_s(\mathcal{S}_s)=\sigma_{\mathbb{C}}(\mathcal{A}_s)\subset\{\pm1\},
\end{aligned}$$

where $s+r=3$ and $s,\ r\geq 1$. Therefore, K satisfies that

$$\begin{aligned}
K &\equiv \sigma_1^{-1}(U(k_1-1))\\
&= \left\{\left(A,\ (t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix}C&0\\0&c\end{pmatrix},\ \begin{pmatrix}X&0\\0&x\end{pmatrix},\ t\right)\ \middle|\ \sigma_1(A)=t_1^{\alpha_1}\cdots t_{a-1}^{\alpha_{a-1}}x^{\alpha_a}t^{\alpha}c\right\}\\
&\equiv \sigma_2^{-1}(U(k_2-1))\\
&= \left\{\left(B,\ (t_1,\ \cdots,\ t_{a-1}),\ \begin{pmatrix}Y&0\\0&y\end{pmatrix},\ \begin{pmatrix}D&0\\0&d\end{pmatrix},\ t\right)\ \middle|\ \sigma_2(B)=t_1^{\beta_1}\cdots t_{a-1}^{\beta_{a-1}}y^{\beta_a}t^{\beta}d\right\},
\end{aligned}$$

where $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$, $(t_1, \cdots, t_{a-1}) \in \prod_{i=1}^{a-1} S(U(l_i) \times U(1))$, and $C,\ Y \in U(k_1 - 1)$, $D,\ X \in U(k_2 - 1)$, $x,\ y,\ c,\ d \in T^1$ such that $x \det X = y \det Y = c \det C = d \det D = 1$.

By the same argument as Section 7.2 in Chapter 2 for $\sigma_1(A) = 1 = \sigma_2(B)$, we see that

$$\alpha_i = \beta_i \in \mathbb{Z} \text{ for } i = 1, \cdots, a-1, \quad \text{and} \quad \alpha_a = \beta_a = 1 \quad \text{and} \quad \alpha = \beta \in \mathbb{N}.$$

Moreover, if $k_1 = 1$ or $k_2 = 1$, then we have that $\alpha = 1$ or $\beta = 1$, respectively. Using the method similar to that demonstrated in Section 3.3, we can also prove

$$\mathcal{S}_1 = \mathcal{S}_2 \subset K_1 \cap K_2,$$

and $\sigma_1|_{\mathcal{S}_1} = \sigma_2|_{\mathcal{S}_2}$ (i.e., the restricted representations are the same representations). Hence, K is as follows:

$$K = \left\{ \left(A,\ (t_1, \cdots, t_{a-1}),\ \begin{pmatrix} Y & 0 \\ 0 & y \end{pmatrix},\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_1(A) t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}} x y t^{\alpha} = 1 \right\}.$$

As the result of the above argument, the tubular neighborhood $X_s = G \times_{K_s} D^{2k_s}$ $(s = 1,\ 2)$ is equivariantly diffeomorphic to the following manifold:

$$\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} D(\mathbb{C}_{\mathfrak{b}}^{k_s}) \right) \times \mathbb{C}P(k_r - 1),$$

where T^{a-1} acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ naturally and on $D(\mathbb{C}_{\mathfrak{b}}^{k_s})$ by $\mathfrak{b} : (t_1, \cdots, t_{a-1}) \mapsto t_1^{\alpha_1} \cdots t_{a-1}^{\alpha_{a-1}}$, $\mathcal{A}(= \mathcal{A}_s \simeq \mathcal{S}_s/\mathcal{S}_s^o)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$ and on $D(\mathbb{C}_{\mathfrak{b}}^{k_s})$ by $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$, and $s + r = 3$ for $s,\ r \geq 1$. Here, $G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a-1} SU(l_i + 1) \times SU(k_s) \times T^1 \times SU(k_r)$ acts on this manifold as follows: $\prod_{j=1}^{b} SO(2m_j + 1)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ naturally; $\prod_{i=1}^{a-1} SU(l_i + 1)$ acts on $\prod_{i=1}^{a-1} S^{2l_i+1}$ naturally; $SU(k_s) \times T^1$ acts on $D(\mathbb{C}_{\mathfrak{b}}^{k_s})$ by $(A,\ t) \mapsto At^{\alpha}$; and $SU(k_r)$ acts on $\mathbb{C}P(k_r - 1)$ naturally.

Next, we analyze attaching maps $f : \partial X_1 \to \partial X_2$. In this case, we have

$$N(K; G)/N(K; G)^o \simeq \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a-1} W_{l_i+1}$$

where $\prod_{j=1}^{b} S(O(2m_j) \times O(1))/SO(2m_j) \simeq \prod_{j=1}^{b} \mathbb{Z}_2$ and

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0. \end{cases}$$

This is the same as the case (1)-(a). Therefore, we can show the Uchida's criterion (2) for $I \circ f$. Hence, by using the similar argument to that demonstrated in Section 3.3, we have that $(M,\ G)$ which satisfies the case (1)-(b) is unique up to essential isomorphism if we fix the representations $\mathfrak{b}$ and $\sigma_{\mathbb{C}}$. Thus, we have that M and G in the case (1)-(b) are as follows (up to essential isomorphism):

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a-1} S^{2l_i+1} \times_{T^{a-1}} P(\mathbb{C}_{\mathfrak{b}}^{k_1} \oplus \mathbb{C}^{k_2}) \right),$$

$$G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a-1} SU(l_i + 1) \times S(U(k_1) \times U(k_2)).$$

This corresponds with the second case of Theorem 3.1.

3.5. The case (1)-(c). In this subsection, we study the case (1)-(c), that is,

- $G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times SO(2k)$,
- $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,
- $K_2 = \mathcal{S}_2 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k)$,

where $\prod_{j=1}^{b} SO(2m_j) \subset \mathcal{S}_s \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1))$, $\dim G/K_s = 2n - 2k$ and $k \geq 2$ for $s = 1,\ 2$.

In order to know the precise structure of K, we analyze the slice representation $\sigma_s : K_s \to SO(2k)$ as well as the cases (1) and (2). By Section 3.2, we have the followings for the slice representation $\sigma_s : K_s \to SO(2k)$:

$$\begin{aligned}
&\sigma_s(S(U(l_i) \times U(1))) = \{I_{2k}\} \subset SO(2k) \text{ for all } i = 1,\ \cdots,\ a;\\
&\sigma_s(SO(2k)) = SO(2k);\\
&\sigma_s(\mathcal{S}_s) = \sigma_{\mathbb{R}}(\mathcal{A}_s) \subset \{\pm I_{2k}\}.
\end{aligned}$$

Therefore, we have that K is the following subgroup:

$$\begin{aligned}
K &\equiv \sigma_1^{-1}(SO(2k-1)) = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} E & 0 \\ 0 & e \end{pmatrix} \right) \,\middle|\, \sigma_1(A)e = 1 \right\}\\
&\equiv \sigma_2^{-1}(SO(2k-1)) = \left\{ \left(B,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} F & 0 \\ 0 & f \end{pmatrix} \right) \,\middle|\, \sigma_2(B)f = 1 \right\},
\end{aligned}$$

where $A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$, $(t_1,\ \cdots,\ t_a) \in \prod_{i=1}^{a} S(U(l_i) \times U(1))$, and $E,\ F \in O(2k-1)$, $e,\ f \in O(1)$ such that $e \det E = f \det F = 1$.

Next, we consider the following part in K:

$$\begin{aligned}
\mathcal{R}_1 &= \left\{ \left(A,\ (1,\ \cdots,\ 1),\ \begin{pmatrix} \sigma_1(A)I_{2k-1} & 0 \\ 0 & \sigma_1(A) \end{pmatrix} \right) \,\middle|\, \sigma_1(A) = \pm 1 \right\}\\
&\equiv \left\{ \left(B,\ (1,\ \cdots,\ 1),\ \begin{pmatrix} \sigma_2(B)I_{2k-1} & 0 \\ 0 & \sigma_2(B) \end{pmatrix} \right) \,\middle|\, \sigma_2(B) = \pm 1 \right\} = \mathcal{R}_2.
\end{aligned}$$

By using $\mathcal{R}_1$, $\mathcal{R}_2$ and the method similar to that demonstrated in Section 3.3 (the case (1)-(a)), we can regard $K_1 = K_2$ in G and $\sigma_1 = \sigma_2$. Hence, K is as follows:

$$K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \left\{ \left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k-1) \times O(1)) \,\middle|\, \sigma_1(A)x = 1 \right\}$$

As the result of the above argument, the tubular neighborhood $X_s = G \times_{K_s} D^{2k}$ $(s = 1,\ 2)$ is equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} D(\mathbb{R}^{2k}) \right), \tag{3.2}$$

where $\mathcal{A}(= \mathcal{A}_s \simeq \mathcal{S}_s/\mathcal{S}_s^o)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$ and on $D(\mathbb{R}^{2k})$ by $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k}\}$. Here, $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k)$ acts on this manifold as follows: $\prod_{i=1}^{a} SU(l_i+1)$ acts on $\prod_{i=1}^{a} \mathbb{C}P(l_i)$ naturally; $\prod_{j=1}^{b} SO(2m_j+1)$ acts on $\prod_{j=1}^{b} S^{2m_j}$ naturally; and $SO(2k)$ acts on $D(\mathbb{R}^{2k})$ naturally.

Next, we analyze attaching maps $f : \partial X_1 \to \partial X_2$. In this case, we have

$$N(K;G)/N(K;G)^o \simeq \prod_{j=1}^{b} \mathbb{Z}_2 \times \prod_{i=1}^{a} W_{l_i} \times W,$$

where $W = S(O(2k-1) \times O(1))/SO(2k-1) = \{I_{2k},\ -I_{2k}\} \simeq \mathbb{Z}_2$, and

$$W_{l_i+1} = \begin{cases} \{I_{l_i+1}\} & \text{if } l_i \geq 2 \text{ or } \alpha_i \neq 0 \\ S_2 & \text{if } l_i = 1 \text{ and } \alpha_i = 0. \end{cases}$$

Therefore, by the same argument of [?, Section 8.4], we can show the Uchida's criterion (2) for $I \circ f$. Hence, by using the similar argument to that demonstrated in Section 3.3, we have that $(M,\ G)$

which satisfies the case (1)-(c) is unique up to essential isomorphism if we fix the representation $\sigma_{\mathbb{R}}$. Thus, we have that M and G in the case (1)-(c) are as follows (up to essential isomorphism):

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k} \oplus \mathbb{R}) \right),$$

$$G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times SO(2k).$$

This corresponds with the third case of Theorem 3.1.

4. Preliminary for the cases (2) and (3)

In this section, we prepare to classify the other cases, i.e., G/K_1 is a torus manifold but G/K_2 is not a torus manifold (the cases (2) and (3)). Henceforth, we assume that G/K_2 is not a torus manifold. In this case, $G/K_2 \cap M^T = \emptyset$, by using Lemma 2.1. It follows that $T \subset G$ (a maximal torus) but $T \not\subset K_2^o$, i.e., rank $G >$ rank K_2^o.

We first prove that $\dim G/K_2^o = \dim G/K_2$ is odd. Because G/K_1 is a torus manifold, we can apply the same argument which is demonstrated in Section 3 to get K (also see Section 4.1). By using K's in Section 3, we can easily show the following formula:

$$\text{rank } G = \text{rank } K_1^o = \text{rank } K^o + 1.$$

Let $T' \subset K^o$ be a maximal torus such that $T' \subset T$. If T' is not a maximal torus in K_2^o, then we have rank $K_2^o =$ rank $K^o + 1$ by using $T' \subset K^o \subset K_2^o \subset G$ and the above formula. However, this gives a contradiction to rank $G >$ rank K_2^o. Therefore, we have rank $K_2^o =$ rank K^o. Due to [**16**, Theorem 1.1], we also have $\dim K_2^o/K^o$ is even. Thus, by considering the fibration: $K_2^o/K^o \to G/K^o \to G/K_2^o$ and $\dim G/K^o = 2n-1$, we have that $\dim G/K_2^o = \dim G/K_2$ is odd. Hence, we can put

$$\dim G/K_2 = 2n - 2k_2 + 1$$

for $k_2 \geq 1$. Remark that if $k_2 = 1$ then this is in the case (2): G/K_2 is an exceptional orbit, otherwise this is in the case (3): G/K_2 is a singular orbit.

Since our G-action on M has codimension 1 orbits, we have that

$$K_2/K \cong S^{2k_2-2}$$

in each case.

In summary, we have the following lemma.

Lemma 4.1. *Suppose a torus manifold $(M,\ T)$ has codimension 1 extended G-actions. If G/K_2 is not a torus manifold, then*

$$\dim G/K_2 = 2n - 2k_2 + 1, \quad K_2/K \cong S^{2k_2-2},$$

for $k_2 \geq 1$.

4.1. Structures of G/K_1 and their tubular neighborhoods of the cases (2) and (3). The main part to classify the cases (2) and (3) is to determine the group K_2 and its inclusion $K_2 \subset G$. From the next section, We will determine $K_2 \subset G$ by making use of the relation $K \subset K_2 \subset G$ and the classification result of the transitive action on S^{2k_2-2}. In this subsection, we recall G, K_1 and K; and in the next subsection, we give an important remark for the attaching map.

Let $\dim G/K_1 = 2n - 2k_1$ for $k_1 \geq 1$. By Lemma 2.2, 2.3 and Remark 2.4, we have $(G,\ K_1)$ as the following two cases:

- **(a):** $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SU(k_1) \times T^1$, and $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k_1) \times T^1$ for $k_1 \geq 1$;
- **(b):** $G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times SO(2k_1)$, and $K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k_1)$ for $k_1 \geq 2$.

We call them *the case* (a) and (b) respectively.

4.1.1. *The case (a).* For the principal isotropy subgroup K of the case (a), we can easily show the following by using the same argument in Section 3.3:

$$K = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_1(A) x t_1^{\alpha_1} \cdots t_a^{\alpha_a} t^{\alpha} = 1 \right\},$$

where $A \in \mathcal{S}_1$, $(t_1,\ \cdots,\ t_a) \in \prod_{i=1}^{a} S(U(l_i) \times U(1))$, $t \in T^1$ and $X \in U(k_1 - 1)$ such that $x \det X = 1$. Here, $\sigma_1(A) \subset \{\pm 1\} \subset S^1$ and $(\alpha_1,\ \cdots,\ \alpha_a,\ \alpha) \in \mathbb{Z}^a \times \mathbb{N}$. If $k_1 = 1$, then we can take $\alpha = 1$. Therefore, a tubular neighborhood $X_1 = G \times_{K_1} D^{2k_1}$ of the case (a) is equivariantly diffeomorphic to the following manifold defined in (3.1):

$$\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}_1} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} D(\mathbb{C}_{\mathfrak{c}}^{k_1}) \right). \tag{4.1}$$

Remark that from this section we write the representation $\mathfrak{a} : T^a \to S^1$ as $\mathfrak{c}$.

4.1.2. *The case (b).* For the principal isotropy subgroup K of the case (b), we can easily show the following by using the same argument in Section 3.5:

$$\begin{aligned} K &= \prod_{i=1}^{a} S(U(l_i) \times U(1)) \\ &\times \left\{ \left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k_1 - 1) \times O(1)) \ \middle|\ \sigma_1(A) x = 1 \right\}. \end{aligned}$$

Hence, a tubular neighborhood $X_1 = G \times_{K_1} D^{2k_1}$ of the case (b) is equivariantly diffeomorphic to the following manifold defined in (3.2):

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}_1} D(\mathbb{R}^{2k_1}) \right). \tag{4.2}$$

4.2. Important remark for the attaching maps of the cases (2) and (3). Before we go to the next section, we give the following important remark for the attaching maps $f : G/K = \partial X_1 \to \partial X_2 = G/K$.

REMARK 4.2. In each above case (a) and (b), we see that $N(K;G)/N(K;G)^o$ is the same as one of the cases (1)-(a), (1)-(b) and (1)-(c). Therefore, we have already proved the Uchida's criterion (2) for $I \circ f$ in Section 3, that is, $I \circ f : \partial X_1 \to \partial X_1$ is extendable to the equivariant map $X_1 \to X_1$. Hence, once we know X_1 and X_2 for the cases (2) and (3), then a G-diffeomorphism $M(f) \cong M(I)$ always holds, i.e., the constructing manifold $X_1 \cup X_2$ is unique. It follows that, for the cases (2) and (3), we may only analyse the structure of $X_2 = G \times_{K_2} D^{2k_2 - 1}$.

In the next six sections (the case (2) in Section 5, 6, 7; the case (3) in Section 8, 9, 10), we will analyze K_2 and its slice representation $\sigma_2 : K_2 \to O(2k_2 - 1)$ of the cases (2) and (3).

5. Main theorem and remarks of the case (2)

The goal of this section is to state the main theorem and give some remarks of the case (2). In this section and the next two sections, we assume that the other singular orbit G/K_2 is an exceptional orbit, i.e., $\dim G/K_2 = \dim G/K = 2n - 1$. Then we have $k_2 = 1$ and $K_2/K \cong S^0$ by Lemma 4.1. Due to [**6**, 3.2 Theorem (ii) in Chapter IV], we have that K is a normal subgroup of K_2, i.e.,

$$K \subset K_2 \subset N(K;G), \tag{5.1}$$

where $N(K;G)$ is the normalizer of K in G.

5.1. Main theorem of the case (2)**.** Before we state the main theorem, we prepare some notations (also see Section 3.1). Let $\mathcal{A}$ be a subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$, where $\prod_{j=1}^{b}\mathbb{Z}_2$ is generated by the antipodal involutions on S^{2m_j} for $j = 1, \cdots, b$. The subgroup Δ in $S^1\times O(1)$ (resp. in $SO(2k_1)\times O(1)$) denotes the diagonal subgroup $\{(1,\ 1),\ (-1,\ -1)\}$ (resp. $\{(I_{2k_1},\ 1),\ (-I_{2k_1},\ -1)\}$), and the subgroup $O(1)$ in $S^1\times O(1)$ (resp. in $SO(2k_1)\times O(1)$) denotes the subgroup $\{(1,\ 1),\ (1,\ -1)\}$ (resp. $\{(I_{2k_1},\ 1),\ (I_{2k_1},\ -1)\}$). Now we may state the main theorem of this case.

Theorem 5.1. *Suppose a torus manifold M has a codimension one extended G-action. If there is an exceptional orbit, then $(M,\ G)$ is essentially isomorphic to*

$$\left(\prod_{j=1}^{b} S^{2m_j}\times_{\mathcal{A}\times\mathbb{Z}_2} N,\quad \prod_{j=1}^{b} SO(2m_j+1)\times H\right),$$

such that $(N,\ H)$ is one of the followings:

	N	H
(a)	$\left(\prod_{i=1}^{a} S^{2l_i+1}\right)\times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1}\oplus\mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1)\times U(k_1)$
(b)	$\prod_{i=1}^{a}\mathbb{C}P(l_i)\times S(\mathbb{R}^{2k_1}\oplus\mathbb{R})$	$\prod_{i=1}^{a} SU(l_i+1)\times SO(2k_1)$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup $\prod_{j=1}^{b}\mathbb{Z}_2$ and on the fibre of N through the following representations:

- **(a):** $\sigma_{\mathbb{C}}:\mathcal{A}\to\{\pm1\}\subset S^1$ *on* $S(\mathbb{C}_{\mathfrak{c}}^{k_1}\oplus\mathbb{R})\cap\mathbb{C}_{\mathfrak{c}}^{k_1}$*;*
- **(b):** $\sigma_{\mathbb{R}}:\mathcal{A}\to\{\pm I_{2k_1}\}\subset SO(2k_1)$ *on* $S(\mathbb{R}^{2k_1}\oplus\mathbb{R})\cap\mathbb{R}^{2k_1}$*;*

respectively, and $\mathbb{Z}_2$ (i.e., the second factor of $\mathcal{A}\times\mathbb{Z}_2$) acts on $\prod_{j=1}^{b} S^{2m_j}$ through the representation $\rho:\mathbb{Z}_2\to\prod_{j=1}^{b}\mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2)\cap\mathcal{A}=\{1\}$ and on the fibre of N through the following representations:

- **(a):** $\sigma_{\mathbb{C}\oplus\mathbb{R}}:\mathbb{Z}_2\to\{\pm1\}\times O(1)\subset S^1\times O(1)$ *on* $\mathbb{C}_{\mathfrak{c}}^{k_1}\oplus\mathbb{R}$*;*
- **(b):** $\sigma_{\mathbb{R}\oplus\mathbb{R}}:\mathbb{Z}_2\to\{\pm I_{2k_1}\}\times O(1)\subset SO(2k_1)\times O(1)$ *on* $\mathbb{R}^{2k_1}\oplus\mathbb{R}$*;*

respectively, such that $\sigma (=\sigma_{\mathbb{C}\oplus\mathbb{R}},\ \sigma_{\mathbb{R}\oplus\mathbb{R}})$ and ρ satisfy one of the followings:

- **(i):** *ρ is non-trivial, and $\sigma(\mathbb{Z}_2)=O(1)$ or $\sigma(\mathbb{Z}_2)=\Delta$;*
- **(ii):** *ρ is trivial, and $\sigma(\mathbb{Z}_2)=\Delta$.*

Here, G-actions of (a), (b) are the same as Theorem 3.1 (a), (c), respectively.

Remark 5.2. All manifolds appeared in Theorem 5.1 (a), (b) are $\mathbb{Z}_2$-quotient of manifolds appeared in Theorem 3.1 (a), (c), respectively. These $\mathbb{Z}_2$-actions are defined by $\rho\times\sigma$ in Theorem 5.1. Moreover, we remark that torus actions of manifolds in Theorem 3.1 (a), (c) have the next property: two singular orbits are same (diffeomorphic to G/H). By the $\mathbb{Z}_2$-quotient defined in Theorem 5.1, these two same singular orbits in Theorem 3.1 (a), (c) go to just one singular orbit G/K_1. Then G/K_1 can be regarded as $\mathbb{Z}_2\backslash G/H$ (if ρ is non-trivial) or G/H (if ρ is trivial).

In order to prove Theorem 5.1, we divide this case (2) into two cases which correspond with the case (a) and (b) for the type of K_1 (see Section 4). We call them *the case* (2)-(a) and (2)-(b), respectively, i.e., the case (2)-(a) is that G/K_2 is an exceptional orbit and G/K_1 satisfies the case (a), and the case (2)-(b) is that G/K_2 is an exceptional orbit and G/K_1 satisfies the case (b).

In Section 6 and 7, we study the case (2)-(a) and (2)-(b), respectively. Before we go to Section 6, we give some technical remarks to study the case (2).

5.2. Remarks of the case (2)**.** We have already analysed X_1 and attaching map $f:\partial X_1\to\partial X_2$ in Section 4.1.1, 4.1.2 and 4.2; therefore, in order to construct a G-manifold $M = X_1\cup_f X_2$, we may only analyse $X_2 = G\times_{K_2} D(\mathbb{R})$, where $D(\mathbb{R}) = D^1\subset\mathbb{R}$ (1-dimensional disk).

Because $K_2/K\simeq S^0$ and G/K_2 is an exceptional orbit, we can easily show that the slice representation $\sigma_2 : K_2\longrightarrow O(1)$ always satisfies the following properties:

- σ_2 is surjective;
- $\ker\sigma_2 = K$.

Hence, a tubular neighborhood $X_2 = G \times_{K_2} D(\mathbb{R})$ is only determined by the inclusion $K_2 \subset G$. Therefore, we may only analyse the inclusion $K \subset K_2 \subset N(K;G)$, (see (5.1)) in the case (2). So we first need to compute $N(K;G)$. In the remainder of this section, we compute $N(K;G)$ in the cases (2)-(a) and (2)-(b).

First we assume that K is in the case (2)-(a) (see Section 4.1.1). Because of the definition of K in the case (2)-(a), we have that $N(K;G)$ is as follows:

$$N(K;G) = \prod_{j=1}^{b} S(O(2m_j) \times O(1)) \times \prod_{i\in I} W_i \times \prod_{i'\in I'} S(U(l_{i'}) \times U(1)) \\ \times S(U(k_1 - 1) \times U(1)) \times T^1,$$

where if $l_i = 1$ and $\alpha_i = 0$ then $i \in I$, otherwise $i' \in I'$ ($I \cup I' = \{1, \cdots, a\}$), and $W_i = N(S(U(1) \times U(1)); SU(2))$ for $i \in I$. However, if $l_i = 1$ and $\alpha_i = 0$ then we can regard $SU(2)$ as $SO(3)$ up to essential isomorphism ($SU(2) \approx SO(3)$). Regarding $\{1, \cdots, b\} \cup I$ as $\{1, \cdots, b\}$ and I' as $\{1, \cdots, a\}$ again, we can write $N(K;G)$ as follows:

$$\begin{aligned} N(K;G) &= \prod_{j=1}^{b} S(O(2m_j) \times O(1)) \\ &\times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(U(k_1 - 1) \times U(1)) \times T^1. \end{aligned} \tag{5.2}$$

Next we assume that K is in the case (2)-(b) (see Section 4.1.2). By the similar argument of the case (2)-(a), we can regard $N(K;G)$ in the case (2)-(b) as follows:

$$\begin{aligned} N(K;G) &= \prod_{j=1}^{b} S(O(2m_j) \times O(1)) \\ &\times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(O(2k_1 - 1) \times O(1)). \end{aligned} \tag{5.3}$$

In the next two sections, we will analyse $K \subset K_2 \subset N(K;G)$.

6. The case (2)-(a)

In this section, we study the case (2)-(a). From Section 4.1.1, we have that

$$G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times SU(k_1) \times T^1,$$

$$K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k_1) \times T^1,$$

$$K = \left\{ \left(A,\ (t_1, \cdots, t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \ \middle|\ \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\}.$$

The elements of K are often denoted by $(A,\ (t_1, \cdots, t_a),\ x,\ t)$ simply.

In order to analyse the inclusion $K \subset K_2 \subset N(K;G)$ (see (5.2) for $N(K;G)$), we first define p_1 and p_2 as the following two natural projections:

$$p_1 : N(K;G) \longrightarrow \prod_{j=1}^{b} S(O(2m_j) \times O(1));$$

$$p_2 : N(K;G) \longrightarrow \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(U(k_1 - 1) \times U(1)) \times T^1.$$

Then we can easily prove the following lemma.

LEMMA 6.1. *For p_1, p_2 and K, the following properties hold:*

(1) *the image of K by p_1 satisfies $p_1(K) = \mathcal{S}_1$;*
(2) *if $\sigma_1(\mathcal{S}_1) = \{1\}$, then $K = \mathcal{S}_1 \times p_2(K)$ and*
$p_2(K) = \{((t_1, \cdots, t_a), x, t) \mid t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1\}$;
(3) *if $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then $(\mathcal{S}_1 \times p_2(K))/K \simeq \mathbb{Z}_2$ and*
$p_2(K) = \{((t_1, \cdots, t_a), x, t) \mid t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = \pm 1\}$.

PROOF. We define $\mathcal{R}_1 \subset K$ as follows (also see Section 3.3):

$$\mathcal{R}_1 = \left\{ \left(A,\ e,\ \begin{pmatrix} J & 0 \\ 0 & \sigma_1(A) \end{pmatrix},\ 1 \right) \middle| A \in \mathcal{S}_1 \right\}$$

where $e \in \prod_{i=1}^{a} S(U(l_i) \times U(1))$ is the identity element and $J \in U(k_1 - 1)$ such that $J = I_{k_1 - 1}$ (if $\sigma_1(A) = 1$) or $\det J = -1$ with $J^2 = I_{k_1-1}$ (if $\sigma_1(A) = -1$). Then $\mathcal{R}_1 \simeq \mathcal{S}_1$. It follows that the first statement holds. The second and third statements are proved by the definition of K and the first property $p_1(K) = \mathcal{S}_1$. □

Let $\mathcal{S}_2$ be a subgroup of $\prod_{j=1}^{b} S(O(2m_j) \times O(1))$ such that $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$. Because $K_2/K \simeq S^0$, for $p_s(K_2)$ and $p_s(K)$ $(s = 1,\ 2)$, one of the following four cases occurs:

(i): $p_1(K_2) = \mathcal{S}_2$, and $p_2(K_2) = p_2(K)$;
(ii): $p_1(K_2) = p_1(K) = \mathcal{S}_1$, and $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$;
(iii): $p_1(K_2) = p_1(K) = \mathcal{S}_1$, and $p_2(K_2) = p_2(K)$;
(iv): otherwise, i.e., $p_1(K_2) = \mathcal{S}_2$, and $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$.

We call the above cases *the case* (2)-(a)-(i), (2)-(a)-(ii), (2)-(a)-(iii) and (2)-(a)-(iv), respectively.

6.1. The case (2)-(a)-(i). Suppose the case (2)-(a)-(i) occurs, that is,

$$p_1(K_2)/p_1(K) = \mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2, \quad \text{and}$$
$$p_2(K_2) = p_2(K) = \{((t_1,\ \cdots,\ t_a),\ x,\ t) \mid t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \in \sigma_1(\mathcal{S}_1) \subset \{\pm 1\}\}.$$

Because $K \subset p_1(K) \times p_2(K)$ and $K_2 \subset p_1(K_2) \times p_2(K_2)$, in this case we have that

$$K \subset \mathcal{S}_1 \times p_2(K), \quad K \subset K_2 \subset \mathcal{S}_2 \times p_2(K). \tag{6.1}$$

First we assume $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then $(\mathcal{S}_1 \times p_2(K))/K \simeq \mathbb{Z}_2$ by Lemma 6.1 (3). Therefore, $\#(\mathcal{S}_2 \times p_2(K))/K = 4$ by $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$ and $K \subset \mathcal{S}_1 \times p_2(K) \subset \mathcal{S}_2 \times p_2(K)$ (where $\#F$ is the number of the finite group F). Hence, by making use of $K_2/K \simeq S^0$ and (6.1), we also have

$$(\mathcal{S}_2 \times p_2(K))/K_2 \simeq \mathbb{Z}_2. \tag{6.2}$$

Because $p_1(K_2) = \mathcal{S}_2$, $K \subset K_2 \cap (\mathcal{S}_1 \times p_2(K)) \subset K_2$ and $K_2/K \simeq S^0$, we also have

$$K_2 \cap (\mathcal{S}_1 \times p_2(K)) = K. \tag{6.3}$$

Now we define the representation $\rho : \mathcal{S}_1 \times p_2(K) \to \mathbb{Z}_2$ as follows:

$$\rho(A,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha}.$$

By the definition of K, we have $\ker \rho = K$. Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K) \to \mathbb{Z}_2$ be a lift of this representation, i.e., the restricted representation $\widetilde{\rho}|_{\mathcal{S}_1 \times p_2(K)}$ coincides with ρ. Because $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$ and $\rho|_{p_2(K)} = \widetilde{\rho}|_{p_2(K)}$, this lift $\widetilde{\rho}$ is only determined by a representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Hence, we have the following lemma.

LEMMA 6.2. *Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K) \to \mathbb{Z}_2$ be a lift of ρ. Then there is a representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$ and $\widetilde{\rho}$ is denoted as follows:*

$$\widetilde{\rho}(B,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = \sigma_2(B) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha}.$$

On the other hand, by (6.2), there is the following sequence:

$$K_2 \xrightarrow{\widetilde{i}} \mathcal{S}_2 \times p_2(K) \xrightarrow{\widetilde{r}} \mathbb{Z}_2,$$

where $\widetilde{i}$ is an inclusion and $\widetilde{r}$ is the surjective representation induced by the projective representation $\mathcal{S}_2 \times p_2(K) \to (\mathcal{S}_2 \times p_2(K))/K_2$. Let r be the restricted representation $\widetilde{r}|_{\mathcal{S}_1 \times p_2(K)}$. Using (6.3), we see that the representation r is induced by the natural surjection $\mathcal{S}_1 \times p_2(K) \to (\mathcal{S}_1 \times p_2(K))/K$, i.e., there is the following restricted sequence:

$$K \xrightarrow{i} \mathcal{S}_1 \times p_2(K) \xrightarrow{r} \mathbb{Z}_2,$$

where i is the natural inclusion. By the definition of K, we have that r can be identified with ρ. Therefore, by Lemma 6.2, we have that $\widetilde{r} = \widetilde{\rho}$ for some $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$. In other wards, there is the representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that

$$\begin{aligned} K_2 &= \ker\widetilde{\rho}(=\ker\widetilde{r}) \\ &= \left\{ \left(B,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \;\middle|\; \sigma_2(B) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\}. \end{aligned} \tag{6.4}$$

Next we assume $\sigma_1(\mathcal{S}_1) = \{1\}$, then $K = \mathcal{S}_1 \times p_2(K)$ by Lemma 6.1 (2). Hence, by using (6.1) and the assumption of the case (2)-(a)-(i), we have

$$K_2 = \mathcal{S}_2 \times p_2(K).$$

We can regard this case as the case that $\sigma_2(\mathcal{S}_2) = \{1\}$ in Eq.(6.4).

Therefore, in the case (2)-(a)-(i), the inclusion $K_2 \subset N(K;G)$ is completely determined by the subgroup $\mathcal{S}_2 \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1))$ and the representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Hence, using (4.1) and Remark 4.2, we can easily check the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism (by computing orbit types of the natural $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k_1)$ action):

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{S}_1/\prod_{j=1}^{b} SO(2m_j)$ and the $\mathcal{A}$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ through the representation $\sigma_1 : \mathcal{A} \to \mathbb{Z}_2$ (this representation is induced by $\sigma_1 : \mathcal{S}_1 \to \mathbb{Z}_2$);
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ by a non-trivial representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b} \mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$ (this corresponds with $p_1(K_2) = \mathcal{S}_2$);
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$, where if σ_1 is trivial then σ_2 is also trivial (this corresponds with that if $\sigma_1(\mathcal{S}_1) = \{1\}$ then $\sigma_2(\mathcal{S}_2) = \{1\}$);
- on $\mathbb{R}$ by the natural representation (this corresponds with the existence of an exceptional orbit).

Remark that $\mathcal{A} \times \mathbb{Z}_2$ acts on $\prod_{j=1}^{b} S^{2m_j}$ freely because ρ is non-trivial and satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$; therefore, M is a manifold, more precisely a fibre bundle over $\prod_{j=1}^{b} S^{2m_j}/(\mathcal{A} \times \mathbb{Z}_2)$ with the fibre $\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})$.

6.2. The case (2)-(a)-(ii). Suppose the case (2)-(a)-(ii) occurs, that is,

$$\begin{aligned} &p_1(K_2) = p_1(K) = \mathcal{S}_1 \quad \text{and} \\ &p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2. \end{aligned}$$

First we prove the following lemma.

LEMMA 6.3. *Suppose that $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$. Then the inclusion $p_2(K_2) \subset p_2(N(K;G))$ is unique. Furthermore, we have*

$$p_2(K_2) = \left\{ \left((t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \;\middle|\; t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \in \mathbb{Z}_m \right\},$$

where $m = 2$ if $\sigma_1(\mathcal{S}_1) = \{1\}$ or $m = 4$ if $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$.

Proof. We first remark that $\sigma_1(\mathcal{S}_1) \subset \{\pm 1\}$, by Section 4.1.1.

Consider the following surjective representation $\sigma : p_2(N(K;G)) = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times S(U(k_1 - 1) \times U(1)) \times T^1 \to S^1$ (induced by the slice representation σ_1):

$$\sigma\left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t\right) = t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \quad (\text{or} \quad t_1^{\alpha_1} \cdots t_a^{\alpha_a} t \quad \text{for} \quad k_1 = 1).$$

Then we see that $p_2(K) = \sigma^{-1}(\sigma_1(\mathcal{S}_1)) \subset p_2(K_2)$ by the definition of K. Therefore, we have that $\sigma(p_2(K)) = \sigma_1(\mathcal{S}_1) \subset \sigma(p_2(K_2))$.

Because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$, we also have that

$$\sigma(p_2(K_2))/\sigma(p_2(K)) \subset \mathbb{Z}_2(\subset \sigma(p_2(N(K;G)))/\sigma(p_2(K)) \simeq S^1).$$

If $\sigma(p_2(K_2))/\sigma(p_2(K)) = \{1\} \subset \mathbb{Z}_2$, then $\sigma(p_2(K_2)) = \sigma(p_2(K)) = \sigma_1(\mathcal{S}_1)$. Hence, we have $p_2(K_2) \subset \sigma^{-1}(\sigma_1(\mathcal{S}_1)) = p_2(K)$. However, this gives a contradiction, because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$. Therefore, we have

$$\sigma(p_2(K_2))/\sigma(p_2(K)) = \mathbb{Z}_2 \subset S^1.$$

It follows that there are the following two cases:

- if $\sigma_1(\mathcal{S}_1) = \sigma(p_2(K)) = \{1\}$, then $\sigma(p_2(K_2)) = \mathbb{Z}_2 = \{\pm 1\} \subset S^1$; hence, $p_2(K_2) \subset \sigma^{-1}(\mathbb{Z}_2)$;
- if $\sigma_1(\mathcal{S}_1) = \sigma(p_2(K)) = \{\pm 1\}$, then $\sigma(p_2(K_2)) = \mathbb{Z}_4 = \{\pm 1, \pm i\} \subset S^1$; hence, $p_2(K_2) \subset \sigma^{-1}(\mathbb{Z}_4)$.

Because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$ and $p_2(K) = \sigma^{-1}(\sigma_1(\mathcal{S}_1))$, we can easily show that $p_2(K_2) = \sigma^{-1}(\mathbb{Z}_2)$ for the case $\sigma_1(\mathcal{S}_1) = \{1\}$ and $p_2(K_2) = \sigma^{-1}(\mathbb{Z}_4)$ for the case $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Hence, we have that the inclusion of $p_2(K_2) \subset p_2(N(K;G))$ is uniquely determined by $\sigma^{-1}(\mathbb{Z}_2)$ or $\sigma^{-1}(\mathbb{Z}_4)$. Thus, we have the statement of this lemma. $\square$

Assume $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Then $m = 4$ by Lemma 6.3. Let $\widetilde{\rho} : \mathcal{S}_1 \times p_2(K_2) \to \mathbb{Z}_4$ be the following representation:

$$\widetilde{\rho}(A, (t_1, \cdots, t_a), x, t) = \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \in \mathbb{Z}_4.$$

We can easily show that $\widetilde{\rho}^{-1}(\mathbb{Z}_2) = \mathcal{S}_1 \times p_2(K)$ and $\ker \widetilde{\rho} = K$. Because $K_2/K \simeq S^0$, we also have $\widetilde{\rho}(K_2) = \mathbb{Z}_2 \subset \mathbb{Z}_4$. Hence, we have $K_2 \subset \widetilde{\rho}^{-1}(\mathbb{Z}_2) = \mathcal{S}_1 \times p_2(K)$. However, this gives a contradiction, because $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$. Therefore, we have $\sigma_1(\mathcal{S}_1) = \{1\}$.

Thus, we have $K = \mathcal{S}_1 \times p_2(K)$ by Lemma 6.1 (2). Because $K_2/K \simeq S^0 \simeq \mathbb{Z}_2 \simeq p_2(K_2)/p_2(K)$ and $K \subset K_2 \subset \mathcal{S}_1 \times p_2(K_2)$, the subgroup $K_2 \subset N(K;G)$ in the case (2)-(a)-(ii) is as follows:

$$K_2 = \mathcal{S}_1 \times p_2(K_2),$$

where $p_2(K_2)$ is the group in Lemma 6.3 with $m = 2$. Hence, with a method similar to that demonstrated in the case (2)-(a)-(i) (Section 6.1), we can easily check the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}(= \mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ trivially (this corresponds with that $\sigma_1(\mathcal{S}_1) = \{1\}$);
- on $\mathbb{R}$ trivially

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ trivially (this corresponds with $p_1(K_2) = \mathcal{S}_1$);
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a non-trivial representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$ (this corresponds with $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$);
- on $\mathbb{R}$ by the natural representation.

Remark that M is also G-equivariantly diffeomorphic to the following manifold (where $G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k_1)$):

$$\prod_{j=1}^{b} S^{2m_j}/\mathcal{A} \times \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} P(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where $P(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R})$ is the real projective space.

6.3. The case (2)-(a)-(iii). Suppose the case (2)-(a)-(iii) occurs, that is,

$$\begin{aligned} &p_1(K_2) = p_1(K) = \mathcal{S}_1 \quad \text{and} \\ &p_2(K_2) = p_2(K). \end{aligned}$$

For this case (iii), the following lemma holds.

LEMMA 6.4. *Suppose that the case (iii) occurs. Then $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, and $p_2(K_2) \subset p_2(N(K;G))$ is*

$$p_2(K_2) = p_2(K) = \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = \pm 1 \right\}.$$

Furthermore, we have $K_2 = \mathcal{S}_1 \times p_2(K)$.

PROOF. If $\sigma_1(\mathcal{S}_1) = \{1\}$, then we have $K = \mathcal{S}_1 \times p_2(K)$ by Lemma 6.1 (2). It follows that $K = p_1(K_2) \times p_2(K_2) \supset K_2$ by the assumption of the case (iii). This gives a contradiction to $K_2/K \simeq S^0$. Hence, we have $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Moreover, we have that

$$p_2(K_2) = p_2(K) = \left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = \pm 1 \right\}$$

by Lemma 6.1 (3).

By the definition of p_1, p_2 and the assumptions of the case (iii), we have the following relation:

$$K \subset K_2 \subset \mathcal{S}_1 \times p_2(K).$$

By the definition of K, we can easily check that $(\mathcal{S}_1 \times p_2(K))/K \simeq \mathbb{Z}_2$. Therefore, we have $K_2 = \mathcal{S}_1 \times p_2(K)$. $\square$

By using Lemma 6.4 and a method similar to that demonstrated in the previous cases, we can easily check that the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}(= \mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ through the non-trivial representation $\sigma_1 : \mathcal{A} \to \mathbb{Z}_2$ (this corresponds with $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$);
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ trivially (this corresponds with $p_1(K_2) = \mathcal{S}_1$);
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a non-trivial representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$ (this is known by using $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$ and $p_2(K_2) = p_2(K)$);
- on $\mathbb{R}$ by the natural representation.

Remark that M is also G-equivariantly diffeomorphic to the following manifold as well as the case (2)-(a)-(ii):

$$\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} P(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right).$$

6.4. The case (2)-(a)-(iv). Suppose the case (2)-(a)-(iv) occurs, that is,

$$p_1(K_2)/p_1(K) = \mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2 \quad \text{and}$$
$$p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2.$$

First, we remark that Lemma 6.3 can be used in this case because of $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$.

Assume $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$. Then we have $m = 4$ for the group $p_2(K_2)$ in Lemma 6.3. Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K_2) \to \mathbb{Z}_4$ be the following representation:

$$\widetilde{\rho}(B,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = \sigma_2(B) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} \in \mathbb{Z}_4,$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_4$ is some representation such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Then we can easily show that $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$, i.e., the image of $\mathcal{S}_2$ is in $\mathbb{Z}_2 \subset \mathbb{Z}_4$, because $\prod_{j=1}^{b} SO(2m_j) \subset \ker \sigma_1 \subset \ker \sigma_2$ and $\mathcal{S}_2 / \prod_{j=1}^{b} SO(2m_j) \subset \prod_{j=1}^{b} \mathbb{Z}_2$. Hence, we have that

$$\ker \widetilde{\rho} \subset \mathcal{S}_2 \times p_2(K). \tag{6.5}$$

Consider the restricted representation $\widetilde{\rho}|_{K_2} : K_2 \to \mathbb{Z}_4$. Then we have the following sequence:

$$K \subset \ker \widetilde{\rho}|_{K_2} = \ker \widetilde{\rho} \cap K_2 \subset K_2$$

because of the definitions of K and σ_2. Therefore, by $K_2/K \simeq S^0$, we have that $\ker \widetilde{\rho}|_{K_2} = K_2$ or K. If $\ker \widetilde{\rho}|_{K_2} = K_2$, then we have $p_2(K_2) = p_2(K)$ by (6.5). This gives a contradiction to $p_2(K_2)/p_2(K) = \mathbb{Z}_2$. Hence, we have $\ker \widetilde{\rho}|_{K_2} = K$. Then we have

$$K_2/K = K_2/\ker \widetilde{\rho}|_{K_2} \simeq \widetilde{\rho}(K_2) = \mathbb{Z}_2 = \{\pm 1\} \subset \mathbb{Z}_4.$$

It also follows that $K_2 \subset \widetilde{\rho}^{-1}(\mathbb{Z}_2) = \mathcal{S}_2 \times p_2(K)$; therefore, $p_2(K_2) = p_2(K)$. This also gives a contradiction to $p_2(K_2)/p_2(K) = \mathbb{Z}_2$. Thus, we have $\sigma_1(\mathcal{S}_1) = \{1\}$. Hence, by Lemma 6.1 (2), we have $K = \mathcal{S}_1 \times p_2(K)$.

Because $K_2/K = K_2/(\mathcal{S}_1 \times p_2(K)) \simeq \mathbb{Z}_2$ and $(\mathcal{S}_2 \times p_2(K_2))/K \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (the assumption of (2)-(a)-(iv)), we have that

$$(\mathcal{S}_2 \times p_2(K_2))/K_2 \simeq \mathbb{Z}_2. \tag{6.6}$$

Again, we define $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K_2) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ as follows:

$$\widetilde{\rho}(B,\ (t_1,\ \cdots,\ t_a),\ x,\ t) = (\sigma_2(B),\ t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha}) \in \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. Then we have $\ker \widetilde{\rho} = K$ by $K = \mathcal{S}_1 \times p_2(K)$ and $\sigma_1(\mathcal{S}_1) = \{1\}$. Because $K_2/K \simeq \mathbb{Z}_2$, we have that $\widetilde{\rho}(K_2) \simeq \mathbb{Z}_2$, i.e., $\widetilde{\rho}(K_2) = \mathbb{Z}_2 \times \{1\}$, $\{1\} \times \mathbb{Z}_2$ or $\Delta = \{(1,\ 1),\ (-1,\ -1)\}$. If $\widetilde{\rho}(K_2) = \mathbb{Z}_2 \times \{1\}$ or $\{1\} \times \mathbb{Z}_2$, then this gives a contradiction to that $p_2(K_2)/p_2(K) = \mathbb{Z}_2$ or $p_1(K_2) = \mathcal{S}_1$, respectively. Therefore, we have that $\widetilde{\rho}(K_2) = \Delta$. Moreover, by using (6.6), the subgroup $K_2 \subset \mathcal{S}_2 \times p_2(K_2)$ can be denoted as follows:

$$\widetilde{\rho}^{-1}(\Delta) = K_2 = \{(B,\ t_1,\ \cdots,\ t_a,\ x,\ t) \mid \sigma_2(B) t_1^{\alpha_a} \cdots t_a^{\alpha_a} x t^{\alpha} = 1\},$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\ker \sigma_2 = \mathcal{S}_1$. Hence $K_2 \subset N(K; G)$ is completely determined by $\mathcal{S}_2$ and σ_2 in the case (2)-(a)-(iv).

With a method similar to that demonstrated in the previous cases, we can easily check the following manifold corresponds with $M = X_1 \cup X_2$ up to essential isomorphism:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}(= \mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ trivially (this corresponds with $\sigma_1(\mathcal{S}_1) = \{1\}$);

- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ by a non-trivial representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b} \mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$ (this corresponds with $p_1(K_2) = \mathcal{S}_2$);
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a non-trivial representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$ (this corresponds with $p_2(K_2)/p_2(K) \simeq \mathbb{Z}_2$);
- on $\mathbb{R}$ by the natural representation.

Remark that M is also G-equivariantly diffeomorphic to the following manifold as well as the previous cases:

$$\prod_{j=1}^{b} S^{2m_j}/\mathcal{A} \times_{\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \right).$$

6.5. Summary of the results from (2)-(a)-(i) to (iv). In summary, we can state the result of the case (2)-(a) as follows. Let $(M,\ G)$ be the pair in the case (2)-(a). Then $(M,\ G)$ is essentially isomorphism to the followings:

$$M = \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}\times\mathbb{Z}_2} \left(\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} S(\mathbb{C}_{\mathfrak{a}}^{k_1} \oplus \mathbb{R}) \right),$$
$$G = \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times U(k_1),$$

where G acts on M naturally, and the $\mathcal{A}(=\mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ through the representation $\sigma_1(=\sigma_{\mathbb{C}}) : \mathcal{A} \to \mathbb{Z}_2$;
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b} S^{2m_j}$ by a representation $\rho : \mathbb{Z}_2 \to \prod_{j=1}^{b} \mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2) \cap \mathcal{A} = \{1\}$;
- on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by a representation $\sigma_2 : \mathbb{Z}_2 \to \{\pm 1\}$;
- on $\mathbb{R}$ by the natural representation $\kappa : \mathbb{Z}_2 \to O(1)$,

where ρ or σ_2 is always non-trivial, i.e., the case that both of two representations ρ and σ_2 are trivial does not occur. This corresponds with the first manifold in Theorem 5.1, where $\sigma_2 \oplus \kappa = \sigma_{\mathbb{C}\oplus\mathbb{R}}$.

7. The case (2)-(b)

In this section, we study the case (2)-(b). From Section 4.1.2, we have that

$$\begin{aligned}
G &= \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1),\\
K_1 &= \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SO(2k_1),\\
K &= \prod_{i=1}^{a} S(U(l_i) \times U(1))\\
&\quad \times \left\{ \left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k_1-1) \times O(1)) \;\middle|\; \sigma_1(A)x = 1 \right\}\\
&= \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times K'.
\end{aligned}$$

An element in $\mathcal{S}_1 \times S(O(2k_1-1)\times O(1))$ is often denoted by $(A,\ x)$. In this case, by Section 5.2, we may only analyze the inclusion of K_2 such that $K \subset K_2 \subset N(K;G)$ and $N(K;G)$ is known as (5.3).

Because $K \subset K_2 \subset N(K;G)$ and $K_2/K \simeq S^0$, we have that

$$K_2 = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times K_2',$$

where $K_2' \subset \prod_{j=1}^{b} S(O(2m_j)\times O(1)) \times S(O(2k_1-1)\times O(1))$ such that $K_2'/K' \simeq \mathbb{Z}_2$.

Let p_1 and p_2 be the following two natural projections:

$$p_1 : \prod_{j=1}^{b} S(O(2m_j)\times O(1)) \times S(O(2k_1-1)\times O(1)) \longrightarrow \prod_{j=1}^{b} S(O(2m_j)\times O(1));$$

$$p_2 : \prod_{j=1}^{b} S(O(2m_j)\times O(1)) \times S(O(2k_1-1)\times O(1)) \longrightarrow S(O(2k_1-1)\times O(1)).$$

Similarly to Lemma 6.1, we can easily prove the following lemma.

LEMMA 7.1. *For p_1, p_2 and K', the following properties hold:*

(1) *the image of K' by p_1 satisfies $p_1(K') = \mathcal{S}_1$;*
(2) *if $\sigma_1(\mathcal{S}_1) = \{1\}$, then $p_2(K') = SO(2k_1-1)$ and $K' = \mathcal{S}_1 \times SO(2k_1-1)$;*
(3) *if $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then $p_2(K') = S(O(2k_1-1)\times O(1))$ and $(\mathcal{S}_1 \times S(O(2k_1-1)\times O(1)))/K' \simeq \mathbb{Z}_2$.*

PROOF. The first statement is proved by making use of the subgroup $\mathcal{R}_1$ in Section 3.5. The second and third statements are proved by the definition of K' and the first property $p_1(K') = \mathcal{S}_1$. □

Because $K_2'/K' \simeq \mathbb{Z}_2$, similarly to the case (2)-(a), one of the following four cases occurs:

(i): $p_1(K_2') = \mathcal{S}_2$, and $p_2(K_2') = p_2(K')$;
(ii): $p_1(K_2') = p_1(K') = \mathcal{S}_1$, and $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$;
(iii): $p_1(K_2') = p_1(K') = \mathcal{S}_1$, and $p_2(K_2') = p_2(K')$;
(iv): otherwise, i.e., $p_1(K_2') = \mathcal{S}_2$, and $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$,

where $\mathcal{S}_2$ is a subgroup of $\prod_{j=1}^{b} S(O(2m_j)\times O(1))$ such that $\mathcal{S}_2/\mathcal{S}_1 \simeq \mathbb{Z}_2$. We call the above cases *the case* (2)-(b)-(i), (2)-(b)-(ii), (2)-(b)-(iii) and (2)-(b)-(iv), respectively.

7.1. The case (2)-(b)-(i). Suppose the case (2)-(b)-(i) occurs, that is,

$$p_1(K_2') = \mathcal{S}_2 \quad \text{and} \quad p_2(K_2') = p_2(K').$$

If $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then K' is defined as $\ker\rho$, where $\rho : \mathcal{S}_1 \times p_2(K') \to \mathbb{Z}_2$ is the following representation:

$$\rho(A, x) = \sigma_1(A)x \in \mathbb{Z}_2.$$

Consider the lift of this representation $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K') \to \mathbb{Z}_2$. Similarly to the proof of Lemma 6.2, we can easily show that this lift is only determined by $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$, i.e., a representation $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K') \to \mathbb{Z}_2$ is denoted by

$$\widetilde{\rho}(B,\ x) = \sigma_2(B)x.$$

On the other hand, there is the following induced representation form $K_2' \subset \mathcal{S}_2 \times p_2(K')$:

$$\mathcal{S}_2 \times p_2(K') \xrightarrow{\widetilde{r}} (\mathcal{S}_2 \times p_2(K'))/K_2' \simeq \mathbb{Z}_2.$$

With a method similar to that demonstrated in Section 6.1, $\widetilde{r}$ can be identified with $\widetilde{\rho}$, i.e., the lift of ρ. Hence, there is a representation $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ such that $\widetilde{r}(B,\ x) = \sigma_2(B)x$. Because

$\ker \widetilde{r} = K_2'$, we have that

$$K_2' = \left\{ \left(B, \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)) \;\middle|\; \sigma_2(B)x = 1 \right\}, \tag{7.1}$$

where $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ is a representation such that $\sigma_2|_{\mathcal{S}_1} = \sigma_1$.

If $\sigma_1(\mathcal{S}_1) = \{1\}$, then we have

$$K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times SO(2k_1 - 1),$$

by Lemma 7.1 (2). Because of the assumptions of the case (2)-(b)-(i), we have that

$$K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_2 \times SO(2k_1 - 1).$$

Therefore, we can regard K_2' of this case as that with $\sigma_2(\mathcal{S}_2) = 1$ in Eq.(7.1).

Hence, we have that $M = X_1 \cup X_2$ is as follows, with the method similar to that demonstrated in Section 6.1:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A} \times \mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(i) (see Section 6.1 by replacing $\mathbb{C}_{\mathfrak{c}}^{k_1}$ as $\mathbb{R}^{2k_1}$), and $G = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times SO(2k_1)$ acts naturally on this manifold.

7.2. The case (2)-(b)-(ii). Suppose the case (2)-(b)-(ii) occurs, that is,

$$p_1(K_2') = p_1(K') = \mathcal{S}_1 \quad \text{and} \quad p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2.$$

If $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$, then we have $p_2(K') = S(O(2k_1 - 1) \times O(1)) \supset p_2(K_2')$ by Lemma 7.1 (3). This gives a contradiction to $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$. Therefore, we have $\sigma_1(\mathcal{S}_1) = \{1\}$. Hence, we have that $K' = \mathcal{S}_1 \times SO(2k_1 - 1)$ by Lemma 7.1 (2).

Because $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$ and $p_2(K') = SO(2k_1 - 1)$, we have the following sequence:

$$K' = \mathcal{S}_1 \times SO(2k_1 - 1) \subset K_2' \subset \mathcal{S}_1 \times p_2(K_2') = \mathcal{S}_1 \times S(O(2k_1 - 1) \times O(1)).$$

Because $K_2'/K' \simeq \mathbb{Z}_2$, we have that $K_2' = \mathcal{S}_1 \times S(O(2k_1 - 1) \times O(1))$ and

$$K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times S(O(2k_1 - 1) \times O(1)),$$

$$K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times SO(2k_1 - 1).$$

Hence, similarly to Section 7.1, we have that $M = X_1 \cup X_2$ is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A} \times \mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(ii) (see Section 6.2 by replacing $\mathbb{C}_{\mathfrak{c}}^{k_1}$ as $\mathbb{R}^{2k_1}$). This manifold is also G-equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j}/\mathcal{A} \times P(\mathbb{R}^{2k_1} \oplus \mathbb{R}),$$

where $G = \prod_{i=1}^{a} SU(l_i + 1) \times \prod_{j=1}^{b} SO(2m_j + 1) \times SO(2k_1)$.

7.3. The case (2)-(b)-(iii). Suppose the case (2)-(b)-(iii) occurs, that is,

$$p_1(K_2') = p_1(K') = \mathcal{S}_1 \quad \text{and} \quad p_2(K_2') = p_2(K').$$

If $\sigma_1(\mathcal{S}_1) = \{1\}$, then we have $K' = \mathcal{S}_1 \times SO(2k_1 - 1) = p_1(K') \times p_2(K') = p_1(K_2') \times p_2(K_2') \supset K_2'$. This gives a contradiction to $K_2'/K' \simeq \mathbb{Z}_2$. Therefore, we have $\sigma_1(\mathcal{S}_1) = \{\pm 1\}$.

Hence, we have the following sequence:

$$K' \subset K_2' \subset p_1(K_2') \times p_2(K_2') = \mathcal{S}_1 \times S(O(2k_1 - 1) \times O(1)).$$

By Lemma 7.1 (3) and $K_2'/K' \simeq \mathbb{Z}_2$, we also have that $K_2' = \mathcal{S}_1 \times S(O(2k_1 - 1) \times O(1))$ and

$$K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times S(O(2k_1 - 1) \times O(1)),$$

$$K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \{(A,\ x) \mid \sigma_1(A)x = 1\}.$$

Hence, similarly to Section 7.1, we have that $M = X_1 \cup X_2$ is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A} \times \mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(iii) (see Section 6.3 by replacing $\mathbb{C}_c^{k_1}$ as $\mathbb{R}^{2k_1}$). This manifold is also G-equivariantly diffeomorphic to the following manifold as well as the case (2)-(b)-(ii):

$$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A}} P(\mathbb{R}^{2k_1} \oplus \mathbb{R}).$$

7.4. The case (2)-(b)-(iv). Suppose the case (2)-(b)-(iv) occurs, that is,

$$p_1(K_2') = \mathcal{S}_2 \quad \text{and} \quad p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2.$$

By the same reason in the case (2)-(b)-(ii) (see Section 7.2), we have $\sigma_1(\mathcal{S}_1) = \{1\}$. Hence, we have that $K' = \mathcal{S}_1 \times SO(2k_1 - 1)$ by Lemma 7.1 (2).

Because $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$ and $p_2(K') = SO(2k_1 - 1)$, we have the following sequence:

$$K' = \mathcal{S}_1 \times SO(2k_1 - 1) \subset K_2' \subset \mathcal{S}_2 \times p_2(K_2') = \mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)).$$

Because $K_2'/K' \simeq \mathbb{Z}_2$, we also have the following inclusion map:

$$i : K_2'/K' \simeq \mathbb{Z}_2 \longrightarrow (\mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)))/K' \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Therefore, there are three types of the inclusion i, i.e., $i(K_2'/K') = \mathbb{Z}_2 \times \{1\}$, $\{1\} \times \mathbb{Z}_2$ or Δ, where Δ is the diagonal subgroup in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Assume $i(K_2'/K') = \mathbb{Z}_2 \times \{1\}$ or $\{1\} \times \mathbb{Z}_2$. This gives a contradiction to $p_2(K_2')/p_2(K') \simeq \mathbb{Z}_2$ or $p_1(K_2')/p_1(K') \simeq \mathbb{Z}_2$, respectively. Therefore, we have that

$$i(K_2'/K') = \Delta. \tag{7.2}$$

Let $\widetilde{\rho} : \mathcal{S}_2 \times p_2(K_2') \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be a representation such that $\widetilde{\rho}(B,\ x) = (\sigma_2(B),\ x)$ for some $\sigma_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ with $\sigma_2|_{\mathcal{S}_1} = \sigma_1$. By (7.2), we have that $K_2' = \widetilde{\rho}^{-1}(\Delta)$. It follows that

$$K_2' = \left\{ \left(B,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}\right) \in \mathcal{S}_2 \times S(O(2k_1 - 1) \times O(1)) \;\middle|\; \sigma_2(B)x = 1 \right\}.$$

Because $K_2 = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times K_2'$ and $K = \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times \mathcal{S}_1 \times SO(2k_1 - 1)$, (similarly to Section 7.1) we have that $M = X_1 \cup X_2$ is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \left(\prod_{j=1}^{b} S^{2m_j} \times_{\mathcal{A} \times \mathbb{Z}_2} S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \right),$$

where the $\mathcal{A}\times\mathbb{Z}_2$-quotient is defined by the same rule as that in the case (2)-(a)-(iv) (see Section 6.4 by replacing $\mathbb{C}_{\mathfrak{c}}^{k_1}$ as $\mathbb{R}^{2k_1}$).

7.5. Summary of the results from (2)-(b)-(i) to (iv). In summary, we can state the result of the case (2)-(b) as follows. Let $(M,\ G)$ be the pair in the case (2)-(b). Then $(M,\ G)$ is essentially isomorphism to the followings:

$$M=\prod_{j=1}^{b}S^{2m_j}\times_{\mathcal{A}\times\mathbb{Z}_2}\left(\prod_{i=1}^{a}\mathbb{C}P(l_i)\times S(\mathbb{R}^{2k_1}\oplus\mathbb{R})\right),$$
$$G=\prod_{j=1}^{b}SO(2m_j+1)\times\prod_{i=1}^{a}SU(l_i+1)\times SO(2k_1),$$

where G acts on M naturally, and the $\mathcal{A}(=\mathcal{A}_1)$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b}S^{2m_j}$ as the subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$;
- on $\mathbb{R}^{2k_1}$ through the representation $\sigma_1(=\sigma_{\mathbb{R}}):\mathcal{A}\to\mathbb{Z}_2=\{\pm I_{2k_1}\}$;
- on $\mathbb{R}$ trivially,

and the $\mathbb{Z}_2$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b}S^{2m_j}$ by a representation $\rho:\mathbb{Z}_2\to\prod_{j=1}^{b}\mathbb{Z}_2$ which satisfies $\rho(\mathbb{Z}_2)\cap\mathcal{A}=\{1\}$;
- on $\mathbb{R}^{2k_1}$ by a representation $\sigma_2:\mathbb{Z}_2\to\{\pm I_{2k_1}\}$;
- on $\mathbb{R}$ by the natural representation $\kappa:\mathbb{Z}_2\to O(1)$,

where ρ or σ_2 is always non-trivial as well as the case (2)-(a). This corresponds with the second manifold in Theorem 5.1, where $\sigma_2\oplus\kappa=\sigma_{\mathbb{R}\oplus\mathbb{R}}$.

8. Main theorem of the case (3) and preparations

From this section, we start to classify the final case, i.e., the case (3): G/K_2 is not a torus manifold but a singular orbit (see Section 2.3). The goal of this section is to state the main theorem and a preparation to classify the case (3). From this section, we assume that the orbit G/K_2 is not a torus manifold but a singular orbit. By Lemma 4.1 and the assumption of this case, we have for $k_2\geq 2$

$$\dim G/K_2=2n-2k_2+1, \tag{8.1}$$

and

$$K_2/K\cong S^{2k_2-2}. \tag{8.2}$$

By using (8.1) and (8.2), the slice representation of K_2 in the case (3) is

$$\sigma_2:K_2\longrightarrow O(2k_2-1). \tag{8.3}$$

In the case (3) as well as the case (2), there are the following two cases:

(3)-(a): *the case* (3)-(a), i.e., G/K_1 satisfies the case (a) (see Section 4.1.1);
(3)-(b): *the case* (3)-(b), i.e., G/K_1 satisfies the case (b) (see Section 4.1.2).

8.1. Main theorem and notations. First we state the main theorem of the case (3). Before we state it, we prepare some notations (also see Section 3.1 and 5.1). Let $\mathcal{A}$ be a subgroup of $\prod_{j=1}^{b}\mathbb{Z}_2$, where $\prod_{j=1}^{b}\mathbb{Z}_2$ is the following group: the first $(b-1)$ factors $\prod_{j=1}^{b-1}\mathbb{Z}_2$ are generated by the antipodal involutions on S^{2m_j} for $j=1,\ \cdots,\ b-1$ and the b-th factor $\mathbb{Z}_2$ is $\{\pm I_{2k_2-1}\}$. The quotient manifold $\left(\prod_{i=1}^{a}S^{2l_i+1}\right)\times_{T^a}S(\mathbb{C}_{\mathfrak{c}}^{k_1}\oplus\mathbb{R}^{2k_2-1})$ is defined similarly as $\left(\prod_{i=1}^{a}S^{2l_i+1}\right)\times_{T^a}S(\mathbb{C}_{\mathfrak{a}}^{k}\oplus\mathbb{R})$ (see Section 3.1), where $\mathbb{C}_{\mathfrak{c}}^{k_1}(\simeq\mathbb{C}^{k_1})$ is a representation space of a representation $\mathfrak{c}:T^a\to S^1$.

Now we may state the main theorem of this section.

Theorem 8.1. *Suppose a torus manifold M has a codimension one extended G-action. If there are two singular orbits and one of them is not a torus manifold, then $(M,\ G)$ is essentially isomorphic to*

$$\left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} N,\quad \prod_{j=1}^{b-1} SO(2m_j+1) \times H\right),$$

such that $(N,\ H)$ is one of the followings:

	N	H
(a)	$(\prod_{i=1}^{a} S^{2l_i+1}) \times_{T^a} S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1) \times U(k_1) \times SO(2k_2-1)$
(b)	$\prod_{i=1}^{a} \mathbb{C}P(l_i) \times S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$	$\prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1) \times SO(2k_2-1)$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times \mathbb{R}^{2k_2-1}$ as a subgroup of $\prod_{j=1}^{b} \mathbb{Z}_2$ and on the fibre of N through the following representations:

- **(a):** $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\} \subset S^1$ *on* $S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}^{2k_2-1}) \cap \mathbb{C}_{\mathfrak{c}}^{k_1}$;
- **(b):** $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k_1}\} \subset SO(2k_1)$ *on* $S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1}) \cap \mathbb{R}^{2k_1}$;

respectively, such that if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \in \mathcal{A} \subset \prod_{j=1}^{b} \mathbb{Z}_2$ then

$$\sigma(1,\ \cdots,\ 1,\ -I_{2k_2-1}) = -1$$

for $\sigma = \sigma_{\mathbb{C}}$ and $\sigma_{\mathbb{R}}$.

Here, G-actions on M are as follows: $\prod SO(2m_j+1)$ and $\prod SU(l_i+1)$ act naturally on $\prod S^{2m_j}$ and $\prod S^{2l_i+1}$, respectively; and $U(k_1)$, $SO(2k_1)$ and $SO(2k_2-1)$ act naturally on $\mathbb{C}_{\mathfrak{c}}^{k_1}$, $\mathbb{R}^{2k_1}$ and $\mathbb{R}^{2k_2-1}$, respectively.

Note that the following facts: if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \notin \mathcal{A}$ then $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times \mathbb{R}^{2k_2-1}$ freely; if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \in \mathcal{A}$ then $\mathcal{A} = \mathcal{A}' \times \{\pm I_{2k_2-1}\}$ and $\mathcal{A}'$ acts on $\prod_{j=1}^{b-1} S^{2m_j}$ freely and $\{\pm I_{2k_2-1}\}$ acts on $S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}^{2k_2-1})$ or $S(\mathbb{R}^{2k_1} \oplus \mathbb{R}^{2k_2-1})$ freely because of the properties of σ described in Theorem 8.1. Therefore, M in Theorem 8.1 is a manifold. Moreover, there is the case that $\mathcal{A} \subset \prod_{j=1}^{b-1} \mathbb{Z}_2$; hence, we do not write manifolds in Theorem 8.1 as manifolds in Theorem 5.1, i.e., manifolds divided by $\mathcal{A} \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $S(\mathbb{C}_{\mathfrak{c}}^{k_1} \oplus \mathbb{R}) \cap \mathbb{R}$ (or $S(\mathbb{R}^{2k_1} \oplus \mathbb{R}) \cap \mathbb{R}$).

In order to prove the above Theorem 8.1, we will use the following notations.

- Natural projections: $p_i : G \to SU(l_i+1)$, $p : G \to SU(k_1)$, $q : G \to T^1$ and $r_j : G \to SO(2m_j+1)$, where $i = 1,\ \cdots,\ a$ and $j = 1,\ \cdots,\ b$.
- Inclusions: $\iota : K \to K_2$ or $\iota : K^o \to K_2^o$, and $\iota_2 : K_2 \to G$ or $\iota_2 : K_2^o \to G$.

8.2. Structure of K_2. Before we start to prove Theorem 8.1, in this subsection, we will prove the following Lemma 8.2.

Let $N_1 \circ N_2$ be $(N_1 \times N_2)/F$ for some finite, normal subgroup $F \subset N_1 \times N_2$, where N_1 and N_2 are connected Lie groups. Then, the following lemma holds.

Lemma 8.2. *For the cases (3)-(a) and (3)-(b), the pair $(K_2^o,\ K^o)$ is isomorphic to*

$$(Spin(2k_2-1) \circ K_2',\ Spin(2k_2-2) \circ K_2')$$

for some Lie group K_2' and $k_2 \geq 2$.

Proof. Let $\widetilde{K_2^o}$ be the covering of K_2^o such that it is a product of simply connected, simple Lie groups and tori (see Section 2.3 in Chapter 1). Since a connected component K_2^o acts on $S^{2k_2-2} \cong K_2/K$ transitively through σ_2 (see (8.3)), there is a factor H in the product group $\widetilde{K_2^o}$ (i.e., $\widetilde{K_2^o} = H \times \widetilde{K_2'}$ for some product group $\widetilde{K_2'}$) such that $H = Spin(2k_2-1)$ or $H = G_2$ for $k_2 = 4$ by the classification result of transitive actions on even dimensional spheres (see Theorem 5.1, 5.2 in Chapter 2), where here G_2 is the exceptional Lie group. Therefore, there is a subgroup K_2' such that $(K_2^o,\ K^o)$ can be denoted as follows:

$$(Spin(2k_2-1) \circ K_2',\ Spin(2k_2-2) \circ K_2');$$
$$(G_2 \circ K_2',\ SU(3) \circ K_2').$$

In order to prove this lemma, we assume

$$(K_2^o,\ K^o) = (G_2 \circ K_2',\ SU(3) \circ K_2'). \tag{8.4}$$

We will prove that this case does not occur.

Taking some covering of K^o in (8.4), we can put

$$\widetilde{K^o} = SU(3) \times \widetilde{K_2'}, \tag{8.5}$$

where $\widetilde{K_2'}$ is a product of simply connected, simple Lie groups and tori. On the other hand, taking a covering of K^o in the cases (3)-(a) and (3)-(b) (see Section 4.1.1, 4.1.2 or (9.1) in Section 9, (10.1) in Section 10), we can put

$$\widetilde{K^o} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times L \times T, \tag{8.6}$$

where T is a torus and

- $L = SU(k_1 - 1)$ for the case (3)-(a),
- $L = Spin(2k_1 - 1)$ for the case (3)-(b).

Because $\dim Spin(x) \neq \dim SU(3)$ for all $x \in \mathbb{N}$, there are the following two cases by (8.5) and (8.6):

(1) $l_a = 3$ and $\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a-1} SU(l_i) \times L \times T$, in the case (3)-(a) or (3)-(b);
(2) $k_1 = 4$ and $\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T$, in the case (3)-(a).

Suppose $l_a = 3$ and $\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a-1} SU(l_i) \times L \times T$. Let $p_a : G \to SU(l_a + 1)$ be the natural projection (see notations in Section 8.1). Then we have that

$$p_a(SU(3) \circ \{e\}) \subset p_a(G_2 \circ \{e\}) \subset p_a(K_2^o) \subset p_a(G) = SU(4)$$

because $SU(3) \circ \{e\} \subset G_2 \circ \{e\} \subset K_2^o \subset G$, where $\{e\} \subset K_2'$ is the identity element in K_2'. Since $SU(3) \circ \{e\} \subset K^o$ and $p_a(K^o) = S(U(3) \times U(1))$ by Section 4.1, we also have that

$$p_a(SU(3) \circ \{e\}) = SU(3).$$

Therefore, $p_a(G_2 \circ \{e\})$ is a non-trivial subgroup in $SU(4)$. Since the restricted representation $p_a|_{G_2 \circ \{e\}}$ is a non-trivial representation and G_2 is a simple Lie group, we also have that

$$\dim p_a(G_2 \circ \{e\}) = \dim G_2 = 14.$$

It follows that there is a subgroup $H \subset SU(4)$ such that $\dim H = 14$ and $SU(4)/H \cong S^1$, because $SU(4)$ is compact and $\dim SU(4) = 15$. However, this gives a contradiction because $SU(4)$-action on S^1 is trivial (see Theorem 5.2, 5.3 in Chapter 2).

Suppose $k_1 = 4$ and $\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T$. In this case, the above argument for $l_a = 3$ can also work for the natural projection $p : G \to SU(k_1)$. It follows that the case $(K_2^o,\ K^o) = (G_2 \circ K_2',\ SU(3) \circ K_2')$ does not occur. Hence, we have Lemma 8.2. □

In the next two sections, we study the cases (3)-(a) and (3)-(b).

9. The case (3)-(a)

In this section, we study the case (3)-(a). From Section 4.1, we have

$$G = \prod_{j=1}^{b} SO(2m_j + 1) \times \prod_{i=1}^{a} SU(l_i + 1) \times SU(k_1) \times T^1,$$

$$K_1 = \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i) \times U(1)) \times SU(k_1) \times T^1,$$

$$K = \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \,\middle|\, \sigma_1(A) t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\}. \tag{9.1}$$

where $A \in \mathcal{S}_1 \subset \prod_{j=1}^{b} S(O(2m_j) \times O(1))$, $(t_1, \cdots, t_a) \in \prod_{i=1}^{a} S(U(l_i) \times U(1))$, $t \in T^1$ and $X \in U(k_1 - 1)$ such that $x \det X = 1$. If $k_1 = 1$, then we can take $\alpha = 1$. Moreover, we have that the finite covering of K^o is as follows by (9.1):

$$\widetilde{K^o} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times SU(k_1 - 1) \times T^1 \quad \text{if } k_1 \geq 2;$$
$$\widetilde{K^o} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \quad \text{if } k_1 = 1.$$

Because of Lemma 8.2, we also have

$$K^o = Spin(2k_2 - 2) \circ K_2' \quad \text{and} \quad \widetilde{K^o} = Spin(2k_2 - 2) \times \widetilde{K_2'}, \tag{9.2}$$
$$K_2^o = Spin(2k_2 - 1) \circ K_2' \quad \text{and} \quad \widetilde{K_2^o} = Spin(2k_2 - 1) \times \widetilde{K_2'}. \tag{9.3}$$

In order to classify the case (3)-(a), we will divide this case into the following two cases:

- $k_2 \geq 3$ (we will discuss in Section 9.1);
- $k_2 = 2$ (we will discuss in Section 9.2).

9.1. The case $k_2 \geq 3$. Assume $k_2 \geq 3$. Comparing coverings $\widetilde{K^o}$ of the above K^o's in (9.1) and (9.2), and using the fact that $Spin(4) \simeq SU(2) \times SU(2)$ and $Spin(6) \simeq SU(4)$, there are the following five cases:

(i): $Spin(2k_2 - 2) = Spin(2m_b)$, and $k_2 = m_b + 1 \geq 3$;
(ii): $Spin(2k_2 - 2) = SU(l_{a-1}) \times SU(l_a)$, and $k_2 = 3$, $l_{a-1} = l_a = 2$;
(iii): $Spin(2k_2 - 2) = SU(l_a) \times SU(k_1 - 1)$, and $k_2 = 3 = k_1$, $l_a = 2$;
(iv): $Spin(2k_2 - 2) = SU(l_a)$ and $k_2 = 4$, $l_a = 4$;
(v): $Spin(2k_2 - 2) = SU(k_1 - 1)$ and $k_2 = 4$, $k_1 = 5$,

First we prove the following lemma.

LEMMA 9.1. *In the above cases, the cases from (ii) to (v) do not occur.*

PROOF. If the case (ii) occurs, then $Spin(2k_2 - 2) = SU(l_{a-1}) \times SU(l_a)$ and $k_2 = 3$, $l_{a-1} = l_a = 2$. Let $\varphi : \widetilde{K_2^o} \to K_2^o$ be the finite covering. By using (9.1), (9.2) and (9.3), we have

$$\begin{aligned} K_2' &= \prod_{j=1}^{b} SO(2m_j) \times \left\{ \left(\begin{pmatrix} A_1 & 0 \\ 0 & t_1 \end{pmatrix}, \cdots, \begin{pmatrix} A_{a-2} & 0 \\ 0 & t_{a-2} \end{pmatrix}, \right.\right. \\ &\quad \left.\left. \begin{pmatrix} r_{a-1} I_2 & 0 \\ 0 & t_{a-1} \end{pmatrix}, \begin{pmatrix} r_a I_2 & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \;\middle|\; t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\} \\ &= \varphi(\widetilde{K_2''} \times T^2), \end{aligned}$$

where $r_{a-1}^{-2} = t_{a-1}$, $r_a^{-2} = t_a$, $\widetilde{K_2''}$ is the product of factors in $\widetilde{K^o}$ except $Spin(2k_2 - 2)$ and $T^2 (\subset T^a)$. By (9.2), (9.3), Section 3.1 in Chapter 1 (i.e., for the factor $H \subset \widetilde{K^o}$ in $H' \subset \widetilde{K_2^o}$ such that $H \subset H'$, if $\widetilde{K^o}$ and $\widetilde{K_2^o}$ are same rank then H and H' are same rank) and the assumption of the case (ii), we have the following commutative diagram:

$$\begin{array}{ccc} (T^2 \times SU(2) \times SU(2)) \times \widetilde{K_2''} & \xrightarrow{\tilde{\iota}} & (T^2 \times Spin(5)) \times \widetilde{K_2''} \\ \varphi \downarrow & & \downarrow \varphi \\ K^o & \xrightarrow{\iota} & K_2^o, \end{array}$$

where $\widetilde{\iota}$ is an inclusion map. By using the above diagram and the definition of K and G, we have the following sequence (also see Section 8.1 about the definitions of notations):

$$\begin{aligned}
& S(U(2) \times U(1)) \times S(U(2) \times U(1)) \\
= \ & (p_{a-1} \times p_a) \circ \iota_2 \circ \iota \circ \varphi(T^2 \times SU(2) \times SU(2)) \\
= \ & (p_{a-1} \times p_a) \circ \iota_2 \circ \varphi \circ \widetilde{\iota}(T^2 \times SU(2) \times SU(2)) \\
\subset \ & (p_{a-1} \times p_a) \circ \iota_2 \circ \varphi(T^2 \times Spin(5)) \\
\subset \ & p_{a-1} \times p_a(G) \\
= \ & SU(3) \times SU(3).
\end{aligned}$$

This sequence implies that there is a non-trivial representation from $T^2 \times Spin(5)$ to $SU(3) \times SU(3)$. Since $Spin(5)$ is a simple Lie group and rank $(T^2 \times Spin(5)) =$ rank $(SU(3) \times SU(3))$, there is some subgroup $H \subset SU(3)$ such that $Spin(5) \approx H$ (because of Section 3.1 in Chapter 1), where $Spin(5) \approx H$ means that $Spin(5)$ and H have the same Lie algebra. This gives a contradiction, because $\dim Spin(5) = \dim H = 10 > 8 = \dim SU(3)$. Hence, the case (ii) does not occur.

With an argument similar to the above for the case (ii), we can also prove that the cases (iii) does not occur.

If the case (iv) occurs, then $Spin(2k_2-2) = SU(l_a)$ and $k_2 = 4, l_a = 4$. With a method similar to that demonstrated in the proof of the case (ii), we have the following commutative diagram:

$$\begin{array}{ccc}
(T^1 \times SU(4)) \times \widetilde{K_2''} & \xrightarrow{\widetilde{\iota}} & (T^1 \times Spin(7)) \times \widetilde{K_2''} \\
\varphi \downarrow & & \downarrow \varphi \\
K^o & \xrightarrow{\iota} & K_2^o.
\end{array}$$

We have the following sequence by using the above diagram and the definitions of K and G:

$$\begin{aligned}
& S(U(4) \times U(1)) \\
= \ & p_a \circ \iota_2 \circ \iota \circ \varphi(T^1 \times SU(4)) \\
= \ & p_a \circ \iota_2 \circ \varphi \circ \widetilde{\iota}(T^1 \times SU(4)) \\
\subset \ & p_a \circ \iota_2 \circ \varphi(T^1 \times Spin(7)) \\
= \ & H \\
\subset \ & p_a(G) = SU(5).
\end{aligned}$$

Because $S(U(4) \times U(1))$, $SU(5)$ and $T^1 \times Spin(7)$ are the same rank Lie groups and $Spin(7)$ is a simple Lie group, we have $\dim H = \dim(T^1 \times Spin(7)) = 22$. Then we see $SU(5)/H$ is a 2-dimensional manifold by $\dim SU(5) = 24$. Moreover, $H = p_a \circ \iota_2 \circ \varphi(T^1 \times Spin(7))$ is connected. Therefore, we have $SU(5)/H$ is a simply connected, compact manifold, because of the homotopy exact sequence of $H \to SU(5) \to SU(5)/H$. Hence, $SU(5)/H \cong S^2$. However, the $SU(5)$-action on S^2 must be trivial (see Theorem 5.2 in Chapter 2). This gives a contradiction. Hence, the case (iv) does not occur.

With a method similar to that demonstrated in the above for the case (iv), we can also prove that the case (v) does not occur. $\square$

Because of the above Lemma 9.1, we have that

$$Spin(2k_2 - 2) = Spin(2m_b) \quad \text{and} \quad k_2 = m_b + 1 \geq 3. \tag{9.4}$$

Now we set

$$\left\{ \left((t_1, \cdots, t_a), \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, t \right) \ \middle| \ t_1^{\alpha_1} \cdots t_a^{\alpha_a} x t^{\alpha} = 1 \right\} = P(\alpha_1, \cdots, \alpha_a, x, \alpha).$$

If $k_1 = 1$, then $x = 1$ and we can assume $\alpha = 1$ up to essential isomorphism. Therefore, $P(\alpha_1, \cdots, \alpha_a, x, \alpha)$ is connected. Then, the following relation holds by using (9.1), (9.2) and

(9.4):

$$\begin{aligned} K^o &= Spin(2k_2-2)\circ K_2' = Spin(2m_b)\circ K_2' \\ &= \prod_{j=1}^{b} SO(2m_j)\times P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha). \end{aligned}$$

Therefore, we can put $K^o = SO(2m_b)\times K_2' = SO(2k_2-2)\times K_2'$ and

$$\begin{aligned} K_2' &= \prod_{j=1}^{b-1} SO(2m_j)\times P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha) \\ &\subset \prod_{j=1}^{b-1} SO(2m_j+1)\times \prod_{i=1}^{a} SU(l_i+1)\times SU(k_1)\times T^1 \\ &= G/SO(2m_b+1) = G/SO(2k_2-1). \end{aligned}$$

By using $K_2^o = Spin(2k_2-1)\circ K_2'$ (by (9.3)) and the above K^o, we have the following covering map φ:

$$\begin{array}{ccc} Spin(2k_2-2)\times K_2' & \longrightarrow & Spin(2k_2-1)\times K_2' \\ \varphi\downarrow & & \downarrow\varphi \\ SO(2k_2-2)\times K_2' & \longrightarrow & K_2^o \end{array}$$

where the top and bottom maps are inclusions. Because the restricted representation $\varphi|_{K_2'}$ is the identity representation, there is the K_2' factor in K_2^o. Therefore, we have that

$$\begin{aligned} K_2^o &= SO(2k_2-1)\times K_2' \\ &= \prod_{j=1}^{b-1} SO(2m_j)\times SO(2k_2-1)\times P(\alpha_1,\ \cdots,\ \alpha_a,\ x,\ \alpha), \end{aligned}$$

and there is some inclusion $SO(2k_2-1)\to K_2$ whose image is the factor $SO(2m_b+1)\subset G$. Because $K_2^o\subset K_2\subset G$, we can put K_2 is as follows:

$$\begin{aligned} & K_2 = SO(2k_2-1)\times K_2'' \\ \subset\ & G = SO(2k_2-1)\times\left(\prod_{j=1}^{b-1} SO(2m_j+1)\times\prod_{i=1}^{a} SU(l_i+1)\times SU(k_1)\times T^1\right), \end{aligned}$$

where $2m_b+1 = 2k_2-1$ and K_2'' is a subgroup of $\prod_{j=1}^{b-1} SO(2m_j+1)\times\prod_{i=1}^{a} SU(l_i+1)\times SU(k_1)\times T^1$ whose connected component is K_2'. By using the argument of Section 8.2, the $SO(2k_2-1)$-factor in K_2 acts transitively on $K_2/K\cong S^{2k_2-2}$. Therefore, for the natural projection $\Phi : K_2 = SO(2k_2-1)\times K_2''\to K_2''$, we have the following relation (see [**24**, Lemma 8.0.2]):

$$\Phi(K_2) = K_2'' = \Phi(K).$$

Hence, we have

$$K_2 = SO(2k_2-1)\times K_2'' = SO(2k_2-1)\times\Phi(K)\subset G.$$

It follows that the inclusion $K_2\subset G$ is completely determined by K (more precisely the projection $\Phi(K)$).

Next, we consider the slice representation $\sigma_2 : K_2\to O(2k_2-1)$. Since the $SO(2k_2-1)$-factor in K_2 acts transitively on $K_2/K\cong S^{2k_2-2}$, the restricted representation $\sigma_2|_{SO(2k_2-1)}$ is the natural isomorphism to $SO(2k_2-1)\subset O(2k_2-1)$. Hence, we have that

$$\sigma_2(K_2'')\subset Z(SO(2k_2-1)) = \{\pm I_{2k_2-1}\}\subset O(2k_2-1).$$

Moreover, by (9.1), we have the following formula for K:

$$
\begin{aligned}
(9.5) \qquad K &= \sigma_2^{-1}(O(2k_2-2)) \\
&= \left\{ \left(A,\ (t_1,\ \cdots,\ t_a),\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},\ t \right) \middle| \ \sigma_1(A)t_1^{\alpha_1}\cdots t_a^{\alpha_a}xt^{\alpha} = 1 \right\} \\
&= \left\{ \left(\begin{pmatrix} A_b & 0 \\ 0 & a_b \end{pmatrix}, Y \right) \middle| \ a_b = \det A_b^{-1} = \det \sigma_2(Y) \right\}. \\
&\subset S(O(2k_2-2)\times O(1)) \times K_2''.
\end{aligned}
$$

Therefore, we can easily show that the following lemma by using (9.5).

Lemma 9.2. *The following two statements are equivalent:*

(1): $\sigma_2(K_2'') = \{I_{2k_2-1}\}$ *(resp.* $\sigma_2(K_2'') = \{\pm I_{2k_2-1}\}$*);*
(2): $K = SO(2k_2-2)\times K_2''$ *(resp.* $r_b(K) = S(O(2m_b)\times O(1))$*).*

Moreover, the following statement holds:

(3): *if* $\sigma_2(K_2'') = \{\pm I_{2k_2-1}\}$, *then we have* $K \neq S(O(2m_b)\times O(1))\times K_2''$.

It follows from (9.5) and Lemma 9.2 (1), (2) that the slice representation $\sigma_2 : K_2 \to O(2k_2-2)$ is also completely determined by K. Therefore, the tubular neighborhood $X_2 = G\times_{K_2} D^{2k_2-1}$ of G/K_2 is completely determined by K, and equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} \left(\left(\prod_{j=1}^{b-1} S^{2m_j} \times D(\mathbb{R}^{2k_2-1}) \right) \times_{\mathcal{A}} S(\mathbb{C}_{\mathfrak{c}}^{k_1}) \right),$$

where T^a-quotient is defined similarly to that in the previous cases (e.g. the case (2)-(a)), $\mathcal{A} \simeq \mathcal{S}_1/\mathcal{S}_1^o$-quotient is defined by the following actions:

- on $\prod_{j=1}^{b-1} S^{2m_j}\times D(\mathbb{R}^{2k_2-1})$ as the subgroup $\mathcal{A} \subset \prod_{j=1}^{b-1}\mathbb{Z}_2\times\{\pm I_{2k_2-1}\}$;
- on $S(\mathbb{C}_{\mathfrak{c}}^{k_1}) \subset \mathbb{C}_{\mathfrak{c}}^{k_1}$ by the representation $\sigma_{\mathbb{C}} : \mathcal{A} \to \{\pm 1\}$ (induced by σ_1).

Moreover, by using (9.5) and Lemma 9.2 (3), if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \in \mathcal{A} \subset \prod_{j=1}^{b-1}\mathbb{Z}_2\times\{\pm I_{2k_2-1}\}$, then $\sigma_{\mathbb{C}}(1,\ \cdots,\ 1,\ -I_{2k_2-1}) = -1$ (because if not so the principal isotropy subgroup is $K = S(O(2m_b)\times O(1))\times K_2''$). It follows that $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j}\times D(\mathbb{R}^{2k_2-1})\times S(\mathbb{C}_{\mathfrak{c}}^{k_1})$ freely; therefore, X_2 is a manifold.

By using Remark 4.2, we can easily check that the pair $(M,\ G)$ of the case (3)-(a) and $k_2 \geq 3$ is as follows (up to essential isomorphism):

$$
\begin{aligned}
M &= \prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} \left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} S(\mathbb{R}^{2k_2-1}\oplus \mathbb{C}_{\mathfrak{c}}^{k_1}) \right); \\
G &= \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b-1} SO(2m_j+1) \times SO(2k_2-1)\times U(k_1),
\end{aligned}
$$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j}\times S(\mathbb{R}^{2k_2-1}\oplus\mathbb{C}_{\mathfrak{c}}^{k_1})$ as follows: on $\mathbb{C}_{\mathfrak{c}}^{k_1}$ by $\sigma_{\mathbb{C}} : \mathcal{A}\to\{\pm1\}$; on $\prod_{j=1}^{b-1} S^{2m_j}\times\mathbb{R}^{2k_2-1}$ as the subgroup $\prod_{j=1}^{b-1}\mathbb{Z}_2\times\{\pm I_{2k_2-1}\}$ such that if $(1,\ \cdots,\ 1,\ -I_{2k_2-1})\in\mathcal{A}$ then $\sigma_{\mathbb{C}}(1,\ \cdots,\ 1,\ -I_{2k_2-1}) = -1$.

This corresponds with the first manifold in Theorem 8.1 for $k_2 \geq 3$.

9.2. The case $k_2 = 2$. Assume $k_2 = 2$. By (9.1) and (9.2), the covering of K^o is as follows:

$$\begin{aligned}
& \widetilde{K^o} = Spin(2k_2-2) \times \widetilde{K_2'} \\
= \ & \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times SU(k_1-1) \times T^1 \quad \text{for } k_1 \geq 2 \quad \text{or} \\
= \ & \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \quad \text{for } k_1 = 1,
\end{aligned}$$

where $l_i \geq 1$ for all $i = 1, \cdots, a$. Comparing the above coverings of K^o's, there are the following three cases:

(i): $Spin(2k_2-2) = Spin(2m_b)$, and $k_2 = m_b + 1 = 2$;
(ii): $Spin(2k_2-2) = T_a$, where T_a is the a-th factor of $T^a = T_1 \times \cdots \times T_a$ $(T_i \simeq T^1)$;
(iii): $Spin(2k_2-2) = T^1$, and $k_1 \geq 2$.

Remark 9.3. The above case (i) is the same as the case (i) in Section 9.1. By the same argument of Section 9.1, we have that (M, G) for the above case (i) is as follows ($k_2 = 2$):

$$M = \prod_{i=1}^{a} S^{2l_i+1} \times_{T^a} \left(\prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} S(\mathbb{R}^{2k_2-1} \oplus \mathbb{C}^{k_1}_{\mathfrak{c}}) \right);$$

$$G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b-1} SO(2m_j+1) \times SO(2k_2-1) \times U(k_1),$$

where T^a and $\mathcal{A}$ quotients are similarly defined as that of M in Section 9.1. This corresponds with the first manifold in Theorem 8.1 for $k_2 = 2$. Hence, in this section, we may only discuss with the other cases: the case (ii) and (iii).

First we prove the following lemma.

Lemma 9.4. *In the above cases, the case (iii) does not occur.*

Proof. By the definition of the case (iii),

$$\widetilde{K_2'} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times SU(k_1-1).$$

By (9.2), (9.3), we can put $\widetilde{K_2^o} = Spin(2k_2-1) \times \widetilde{K_2'}$. Therefore, with a method similar to that demonstrated in the proof of Lemma 9.1, there is the following commutative diagram ($k_2 = 2$):

$$\begin{array}{ccc}
Spin(2k_2-2) \times \widetilde{K_2'} & \xrightarrow{\widetilde{\iota}} & Spin(2k_2-1) \times \widetilde{K_2'} \\
\varphi \downarrow & & \downarrow \varphi \\
K^o & \xrightarrow{\iota} & K_2^o,
\end{array}$$

where φ is the finite covering, and $\widetilde{\iota}$ and ι are inclusion maps. Hence, the following sequence holds by the commutativity of the above diagram and the definition of G (also see notations in Section 8.1):

$$\begin{aligned}
q \circ \iota_2 \circ \iota \circ \varphi(Spin(2)) &= q \circ \iota_2 \circ \varphi \circ \widetilde{\iota}(Spin(2)) \\
&\subset q \circ \iota_2 \circ \varphi(Spin(3)) \\
&\subset q(G) = T^1,
\end{aligned}$$

On the other hand, by (9.1), we have

$$q \circ \iota_2 \circ \iota \circ \varphi(Spin(2)) = T^1 = q(G) = q \circ \iota_2 \circ \iota(K^o).$$

Consequently, we have $q \circ \iota_2 \circ \phi(Spin(3)) = T^1$. However, this gives a contradiction, because there is no non-trivial representation from $Spin(3)$ to T^1. It follows that the case (iii) does not occur. $\square$

By Remark 9.3 and Lemma 9.4, we may only study the case (ii) in this subsection.

Assume the case (ii) occurs, that is, $Spin(2k_2-2) = T_a$, where T_a is the a-th factor of $T^a = T_1 \times \cdots \times T_a$ ($T_i \simeq T^1$). Let $\pi_j : Spin(2m_j) \to SO(2m_j)$ be the double covering. In order to study this case, we divide this case into two parts:

- the case $k_1 = 1$;
- the case $k_1 \geq 2$.

9.2.1. *The case* $k_1 = 1$. First, we assume $k_1 = 1$. Then, we have the following finite covering:

$$\pi : \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a = \widetilde{K_2'} \times T_a \longrightarrow K^o$$

such that

$$\begin{aligned} \pi(A_j,\ B_i,\ t_i) &= \left(\pi_j(A_j), \begin{pmatrix} B_i t_i^{-1/l_i} & 0 \\ 0 & t_i \end{pmatrix}, t_1^{-\alpha_1}\cdots t_a^{-\alpha_a}\right) \\ &\in K^o \subset \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times T^1, \end{aligned}$$

for the element $(A_j,\ B_i,\ t_i) \in \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a$.

If $\alpha_a \neq 0$, then K^o has the following subgroup:

$$\begin{aligned} &\pi(T_a) = \pi(Spin(2k_2-2)) = \left\{ \left(\begin{pmatrix} I_{l_a} t_a^{-1/l_a} & 0 \\ 0 & t_a \end{pmatrix}, t_a^{-\alpha_a} \right) \,\middle|\, t_a \in T_a \right\} \\ \subset\ & (S(U(l_a)\times U(1)) \times T^1) \cap K^o, \end{aligned}$$

where I_{l_a} is the identity element in $U(l_a)$. Therefore, for the following commutative diagram (by (9.2) and (9.3)):

$$\begin{array}{ccc} T_a \times \widetilde{K_2'} & \xrightarrow{\widetilde{\iota}} & Spin(3) \times \widetilde{K_2'} \\ \pi \downarrow & & \downarrow \widetilde{\pi} \\ K^o & \xrightarrow{\iota} & K_2^o, \end{array}$$

we have the following sequence (also see notations in Section 8.1):

$$\begin{aligned} T^1 = q\circ\iota_2\circ\iota\circ\pi(T_a) &= q\circ\iota_2\circ\iota\circ\pi(Spin(2k_2-2)) \\ &= q\circ\iota_2\circ\widetilde{\pi}\circ\widetilde{\iota}(Spin(2k_2-2)) \\ &\subset q\circ\iota_2\circ\widetilde{\pi}(Spin(2k_2-1)) \\ &\subset q(G) = T^1. \end{aligned}$$

This gives a contradiction with a method similar to that demonstrated in the proof of Lemma 9.4. Hence, we have $\alpha_a = 0$.

Because $\alpha_a = 0$, we have that $\pi(T_a \times SU(l_a)) = S(U(l_a)\times U(1))$. Therefore, there is the following sequence:

$$\begin{aligned} S(U(l_a)\times U(1)) &= p_a\circ\iota_2\circ\iota\circ\pi(T_a \times SU(l_a)) \\ &= p_a\circ\iota_2\circ\widetilde{\pi}\circ\widetilde{\iota}(T_a\times SU(l_a)) \\ &\subset p_a\circ\iota_2\circ\widetilde{\pi}(Spin(3)\times SU(l_a)) \\ &\subset p_a(G) = SU(l_a+1). \end{aligned}$$

Because $Spin(3)$ and $SU(l_a)$ are simple and $\dim(SU(l_a)\times Spin(3)) = l_a^2+2$ and $\dim S(U(l_a)\times U(1)) = l_a^2$, we have that $p_a\circ\iota_2\circ\widetilde{\pi}(SU(l_a)\times Spin(3)) \neq S(U(l_a)\times U(1))$. Since $S(U(l_a)\times U(1))$ is a maximal rank, maximal subgroup in $SU(l_a+1)$ (see [**38**]), we have

$$p_a\circ\iota_2\circ\widetilde{\pi}(SU(l_a)\times Spin(3)) = SU(l_a+1).$$

Comparing their dimension ($\dim(SU(l_a)\times Spin(3)) = l_a^2+2$ and $\dim SU(l_a+1) = l_a^2+2l_a$), we have

$$l_a = 1$$

(remark $Spin(3) \simeq SU(2)$). Hence, in the case $k_1 = 1$, we can regard

$$\begin{aligned}
& (SU(l_a+1),\ S(U(l_a)\times U(1))) \\
= \;& (Spin(2k_2-1),\ Spin(2k_2-2)) \\
= \;& (Spin(2m_{b+1}+1),\ Spin(2m_{b+1}))
\end{aligned}$$

where $m_{b+1} = 1$. Therefore, by regarding $b+1$ as b and $a-1$ as a, we can regard the case (ii) with $k_1 = 1$ as the case (i). Hence, this case corresponds with the first case of Theorem 8.1 which satisfies $k_1 = 1$ and $k_2 = 2$ by using Remark 9.3.

9.2.2. *The case $k_1 \geq 2$.* Next, we study the other case: the case $k_1 \geq 2$.

If $k_1 \geq 2$, then we have the following finite coverings:

$$\pi : \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times SU(k_1-1) \times T^1 \to K^o$$

such that,

$$\begin{aligned}
\pi(A_j, B_i, t_i, Y, t) &= \left(\pi_j(A_j), \begin{pmatrix} B_i t_i^{-1/l_i} & 0 \\ 0 & t_i \end{pmatrix}, \begin{pmatrix} Y x^{-1/(k_1-1)} & 0 \\ 0 & x \end{pmatrix}, t\right) \\
\in \;& \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i \times U(1)) \times S(U(k_1-1)\times U(1)) \times T^1,
\end{aligned}$$

where $x = t_1^{-\alpha_1}\cdots t_a^{-\alpha_a} t^{-\alpha}$.

If $\alpha_a = 0$, then we can easily show that $l_a = 1$ and this case corresponds with the first case of Theorem 8.1 which satisfies $k_1 \geq 2$ and $k_2 = 2$ with the method similar to that demonstrated in the previous section (Section 9.2.1). Therefore, we may only discuss with the case $\alpha_a \neq 0$.

Assume $\alpha_a \neq 0$. We will prove this case does not occur. First, we prove the following lemma.

LEMMA 9.5. *If $k_1 \geq 2$ and $\alpha_a \neq 0$, then we can put $l_a = 1$, $k_1 = 2$ and $\alpha_a = \pm 1$.*

PROOF. If $k_1 \geq 2$ and $\alpha_a \neq 0$, then K^o has the following subgroups ($k_2 = 2$):

$$\begin{aligned}
(9.6)\qquad & \pi(SU(l_a)\times Spin(2k_2-2)) \\
= \;& \left\{\left(\begin{pmatrix} B_a t_a^{-1/l_a} & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} I_{k_1-1} t_a^{\alpha_a/(k_1-1)} & 0 \\ 0 & t_a^{-\alpha_a} \end{pmatrix}\right)\right\} \\
(9.7)\qquad & \pi(Spin(2k_2-2)\times SU(k_1-1)) \\
= \;& \left\{\left(\begin{pmatrix} I_{l_a} t_a^{-1/l_a} & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} Y t_a^{\alpha_a/(k_1-1)} & 0 \\ 0 & t_a^{-\alpha_a} \end{pmatrix}\right)\right\}
\end{aligned}$$

where $t_a \in T_a = Spin(2k_2-2)$, $B_a \in SU(l_a)$, $Y \in SU(k_1-1)$, and I_{k_1-1}, I_{l_a} are identity elements in $U(k_1-1)$ and $U(l_a)$, respectively. Moreover, there is the following commutative diagram by (9.2) and (9.3):

$$\begin{array}{ccc}
(SU(l_a)\times Spin(2)\times SU(k_1-1))\times \widetilde{K_2''} & \xrightarrow{\tilde{\iota}} & (SU(l_a)\times Spin(3)\times SU(k_1-1))\times \widetilde{K_2''} \\
\pi\downarrow & & \downarrow\tilde{\pi} \\
K^o & \xrightarrow{\iota} & K_2^o,
\end{array}$$

where $\widetilde{K_2''} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a-1} SU(l_i) \times T^{a-1} \times T^1$. Then, we have the following sequence by (9.6):

$$\begin{aligned} S(U(l_a) \times U(1)) &= p_a \circ \iota_2 \circ \iota \circ \pi(SU(l_a) \times Spin(2)) \\ &= p_a \circ \iota_2 \circ \widetilde{\pi} \circ \widetilde{\iota}(SU(l_a) \times Spin(2)) \\ &\subset p_a \circ \iota_2 \circ \widetilde{\pi}(SU(l_a) \times Spin(3)) \\ &\subset p_a(G) = SU(l_a + 1). \end{aligned}$$

With a method similar to that demonstrated in the proof of $l_a = 1$ in Section 9.2.1, we have $l_a = 1$. Moreover, by (9.7), we have the following sequence:

$$\begin{aligned} S(U(k_1 - 1) \times U(1)) &= p \circ \iota_2 \circ \iota \circ \pi(Spin(2) \times SU(k_1 - 1)) \\ &= p \circ \iota_2 \circ \widetilde{\pi} \circ \widetilde{\iota}(Spin(2) \times SU(k_1 - 1)) \\ &\subset p \circ \iota_2 \circ \widetilde{\pi}(Spin(3) \times SU(k_1 - 1)) \\ &\subset p(G) = SU(k_1). \end{aligned}$$

Similarly, we have $k_1 = 2$.

Moreover, we can easily show that $p \circ \iota_2 \circ \widetilde{\pi} : Spin(3) \to SU(k_1) = SU(2)$ is an isomorphic map. By considering the restricted representation to $Spin(2)$ of this isomorphic map and using (9.7), we also have $\alpha_a = \pm 1$. $\square$

By Lemma 9.5, we have $l_a = 1$, $k_1 = 2$ and $|\alpha_a| = 1$; moreover, we have the following commutative diagram (see the proof of Lemma 9.5):

$$\begin{array}{ccc} Spin(2) \times \widetilde{K_2''} & \xrightarrow{\widetilde{\iota}} & Spin(3) \times \widetilde{K_2''} \\ \pi \downarrow & & \downarrow \widetilde{\pi} \\ K^o & \xrightarrow{\iota} & K_2^o. \end{array}$$

Then, we have the following sequence:

$$\begin{aligned} H &= (p_a \times p) \circ \iota_2 \circ \iota \circ \pi(Spin(2)) \\ &= (p_a \times p) \circ \iota_2 \circ \widetilde{\pi} \circ \widetilde{\iota}(Spin(2)) \\ &\subset (p_a \times p) \circ \iota_2 \circ \widetilde{\pi}(Spin(3)) \\ &\subset p_a \times p(G) = SU(l_a + 1) \times SU(k_1) = SU(2) \times SU(2). \end{aligned}$$

Here, H is one of the followings, because of (9.6) and (9.7) in the proof of Lemma 9.5 and $|\alpha_a| = 1$:

$$\Delta = \left\{ \begin{pmatrix} t_a^{-1} & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} t_a^{-1} & 0 \\ 0 & t_a \end{pmatrix} \right\} \subset SU(2) \times SU(2) \text{ for } \alpha_a = -1;$$
$$\Delta' = \left\{ \begin{pmatrix} t_a^{-1} & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} t_a & 0 \\ 0 & t_a^{-1} \end{pmatrix} \right\} \subset SU(2) \times SU(2) \text{ for } \alpha_a = 1.$$

Since Δ and Δ' are conjugate in G, we can take $\alpha_a = -1$ and $H = \Delta$. Because $Spin(3) \simeq SU(2)$ and $H = \Delta$, we can easily show that

$$(p_a \times p) \circ \iota_2 \circ \widetilde{\pi}(Spin(3)) = \{(X,\ X) \mid X \in SU(2)\} \subset SU(2) \times SU(2),$$

i.e., the diagonal subgroup. By the definition of π and the above argument, we see the followings:

$$\begin{aligned} & \iota \circ \pi(Spin(2) \times T^1) \\ = \ & \left\{ \left(\begin{pmatrix} t_a^{-1} & 0 \\ 0 & t_a \end{pmatrix}, \begin{pmatrix} t_a^{-1} t^{\alpha} & 0 \\ 0 & t_a t^{-\alpha} \end{pmatrix}, t \right) \in SU(2) \times SU(2) \times T^1 \right\} \\ \subset \ & \widetilde{\pi}(Spin(3) \times T^1) \\ = \ & \left\{ (X,\ X,\ t) \in SU(2) \times SU(2) \times T^1 \right\}. \end{aligned}$$

It follows that $\alpha = 0$. However, this gives a contradiction because $\alpha \in \mathbb{N}$ (see Section 4.1.1). Therefore, this case (the case $k_1 \geq 2$ and $\alpha_a \neq 0$) does not occur.

10. The case (3)-(b)

In this section, we study the case (3)-(b). From Section 4.1, we have

$$\begin{aligned}
G &= \prod_{j=1}^{b} SO(2m_j+1) \times \prod_{i=1}^{a} SU(l_i+1) \times SO(2k_1),\\
K_1 &= \mathcal{S}_1 \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SO(2k_1),\\
K &= \prod_{i=1}^{a} S(U(l_i)\times U(1)) \qquad\qquad (10.1)\\
&\quad \times \left\{ \left(A, \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \right) \in \mathcal{S}_1 \times S(O(2k_1-1)\times O(1)) \,\middle|\, \sigma_1(A)x = 1 \right\},
\end{aligned}$$

where $k_1 \geq 2$. From Lemma 8.2, we also have

$$K^o = Spin(2k_2-2)\circ K_2', \tag{10.2}$$

$$K_2^o = Spin(2k_2-1)\circ K_2' \tag{10.3}$$

Therefore, we have the covering of K^o as follows by (10.1) and (10.2):

$$\begin{aligned}
\widetilde{K^o} &= Spin(2k_2-2)\times \widetilde{K_2'}\\
&= \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a} SU(l_i) \times T^a \times Spin(2k_1-1).
\end{aligned}$$

Comparing the above covering of K^o's and using the fact that $Spin(2)\simeq T^1$, $Spin(3)\simeq SU(2)$, $Spin(4)\simeq SU(2)\times SU(2)$ and $Spin(6)\simeq SU(4)$, there are the following five cases:

- **(i):** $Spin(2k_2-2) = Spin(2m_b)$ and $k_2 = m_b+1 \geq 2$;
- **(ii):** $Spin(2k_2-2) = T_a$ and $k_2 = 2$, where T_a is the a-th factor of $T^a = T_1\times\cdots\times T_a$ ($T_i \simeq T^1$);
- **(iii):** $Spin(2k_2-2) = SU(l_{a-1})\times SU(l_a)$ and $l_a = l_{a-1} = 2$, $k_2 = 3$;
- **(iv):** $Spin(2k_2-2) = SU(l_a)\times Spin(2k_1-1)$ and $l_a = 2$, $k_1 = 2$, $k_2 = 3$;
- **(v):** $Spin(2k_2-2) = SU(l_a)$ and $l_a = 4$, $k_2 = 4$.

Similarly to Lemma 9.1, we can show the following lemma.

LEMMA 10.1. *In the above cases, the cases from (iii) to (v) do not occur.*

PROOF. For the cases (iii) and (iv), we can apply a method similar to the proof of that the case (ii) does not occur in Lemma 9.1. For the case (v), we can apply a method similar to the proof of that the case (iv) does not occur in Lemma 9.1. So we may omit the detail of the proof. □

From the next section, we study the cases (i) and (ii): we call them (3)-(b)-(i) and (3)-(b)-(ii), respectively.

10.1. The case (3)-(b)-(i). Suppose the case (3)-(b)-(i) occurs, that is,

$$Spin(2k_2-2) = Spin(2m_b) \quad \text{and} \quad k_2 = m_b+1 \geq 2.$$

It follows from (10.1) and (10.2) that the following relation holds:

$$\begin{aligned}
Spin(2k_2-2)\circ K_2' &= Spin(2m_b)\circ K_2'\\
&= \prod_{j=1}^{b} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SO(2k_1-1) = K^o.
\end{aligned}$$

Therefore, we can put $K^o = SO(2m_b)\times K_2' = SO(2k_2-2)\times K_2'$ and

$$K_2' = \prod_{j=1}^{b-1} SO(2m_j) \times \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times SO(2k_1-1).$$

Because $K_2^o = Spin(2k_2-1)\circ K_2'$ (by (10.3)), by using the same argument in Section 9.1, we have

$$K_2^o = \prod_{i=1}^{a} S(U(l_i)\times U(1)) \times \prod_{j=1}^{b-1} SO(2m_b)\times SO(2k_2-1)\times SO(2k_1-1),$$

and there is the inclusion $SO(2k_2-1)\to K_2$ such that its image is $SO(2m_b+1)\subset G$. Because $K_2^o\subset K_2\subset G$, we can put K_2 is as follows:

$$\begin{aligned} & K_2 = SO(2k_2-1)\times K_2'' \\ \subset\quad & G = SO(2k_2-1)\times\left(\prod_{j=1}^{b-1} SO(2m_j+1)\times\prod_{i=1}^{a} SU(l_i+1)\times SO(2k_1)\right), \end{aligned}$$

where $2m_b+1=2k_2-1$ and K_2'' is a subgroup of $\prod_{j=1}^{b-1} SO(2m_j+1)\times\prod_{i=1}^{a} SU(l_i+1)\times SO(2k_1)$. By using the same argument of Section 9.1, we have that

$$K_2 = SO(2k_2-1)\times\Phi(K)\subset G \quad\text{such that}\quad \Phi(K)=\Phi(K_2)=K_2'',$$

where $\Phi : G\to G/SO(2k_2-1)=G/SO(2m_b+1)$ is the natural projection. It follows that the inclusion $K_2\subset G$ is completely determined by K.

Next, we consider the slice representation $\sigma_2 : K_2\to O(2k_2-1)$. By the same reason demonstrated in Section 9.1, the restricted slice representation $\sigma_2|_{SO(2k_2-1)} : SO(2k_2-1)\to O(2k_2-1)$ is the natural inclusion. Hence, we have that

$$\sigma_2(K_2'')\subset Z(SO(2k_2-1))=\{\pm I_{2k_2-1}\}\subset O(2k_2-1).$$

Moreover, by (10.1), we have the following formula for K:

$$\begin{aligned} & K=\sigma_2^{-1}(O(2k_2-2)) \\ = \quad & \left\{\left(\begin{pmatrix} B & 0\\ 0 & b\end{pmatrix}, Y\right)\in S(O(2k_2-2)\times O(1))\times K_2'' \;\middle|\; b=\det B^{-1}=\det\sigma_2(Y)\right\} \\ = \quad & \prod_{i=1}^{a} S(U(l_i)\times U(1))\times\left\{\left(A,\ \begin{pmatrix} X & 0\\ 0 & x\end{pmatrix}\right)\in\mathcal{S}_1\times S(O(2k_1-1)\times O(1)) \;\middle|\; \sigma_1(A)x=1\right\}. \end{aligned}$$

Therefore, we can easily show that the following lemma by using the above formula for K.

LEMMA 10.2. *The following two statements are equivalent:*

(1): $\sigma_2(K_2'')=\{I_{2k_2-1}\}$ *(resp.* $\sigma_2(K_2'')=\{\pm I_{2k_2-1}\}$*);*
(2): $K=SO(2k_2-2)\times K_2''$ *(resp.* $r_b(K)=S(O(2m_b)\times O(1))$*).*

Moreover, the following statement holds:

(3): *if* $\sigma_2(K_2'')=\{\pm I_{2k_2-1}\}$ *then* $K\neq S(O(2m_b)\times O(1))\times K_2''$*.*

It follows from this Lemma 10.2 and the above formula for K that the slice representation $\sigma_2 : K_2\to O(2k_2-2)$ is also completely determined by K. Therefore, the tubular neighborhood of G/K_2 is completely determined by K and equivariantly diffeomorphic to the following manifold:

$$\prod_{i=1}^{a}\mathbb{C}P(l_i)\times\left(\prod_{j=1}^{b-1} S^{2m_j}\times D(\mathbb{R}^{2k_2-1})\right)\times_{\mathcal{A}} S(\mathbb{R}^{2k_1}),$$

where $\mathcal{A}\simeq\mathcal{S}_1/\mathcal{S}_1^o$-quotient is defined by the following actions: on $\prod_{j=1}^{b-1} S^{2m_j}\times D(\mathbb{R}^{2k_2-1})$ as the subgroup $\mathcal{A}\subset\prod_{j=1}^{b-1}\mathbb{Z}_2\times\{\pm I_{2k_2-1}\}$; and on $S(\mathbb{R}^{2k_1})\subset\mathbb{R}^{2k_1}$ by the representation $\sigma_{\mathbb{R}} : \mathcal{A}\to\{\pm I_{2k_1}\}$ (induced by σ_1). Moreover, by using Lemma 10.2 (3), if $(1,\ \cdots,\ 1,\ -I_{2k_2-1})\in\mathcal{A}\subset\prod_{j=1}^{b-1}\mathbb{Z}_2\times\{\pm I_{2k_2-1}\}$, then $\sigma_{\mathbb{R}}(1,\ \cdots,\ 1,\ -I_{2k_2-1})=-I_{2k_1}$. It follows that $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j}\times D(\mathbb{R}^{2k_2-1})\times S(\mathbb{R}^{2k_1})$ freely; therefore, X_2 is a manifold.

From Remark 4.2, we can easily check that the pair of $(M,\ G)$ of the case (3)-(b)-(i) is as follows:

$$M = \prod_{i=1}^{a} \mathbb{C}P(l_i) \times \prod_{j=1}^{b-1} S^{2m_j} \times_{\mathcal{A}} S(\mathbb{R}^{2k_2-1} \oplus \mathbb{R}^{2k_1});$$
$$G = \prod_{i=1}^{a} SU(l_i+1) \times \prod_{j=1}^{b-1} SO(2m_j+1) \times SO(2k_2-1) \times SO(2k_1),$$

where $\mathcal{A}$ acts on $\prod_{j=1}^{b-1} S^{2m_j} \times S(\mathbb{R}^{2k_2-1} \oplus \mathbb{R}^{2k_1})$ as follows:

- on $\mathbb{R}^{2k_1}$ by $\sigma_{\mathbb{R}} : \mathcal{A} \to \{\pm I_{2k_1}\}$;
- on $\prod_{j=1}^{b-1} S^{2m_j} \times \mathbb{R}^{2k_2-1}$ as a subgroup $\prod_{j=1}^{b-1} \mathbb{Z}_2 \times \{\pm I_{2k_2-1}\}$ such that if $(1,\ \cdots,\ 1,\ -I_{2k_2-1}) \in \mathcal{A}$, then $\sigma_{\mathbb{R}}(1,\ \cdots,\ 1,\ -I_{2k_2-1}) = -I_{2k_1}$.

This corresponds with the second manifold in Theorem 8.1.

10.2. The case (3)-(b)-(ii). Suppose the case (3)-(b)-(ii) occurs, that is,

$$Spin(2k_2-2) = T_a \quad \text{and} \quad k_2 = 2,$$

where T_a is the a-th factor of $T^a = T_1 \times \cdots \times T_a$ ($T_i \simeq T^1$). Let $\psi : \widetilde{K^o} \to K^o$ be the finite covering projection, where $\widetilde{K^o}$ is a product of Lie groups. By (10.1), we can easily show that

$$\psi(Spin(2k_2-2)) = \left\{ \begin{pmatrix} I_{l_a} t_a^{-1/l_a} & 0 \\ 0 & t_a \end{pmatrix} \;\middle|\; t_a \in T_a = Spin(2k_2-2) \right\}.$$

Moreover, there is the following commutative diagram by (10.2) and (10.3):

$$\begin{array}{ccc} (SU(l_a) \times Spin(2k_2-2)) \times \widetilde{K_2''} & \xrightarrow{\widetilde{\iota}} & (SU(l_a) \times Spin(2k_2-1)) \times \widetilde{K_2''} \\ \psi \downarrow & & \downarrow \widetilde{\psi} \\ K^o & \xrightarrow{\iota} & K_2^o, \end{array}$$

where $\widetilde{K_2''} = \prod_{j=1}^{b} Spin(2m_j) \times \prod_{i=1}^{a-1} SU(l_i) \times T^{a-1} \times Spin(2k_1-1)$. Because $p_a(K^o) = S(U(l_a) \times U(1))$ (by (10.1)), we have the following sequence ($k_2 = 2$):

$$\begin{aligned} & S(U(l_a) \times U(1)) \\ = \;& p_a \circ \iota_2 \circ \iota \circ \psi(SU(l_a) \times Spin(2k_2-2)) \\ = \;& p_a \circ \iota_2 \circ \widetilde{\psi} \circ \widetilde{\iota}(SU(l_a) \times Spin(2k_2-2)) \\ \subset \;& p_a \circ \iota_2 \circ \widetilde{\psi}(SU(l_a) \times Spin(2k_2-1)) \\ \subset \;& p_a(G) = SU(l_a+1). \end{aligned}$$

Therefore, with a method similar to the proof of $l_a = 1$ in Section 9.2.1, we also have $l_a = 1$ (remark $Spin(3) \simeq SU(2)$). Hence, we have

$$\begin{aligned} K_2^o &= \prod_{j=1}^{b} SO(2m_b) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times SU(2) \times SO(2k_1-1) \\ &= \prod_{j=1}^{b} SO(2m_b) \times \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times Spin(3) \times SO(2k_1-1). \end{aligned}$$

By the similar argument of Section 10.1 and using (10.1), we have that

$$\begin{aligned} K_2 &= \prod_{i=1}^{a-1} S(U(l_i) \times U(1)) \times Spin(3) \\ &\quad \times \left\{ \left(A,\ \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}\right) \in \mathcal{S}_1 \times S(O(2k_1-1) \times O(1)) \;\middle|\; \sigma_1(A)x = 1 \right\}. \end{aligned}$$

Similarly, the slice representation $\sigma_2 : K_2 \to O(3)$ is the natural representation from the $Spin(3)$-factor in K_2 to $SO(3) \subset O(3)$, and from the other factors K_2'' in K_2 to $ZSO(3) = \{\pm I_3\}$. However, by $\sigma_2^{-1}(O(2)) = K \supset K_2'' = \psi(\widetilde{K_2''})$ (see (10.1)) and $O(2) \cap \{\pm I_3\} = \{I_3\}$, we have that $\sigma_2(K_2'') = \{I_3\}$. Therefore, $K_2 \subset G$ and σ_2 are completely determined by K.

In this case, by constructing the G-manifold, we can regard $Spin(3)$ as $SO(3) = SO(2m_{b+1}+1)$ in G up to essential isomorphism. By regarding $b+1$ as b and $a-1$ as a, we can easily show that this case is the same as the case (3)-(b)-(i) with $k_2 = 2$ and $\sigma_2(K_2'') = \{I_3\}$. Hence, this corresponds with the second manifold in Theorem 8.1 such that $k_2 = 2$ and $\sigma_2 : K_2 \to SO(3) \subset O(3)$.

CHAPTER 4

Topological classification of torus manifolds which have codimension one extended G-actions

This chapter is the joint work with Suyoung Choi.

1. Introduction of Chapter 4

A *toric variety* of dimension n is a normal algebraic variety on which an algebraic torus $(\mathbb{C}^*)^n$ acts with a dense orbit. We call a compact non-singular toric variety a *toric manifold*. We regard the compact torus T^n as the standard subgroup in $(\mathbb{C}^*)^n$. The orbit space of a toric manifold with T^n can be identified with the simple polytope and the action of T^n on a toric manifold is *locally standard*, that is, locally modelled by the standard action on $\mathbb{C}^n$. Davis and Januszkiewicz first introduced the notion of a toric manifold as a topological counterpart, which is now called a *quasitoric manifold*, by taking these two characteristic properties as the starting point (see [**7, 11**]). A quasitoric manifold is a smooth closed manifold of dimension $2n$ with a locally standard T^n-action which has a simple polytope as an orbit space. Obviously not all quasitoric manifolds belong to the family of toric manifolds. For instance, $\mathbb{C}P^2\sharp\mathbb{C}P^2$ is a quasitoric manifold with an appropriate action of T^2 but not a toric manifold because it does not allow an almost complex structure. Moreover, the family of quasitoric manifolds does not contain the family of toric manifolds entirely but the theory can be extended to a certain family of manifolds containing both toric manifolds and quasitoric manifolds. In fact, Hattori and Masuda introduced a *torus manifold*(or *unitary toric manifold* in the earlier terminology) in [**17, 29**], which is an oriented closed smooth manifold of dimension $2n$ with an effective action of T^n with a non-empty fixed point set. Among the definition of torus manifolds, if a torus manifold M satisfies the following two conditions:

(1) $H^{odd}(M)=0$ (resp. $H^*(M)$ is generated by $H^2(M)$ as ring);
(2) M, M_i's and connected components of any multiple intersection of M_i's are all simply connected,

where M_i's are characteristic submanifolds of M, then the orbits space is a homotopy cell (resp. homotopy polytope) and the converse also holds (see [**33**]). Thus Masuda and Suh believe in [**35**] that the toric theory can be developed to the family of torus manifolds satisfying the two conditions above in the topological category in a nice way.

On the other hand, the topological classification of these manifolds has recently attracted the attention of toric topologists. Masuda and Suh gave several problems in [**35**]. Of special interest is the following problem which is now called a *cohomological rigidity problem* for toric manifolds:

PROBLEM 1.1. Are toric manifolds diffeomorphic (or homeomorphic) if their cohomology rings are isomorphic as graded rings?

They also ask the problem for quasitoric manifolds and torus manifolds satisfying the two conditions above. In general, a cohomology ring of a given manifold does not determine its topological type. But in the category of toric or quasitoric manifolds, several mathematicians can find partial affirmative solutions in [**8, 9, 34**]. We do not know any counter example to Problem 1.1.

In the present chapter, we give a negative answer to Problem 1.1 for the family of torus manifolds each of whose orbit space is a homotopy cell. In order to give a negative answer, we study a topological classification of torus manifolds which have extended G-actions, where G is a compact, connected Lie group with maximal torus T^n. In Chapter 1, 2, 3, we have classified the torus manifolds which have extended G-actions with codimension 0 or 1 principal orbits. The (simply connected) torus manifold which has a codimension 0 extended G-action (i.e., transitive G-action) is a product of complex projective spaces and spheres. This is nothing but a product of projective spaces and spheres from the (non-equivariant) topological point of view. Thus, this chapter will be restricted to consideration of the classification of the torus manifolds which have extended G-actions with codimension 1 principal orbits, and the main theorem of this chapter is as follows:

THEOREM 9. Let $M_1, M_2 \in \mathfrak{M}$ with $a = 1, b = 0$. The following statements hold for these manifolds.

(1) If there is an isomorphism between $H^*(M_1)$ and $H^*(M_2)$ then $M_1 \cong M_2$ up to homeomorphism (i.e., the cohomological rigidity holds) if and only if M_1 and M_2 are in the class in $\mathfrak{M}$ which constructed by the following manifolds:

$$\begin{aligned}
&S^{2\ell+1} \times_{S^1} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2});\\
&S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \ell > 1 \text{ and } k \leq \ell;\\
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \text{for} \quad k \equiv_2 0 \ (k > 1);\\
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad k_1 \equiv_2 0.
\end{aligned}$$

(2) If there is an isomorphism between $H^*(M_1)$ and $H^*(M_2)$ and their Pontrjagin classes are same $p(M_1) = p(M_2)$, then $M_1 \cong M_2$ up to homeomorphism (i.e., the cohomological rigidity does not hold) if and only if M_1 and M_2 are in the class in $\mathfrak{M}$ which constructed by the following manifolds:

$$\begin{aligned}
&S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \text{for} \quad k > \ell > 1;\\
&S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad \ell > 1;
\end{aligned}$$

(3) If there is an isomorphism between $H^*(M_1)$ and $H^*(M_2)$ and their Stiefel-Whitney classes are same $w(M_1) = w(M_2)$, then $M_1 \cong M_2$ up to homeomorphism (i.e., the cohomological rigidity does not hold) if and only if M_1 and M_2 are in the class in $\mathfrak{M}$ which constructed by the following manifolds:

$$\begin{aligned}
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \text{for} \quad k \equiv_2 1 \ (k > 1);\\
&S^{3} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad k_1 \equiv_2 1.
\end{aligned}$$

Therefore, we have the following corollary.

COROLLARY 10. In the above theorem, the set of (2) and (3) in torus manifolds do not satisfy the cohomological rigidity. It follows that the Problem 1.1, for the family of torus manifolds each of whose orbit space is a homotopy cell, is negative.

This chapter is organized as follows. In Section 2, we recall the results of Chapter 2, 3 and prepare notations. In Section 3, we compute cohomology rings and characteristic classes of these manifolds. In order to compute them, we use the Davis-Januszkiewicz formula for the topological invariants of quasitoric manifolds and the classical facts of bundles. In Section 4, we give the main result of this chapter. In Section 5, we prove the main result.

2. Notations

Let $\widetilde{\mathfrak{M}}$ be the set of simply connected torus manifolds M^{2n} which have extended G-actions with codimension 1 principal orbits, where G is a compact, connected Lie group with maximal torus T^n. Due to the main results in Chapter 2, 3, $\widetilde{\mathfrak{M}}$ consists of the following three types of manifolds:

$$\begin{aligned}
\text{TYPE 1} \quad &: \quad \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2\ell_i+1} \times_{(S^1)^a} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) \right); \\
\text{TYPE 2} \quad &: \quad \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2\ell_i+1} \times_{(S^1)^a} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \right); \\
\text{TYPE 3} \quad &: \quad \prod_{j=1}^{b} S^{2m_j} \times \left(\prod_{i=1}^{a} S^{2\ell_i+1} \times_{(S^1)^a} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \right),
\end{aligned}$$

where $P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) = (\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2} - \{0\})/\mathbb{C}^*$ is a complex projective space, $S(\mathbb{C}^l \oplus \mathbb{R}^m) \subset \mathbb{C}^l \oplus \mathbb{R}^m$ is a sphere, and $(S^1)^a$ acts on $\prod_{i=1}^{a} S^{2\ell_i+1}$ naturally and on $\mathbb{C}_\rho^{k_1}$ through the following representation $\rho : (S^1)^a \to S^1$:

$$\rho(t_1, \cdots, t_a) = t_1^{\alpha_1} \cdots t_a^{\alpha_a}$$

for $\alpha_i \in \mathbb{Z}$ $(i = 1, \cdots, a)$.

From now on, we assume that $a = 1$ and $b = 0$ and $\mathfrak{M}$ denotes the subset of $\widetilde{\mathfrak{M}}$ which satisfies $a = 1$ and $b = 0$. Let $\mathfrak{M}_i$ $(i = 1, 2, 3)$ be the subset of $\mathfrak{M}$ of TYPE i with $a = 1$ and $b = 0$. By the definition of $\mathfrak{M}_i$, the element $N_i \in \mathfrak{M}_i$ is as follows:

$$\begin{aligned}
N_1 &= S^{2\ell+1} \times_{S^1} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}); \\
N_2 &= S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}); \\
N_3 &= S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}).
\end{aligned}$$

We may regard $\rho \in \mathbb{Z}$ by defining $\rho : S^1 \to S^1$ as $t \mapsto t^\rho$. First we compute the cohomology rings and characteristic classes of N_i's. In order to compute them, we use the following standard notations: $H^*(X)$ is the cohomology ring of X over $\mathbb{Z}$-coefficient; $w(X)$ (resp. $w_i(X)$) is the total (resp. i-th) Stiefel-Whitney class of X; and $p(X)$ (resp. $p_i(X)$) is the total (resp. i-th) Pontrjagin class of X. Moreover, $\mathbb{Z}[x_1, \cdots, x_m]$ denotes the polynomial ring generated by x_j $(j = 1, \cdots, m)$, and $< f_1(x_1, \cdots, x_m), \cdots, f_s(x_1, \cdots, x_m) >$ denotes the ideal in $\mathbb{Z}[x_1, \cdots, x_m]$ generated by the polynomials $f_j(x_1, \cdots, x_m)$ $(j = 1, \cdots, s)$. $E(\eta)$ denotes the total space for the fibre bundle η.

3. Topological invariants

In this section, we will compute the following three topological invariants of N_i $(i = 1, 2, 3)$: cohomology rings $H^*(N_i)$; Stiefel-Whiteny classes $w(N_i)$; and Pontrjagin classes $p(N_i)$.

3.1. Topological invariants of N_1.

In this subsection, we will compute topological invariants of $N_1 = S^{2\ell+1} \times_{S^1} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2})$. In order to compute them, we first recall the torus action on N_1. The torus action on N_1 is defined as follows $(k_1, k_2 \geq 1)$:

$$\begin{aligned}
& (a_1, \cdots, a_l, b_1, \cdots, b_{k_1}, c_1, \cdots, c_{k_2-1}) \cdot [(x_0, \cdots, x_l), [y_1; \cdots; y_{k_1}; y'_1; \cdots; y'_{k_2}]] \\
= \quad & [(x_0, a_1x_1, \cdots, a_lx_l), [b_1y_1; \cdots; b_{k_1}y_{k_1}; c_1y'_1; \cdots; c_{k_2-1}y'_{k_2-1}; y'_{k_2}]],
\end{aligned}$$

where $a_i, b_j, c_k \in S^1$ and $(x_0, \cdots, x_\ell) \in S^{2\ell+1} \subset \mathbb{C}^{\ell+1}$, $[y_1; \cdots; y_{k_1}; y'_1; \cdots; y'_{k_2}] \in P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2})$. Hence, we can easily check that this manifold N_1 is a quasitoric manifold over $\Delta^l \times \Delta^{k_1+k_2-1}$ (product of two simplices) whose dimension is $2l + 2k_1 + 2k_2 - 2$. Therefore, we can use the *Davis-Januszkiewicz formula* in [**11**, Theorem 4.14, Corollary 6.8] for topological invariants of quasitoric manifolds.

3.1.1. *The Davis-Januszkiewicz formula.* Next we quickly review the Davis-Januszkiewicz formula for topological invariants (see [**7, 11**] for detail). The equivariant cohomology of quasitoric manifolds M^{2n} can be described as follows:

$$H^*(ET \times_T M;\ \mathbb{Z}) = H^*_{T^n}(M^{2n};\ \mathbb{Z}) \simeq \mathbb{Z}[v_1,\ \cdots,\ v_m]/\mathcal{I},$$

where v_j ($\deg v_j = 2$, $j = 1, \cdots, m$) is the equivariant Poincaré dual of codimension two invariant submanifold M_j in M^{2n} (*characteristic submanifolds*) and $\mathcal{I}$ is an ideal of the polynomial ring $\mathbb{Z}[v_1,\ \cdots,\ v_m]$ generated by $\{\prod_{j\in I} v_j \mid \cap_{j\in I} M_j = \emptyset\}$. Let $p : ET \times_T M \to BT$ be the natural projection. Then, we can define the induced homomorphism $p^* : H^*(BT;\ \mathbb{Z}) = \mathbb{Z}[t_1,\ \cdots,\ t_n] \to H^*(ET\times_T M;\ \mathbb{Z}) = H^*_{T^n}(M^{2n};\ \mathbb{Z})$. Moreover, we see that $p^*(t_i)$ ($i = 1, \cdots, n$) can be described as the information of T-action on M, i.e., *the characteristic function* $\Lambda = (\lambda_1 \cdots \lambda_m) \in M(n,\ m;\ \mathbb{Z})$ ($n\times m$ integer matrix) and $\lambda_j \in \mathbb{Z}^n$ ($j = 1, \cdots, m$) corresponds with the generator of Lie algebra of isotropy subgroup of characteristic submanifold M_j. This $H^*_T(M)$ is isomorphic to *the face ring* of the polytope M/T. Put $\lambda_j = (\lambda_{1j}\ \cdots\ \lambda_{nj})^t \in \mathbb{Z}^n$. Then we can describe $p^*(t_i)$ ($i = 1, \cdots, n$) as follows:

$$p^*(t_i) = \sum_{j=1}^{m} \lambda_{ij} v_j.$$

Let $\mathcal{J}$ be the ideal in $\mathbb{Z}[v_1,\ \cdots,\ v_m]$ generated by $p^*(t_j)$ for all $j = 1, \cdots, n$. Then the ordinary cohomology of quasitoric manifolds are described as follows:

$$H^*(M^{2n};\ \mathbb{Z}) \simeq \mathbb{Z}[v_1,\ \cdots,\ v_m]/\mathcal{I} + \mathcal{J}. \tag{3.1}$$

Moreover, for an inclusion $j : M \to ET \times_T M$, the Stiefel-Whitney class and the Pontrjagin class[1] can be described as follows:

$$w(M) = j^* \prod_{i=1}^{m} (1 + v_i), \tag{3.2}$$

$$p(M) = j^* \prod_{i=1}^{m} (1 + v_i^2). \tag{3.3}$$

3.1.2. *Topological invariants of* N_1. Now we may compute the topological invariants of N_1. In order to use the Davis-Januszkiewicz formula, we need to compute the characteristic functions of N_1. By the torus action on N_1, the characteristic function of N_1 is as follows:

$$\begin{pmatrix} I_\ell & 0 & 0 & \mathbf{1} & 0 \\ 0 & I_{k_1} & 0 & \rho\mathbf{1} & \mathbf{1} \\ 0 & 0 & I_{k_2-1} & 0 & \mathbf{1} \end{pmatrix} \in M(n,\ n+2;\ \mathbb{Z}), \tag{3.4}$$

where $n = l + k_1 + k_2$ and $\mathbf{1} = (1,\ \cdots,\ 1)^t \in M(1,\ r;\ \mathbb{Z})$ for $r = \ell, k_1, k_2$.

Now we may compute the topological invariants of N_1. Because the equivariant cohomology ring $H^*_{T^n}(N_1)$ is the face ring of $\Delta^l \times \Delta^{k_1+k_2-1}$, we have that

$$H^*_{T^n}(N_1) \simeq \mathbb{Z}[v_1,\ \cdots,\ v_{\ell+1},\ w_1,\ \cdots,\ w_{k_1+k_2}]/\mathcal{I}, \tag{3.5}$$

where $\deg v_i = \deg w_j = 2$, and $\mathcal{I}$ is generated by

$$v_1 \cdots v_{\ell+1} \quad \text{and} \quad w_1 \cdots w_{k_1+k_2}.$$

Now $\mathcal{J}$ in $H^*_{T^n}(N_1)$ is as follows by (3.4):

$$\begin{aligned} &v_1 = -v_{\ell+1},\ \cdots,\ v_\ell = -v_{\ell+1}, \\ &w_1 = -\rho v_{\ell+1} - w_{k_1+k_2},\ \cdots,\ w_{k_1} = -\rho v_{\ell+1} - w_{k_1+k_2}, \\ &w_{k_1+1} = -w_{k_1+k_2},\ \cdots,\ w_{k_1+k_2-1} = -w_{k_1+k_2}. \end{aligned} \tag{3.6}$$

[1] In [**11**, Corollary 6.8], the Pontrjagin class of quasitoric manifolds (*toric manifolds* in [**11**]) is $j^* \prod_{i=1}^m (1 - v_i^2)$. However, this formula coincides with $1 - p_1(M) + p_2(M) - \cdots$. Therefore, by [**35**], the Pontrjagin class of quasitoric manifolds must be $p(M) = 1 + p_1(M) + p_2(M) + \cdots = j^* \prod_{i=1}^m (1 + v_i^2)$

By using (3.1), (3.5) and (3.6), we have the following formula:

$$\begin{aligned} H^*(N_1) &\simeq \mathbb{Z}[v_1, \cdots, v_{l+1}, w_1, \cdots, w_{k_1+k_2}]/\mathcal{I}+\mathcal{J} \\ &\simeq \mathbb{Z}[v_{\ell+1}, w_{k_1+k_2}]/<(-1)^\ell(v_{\ell+1})^{\ell+1}, (-w_{k_1+k_2}-\rho v_{\ell+1})^{k_1}(-w_{k_1+k_2})^{k_2}> \\ &\simeq \mathbb{Z}[x,y]/<x^{\ell+1}, y^{k_2}(y+\rho x)^{k_1}>, \end{aligned}$$

where $\deg x = \deg z = 2$.

Because of (3.2), (3.3) and (3.6), we have the characteristic classes of N_1 as follows:

$$\begin{aligned} p(N_1) &= (1+v_{\ell+1}^2)^{\ell+1}(1+(\rho v_{\ell+1}+w_{k_1+k_2})^2)^{k_1}(1+w_{k_1+k_2}^2)^{k_2} \\ &= (1+x^2)^{\ell+1}(1+(\rho x+y)^2)^{k_1}(1+y^2)^{k_2}; \\ w(N_1) &\equiv_2 (1+v_{\ell+1})^{\ell+1}(1+\rho v_{\ell+1}+w_{k_1+k_2})^{k_1}(1+w_{k_1+k_2})^{k_2} \\ &\equiv_2 (1+x)^{\ell+1}(1+\rho x+y)^{k_1}(1+y)^{k_2}. \end{aligned}$$

In summary, we have the following proposition.

PROPOSITION 3.1. Topological invariants of N_1 are as follows:

$$\begin{aligned} H^*(N_1) &= \mathbb{Z}[x,y]/<x^{\ell+1}, y^{k_2}(y+\rho x)^{k_1}>; \\ p(N_1) &= (1+x^2)^{\ell+1}(1+(\rho x+y)^2)^{k_1}(1+y^2)^{k_2}; \\ w(N_1) &\equiv_2 (1+x)^{\ell+1}(1+\rho x+y)^{k_1}(1+y)^{k_2}; \end{aligned}$$

where $\deg x = \deg y = 2$ and $\ell, k_1, k_2 \in \mathbb{N}$.

3.2. Topological invariants of N_2 and N_3. For $N_2 = S^{2\ell+1}\times_{S^1} S(\mathbb{C}_\rho^k \oplus \mathbb{R})$ and $N_3 = S^{2\ell+1}\times_{S^1} S(\mathbb{C}_\rho^{k_1}\oplus\mathbb{R}^{2k_2+1})$, in this subsection, we prove the following propositions.

PROPOSITION 3.2. Topological invariants of N_2 are as follows:

$$\begin{aligned} H^*(N_2) &= \mathbb{Z}[x,z]/<x^{\ell+1}, z(z+(\rho x)^k)>; \\ p(N_2) &= (1+x^2)^{\ell+1}(1+\rho^2x^2)^k; \\ w(N_2) &\equiv_2 (1+x)^{\ell+1}(1+\rho x)^k : \end{aligned}$$

where $\deg x = 2$, $\deg z = 2k$ and $\ell, k \in \mathbb{N}$.

PROPOSITION 3.3. Topological invariants of N_3 are as follows:

$$\begin{aligned} H^*(N_3) &= \mathbb{Z}[x,z]/<x^{\ell+1}, z^2>; \\ p(N_3) &= (1+x^2)^{\ell+1}(1+\rho^2x^2)^{k_1}; \\ w(N_3) &\equiv_2 (1+x)^{\ell+1}(1+\rho x)^{k_1}; \end{aligned}$$

where $\deg x = 2$, $\deg z = 2(k_1+k_2)$ and $\ell, k_1, k_2 \in \mathbb{N}$.

Now N_2 has the following fibration:

$$S^{2k} = S(\mathbb{C}_\rho^k \oplus \mathbb{R}) \longrightarrow N_2 \xrightarrow{\pi} S^{2\ell+1}/S^1 \cong \mathbb{C}P(\ell),$$

that is, N_2 is a sphere bundle over a complex projective space. Therefore, we can use the following lemma ([**13**, Lemma 4]):

LEMMA 3.4. *Let $\pi : E \to M$ be a smooth, oriented r-sphere bundle over an oriented manifold M which has a section $s : M \to E$. Let the normal bundle ν of the embedding s be oriented by π, and let $\chi(\nu) \in H^r(M)$ be the Euler class of ν with respect to this orientation. Then there exists a unique class $z \in H^r(E)$ such that*

$$s^*(z) = 0 \in H^*(M) \text{ and } <i^*(z), [S^r]>= 1.$$

Furthermore

(1) *$H^*(E)$, as a module over $H^*(M)$, has the basis $\{1, z\}$ subject to the relation*

$$z^2 + \pi^*(\chi(\nu))z = 0;$$

(2) *the induced cohomology map J^* acts identically on the subset* $\mathrm{Im}\,\pi^* \subset H^*(E)$ *and*

$$J^*(z) = (-1)^{r-1}z - \pi^*(\chi(\nu)).$$

We define the section of $\pi : N_2 \to \mathbb{C}P(\ell)$ as follows:

$$s : \mathbb{C}P(\ell) \ni [z_0;\ \cdots;\ z_\ell] \mapsto ([z_0;\ \cdots;\ z_\ell],\ (0,\ \cdots,\ 0,\ 1)) \in N_2,$$

where $(0,\ \cdots,\ 0,\ 1) \in S(\mathbb{C}_\rho^k \oplus \mathbb{R})$. We can easily check this map is well-defined because $(0,\ \cdots,\ 0,\ 1) \in S(\mathbb{C}_\rho^k \oplus \mathbb{R})$ is one of the fixed points of the S^1-action on $S(\mathbb{C}_\rho^k \oplus \mathbb{R})$. Then the normal bundle of this section is isomorphic to the following bundle ξ_ρ:

$$\mathbb{C}^k \longrightarrow S^{2\ell+1} \times_{S^1} \mathbb{C}_\rho^k \longrightarrow \mathbb{C}P(\ell).$$

Then, we have that

$$\xi_\rho \cong \gamma_\rho^{\oplus k}$$

where $E(\gamma_\rho) = S^{2\ell+1} \times_{S^1} \mathbb{C}_\rho$ such that S^1 acts on $\mathbb{C}_\rho$ by the representation $t \mapsto t^\rho$. If $\rho = 1$, then γ_ρ is isomorphic to the canonical line bundle over $\mathbb{C}P(\ell)$. Hence, $\gamma_\rho = \gamma^{\otimes\rho}$ for the canonical line bundle $\gamma(=\gamma_1)$. Therefore, the Euler class of ξ_ρ is

$$\chi(\xi_\rho) = c_k(\xi_\rho) = c_k(\gamma_\rho^{\oplus k}) = c_1(\gamma_\rho)^k = c_1(\gamma^{\otimes\rho})^k = (\rho c)^k = \rho^k c^k,$$

where $c \in H^2(\mathbb{C}P(\ell))$ is the generator (determined by $c_1(\gamma)$) of the cohomology ring $H^*(\mathbb{C}P(\ell))$. Because π^* is injective by using $H^{odd}(\mathbb{C}P(\ell)) = 0$ and $H^{odd}(S^{2k}) = 0$ (see [**38**]), there are the following relations in the cohomology ring $H^*(N_2)$:

$$x^{\ell+1} = 0;$$
$$z^2 + \rho^k x^k z = 0;$$

for $x = \pi^*(c) \in H^2(N_2)$ and some $z \in H^{2k}(N_2)$ by using $H^*(\mathbb{C}P(\ell)) = \mathbb{Z}[c]/ < c^{\ell+1} >$ and Lemma 3.4. Making use of the Serre spectral sequence for the bundle $\pi : N_2 \to \mathbb{C}P(\ell)$, there is an epimorphism $\mathbb{Z}[x,\ z] \to H^*(N_2)$, and additively the cohomology of $H^*(N_2)$ coincides with that of $\mathbb{C}P(\ell) \times S^{2k}$. Hence, there is no other relations except those mentioned in the above arguments. Thus, we have the cohomology formula in Proposition 3.2.

In order to compute characteristic classes, we regard $N_2 = S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^k \oplus \mathbb{R})$ as the unit sphere bundle of the following vector bundle over $\mathbb{C}P(\ell)$:

$$\xi = \xi_\rho \oplus \underline{\mathbb{R}} \cong \gamma_\rho^{\oplus k} \oplus \underline{\mathbb{R}}, \tag{3.7}$$

where $\underline{\mathbb{R}}$ is the trivial line bundle, $E(\xi) = S^{2\ell+1} \times_{S^1} (\mathbb{C}_\rho^k \oplus \mathbb{R})$. We often denote N_2 as $S(\xi)$.

Now $\mathcal{T}$ denotes the tangent bundle of $E(\xi)$. Then, there is the following pull-back diagram:

$$\begin{array}{ccc} \iota^*\mathcal{T} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ S(\xi) & \xrightarrow{\iota} & E(\xi) \end{array}$$

where $\iota : (N_2 =)S(\xi) \to E(\xi)$ is the natural inclusion, and the following relation holds:

$$\mathcal{T}|_{N_2} = \iota^*\mathcal{T} = \tau_2 \oplus \nu_2,$$

where τ_2 is the tangent bundle of $N_2 = S(\xi)$ and ν_2 is the normal bundle of the inclusion $\iota : N_2 = S(\xi) \to E(\xi)$. Note that ν_2 is a real 1-dimensional bundle by the equation $\dim E(\xi) - \dim S(\xi) = 1$. We prove the following lemma for ν_2.

LEMMA 3.5. *ν_2 is the trivial real line bundle, i.e., $E(\nu_2) = N_2 \times \mathbb{R}$.*

PROOF. By the classification theorem of vector bundles, the number of line bundle over a connected CW complex X is the number of $[X,\ B\mathbb{Z}_2]$, where $[X,\ B\mathbb{Z}_2]$ is a set of all continuous map from X to a classified space $B\mathbb{Z}_2$ up to homotopy. Moreover, the following relation is known: $[X,\ B\mathbb{Z}_2] \approx H^1(X;\ \mathbb{Z}_2)$, i.e., two numbers are same (e.g. see [**47**, Chapter 8, Section 1, 10 Theorem]). Because N_2 is simply connected, the line bundle over N_2 is always trivial. $\square$

Hence, we have

$$\iota^* p(\mathcal{T}) = p(\iota^*\mathcal{T}) = p(\tau_2 \oplus \nu_2) = p(\tau_2) = p(N_2), \tag{3.8}$$

$$\iota^* w(\mathcal{T}) = w(\iota^*\mathcal{T}) = w(\tau_2 \oplus \nu_2) = w(\tau_2) = w(N_2). \tag{3.9}$$

We also remark $\iota^* : H^*(E(\xi)) \to H^*(S(\xi))$ is injective, because $\pi : S(\xi) \to \mathbb{C}P(\ell)$ is decomposed into $\pi = \widetilde{\pi} \circ \iota$ where $\widetilde{\pi} : E(\xi) \to \mathbb{C}P(\ell)$ and π^* is injective by the Serre spectoral sequence. In order to prove Proposition 3.2, we compute $p(\mathcal{T})$ and $w(\mathcal{T})$.

Let $\widetilde{s}$ be the zero section of $\widetilde{\pi} : E(\xi) \to \mathbb{C}P(\ell)$. Consider the following pull-back diagram:

$$\begin{array}{ccc} \widetilde{s}^*\mathcal{T} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow \\ \mathbb{C}P(\ell) & \xrightarrow{\widetilde{s}} & E(\xi). \end{array}$$

Because the normal bundle $\nu(\mathbb{C}P(\ell))$ of the image of $\widetilde{s}$ is isomorphic to ξ, we have that

$$\widetilde{s}^*\mathcal{T} \cong \tau(\mathbb{C}P(\ell)) \oplus \nu(\mathbb{C}P(\ell)) \cong \tau(\mathbb{C}P(\ell)) \oplus \xi,$$

where $\tau(\mathbb{C}P(\ell))$ is the tangent bundle over $\mathbb{C}P(\ell)$. Therefore, we have

$$\widetilde{s}^*(p(\mathcal{T})) = p(\widetilde{s}^*\mathcal{T}) = p(\tau(\mathbb{C}P(\ell)) \oplus \xi) = p(\tau(\mathbb{C}P(\ell)))p(\xi) = (1+c^2)^{\ell+1}(1+\rho^2c^2)^k,$$

$$\widetilde{s}^*(w(\mathcal{T})) = w(\widetilde{s}^*\mathcal{T}) = w(\tau(\mathbb{C}P(\ell)) \oplus \xi) = w(\tau(\mathbb{C}P(\ell)))w(\xi) = (1+c)^{\ell+1}(1+\rho c)^k.$$

Because $\widetilde{s}^* : H^*(E(\xi)) \to H^*(\mathbb{C}P(\ell)) \simeq \mathbb{Z}[c]/ < c^{\ell+1} >$ induces the isomorphism and $\widetilde{s}^* = (\widetilde{\pi}^*)^{-1}$, we have

$$p(\mathcal{T}) = (1+\widetilde{\pi}^*(c)^2)^{\ell+1}(1+\rho^2\widetilde{\pi}^*(c)^2)^k,$$

$$w(\mathcal{T}) = (1+\widetilde{\pi}^*(c))^{\ell+1}(1+\rho\widetilde{\pi}^*(c))^k.$$

Hence, we have that

$$p(N_2) = (1+x^2)^{\ell+1}(1+\rho^2x^2)^k,$$

$$w(N_2) = (1+x)^{\ell+1}(1+\rho x)^k,$$

by using $\iota^* \circ \widetilde{\pi}^*(c) = \pi^*(c) = x$, (3.8) and (3.9). Thus, we have the characteristic classes in Proposition 3.2.

With a method similar to that demonstrated in the proof of Proposition 3.2, for $N_3 = S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1})$, we can prove Proposition 3.3.

4. Main theorem and Preliminary

In this section, we state the main theorem and prepare to prove it. First we state the main theorem.

THEOREM 4.1. *Let $M_1, M_2 \in \mathfrak{M}$ with $a = 1, b = 0$. The following statements hold for these manifolds.*

(1) *If there is an isomorphism between $H^*(M_1)$ and $H^*(M_2)$ then $M_1 \cong M_2$ up to homeomorphism (i.e., the cohomological rigidity holds) if and only if M_1 and M_2 are in the class in $\mathfrak{M}$ which constructed by the following manifolds:*

$$S^{2\ell+1} \times_{S^1} P(\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2});$$

$$S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \ell > 1 \text{ and } k \leq \ell;$$

$$S^3 \times_{S^1} S(\mathbb{C}_\rho^{k} \oplus \mathbb{R}) \quad \text{for} \quad k \equiv_2 0 \ (k > 1);$$

$$S^3 \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad k_1 \equiv_2 0.$$

(2) *If there is an isomorphism between $H^*(M_1)$ and $H^*(M_2)$ and their Pontrjagin classes are same $p(M_1) = p(M_2)$, then $M_1 \cong M_2$ up to homeomorphism (i.e., the cohomological*

rigidity does not hold) if and only if M_1 and M_2 are in the class in $\mathfrak{M}$ which constructed by the following manifolds:

$$S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^k \oplus \mathbb{R}) \quad \text{for} \quad k > \ell > 1;$$
$$S^{2\ell+1} \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad \ell > 1.$$

(3) *If there is an isomorphism between $H^*(M_1)$ and $H^*(M_2)$ and their Stiefel-Whitney classes are same $w(M_1) = w(M_2)$, then $M_1 \cong M_2$ up to homeomorphism (i.e., the cohomological rigidity does not hold) if and only if M_1 and M_2 are in the class in $\mathfrak{M}$ which constructed by the following manifolds:*

$$S^3 \times_{S^1} S(\mathbb{C}_\rho^k \oplus \mathbb{R}) \quad \text{for} \quad k \equiv_2 1 \ (k > 1);$$
$$S^3 \times_{S^1} S(\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}) \quad \text{for} \quad k_1 \equiv_2 1.$$

Due to the definition of $N_1 \in \mathfrak{M}_1$, this manifold N_1 is the projectify of the vector bundle η_ρ:

$$\mathbb{C}^{k_1+k_2} \to S^{2\ell+1} \times_{S^1} (\mathbb{C}_\rho^{k_1} \oplus \mathbb{C}^{k_2}) \to \mathbb{C}P(\ell).$$

Now we have

$$\eta_\rho \cong (\gamma^{\otimes \rho})^{\oplus k_1} \oplus \underline{\mathbb{C}}^{\oplus k_2}$$

over $\mathbb{C}P(\ell)$, where $\underline{\mathbb{C}}$ is the trivial complex line bundle, $E(\gamma) = S^{2\ell+1} \times_{S^1} \mathbb{C}_1$ and $\gamma^{\otimes(-1)} = \bar{\gamma}$ (i.e., orientation reversing). Thus, $\mathfrak{M}_1$ consists of 2-stage generalized Bott towers. Therefore, we may use the following theorem.

THEOREM 4.2 ([**8**]). *Top manifolds of 2-stage generalized Bott towers are diffeomorphic if and only if their integral cohomology rings are isomorphic.*

Due to the definition of $N_2 \in \mathfrak{M}_2$ and $N_3 \in \mathfrak{M}_3$, they are sphere bundles over projective spaces $\mathbb{C}P(\ell)$. Then $(n-1)$-sphere bundles over a complex K are generated by mappings of K into some Grassmannian manifold $G_n = BO(n)$. By [**49**], they are one-to-one correspondence with the homotopy classes.

THEOREM 4.3 ([**49**]). *The problem of classifying the $(n-1)$-sphere bundles over a complex K is equivalent to the problem of enumerating the homotopy classes of maps of K in $G_{n+\ell,n}$ for any $\ell > \dim K$, where $G_{n,k}$ is the Grassmannian manifold of oriented k-planes through the origin in a euclidean space of dimension n.*

Note that for $M \in \mathfrak{M}$, $H^*(M) = \mathbb{Z}[x,w]/ < x^{\ell+1}, f(x,w) >$, where f is a homogeneous polynomial and $\deg x = 2$ and $w = y$ for Proposition 3.1 or $w = z$ for Proposition 3.2, 3.3. We list up for each case:

Proposition 3.1: $f(x,\ y) = y^{k_2}(y+\rho x)^{k_1}$, $\deg y = 2$ for $k_1,\ k_2 \in \mathbb{N}$;
Proposition 3.2: $f(x,\ z) = z(z+(\rho x)^k)$, $\deg z = 2k$ for $k \in \mathbb{N}$;
Proposition 3.3: $f(x,\ z) = z^2$, $\deg z = 2k_1 + 2k_2$ for $k_1,\ k_2 \in \mathbb{N}$.

This implies that the cohomology ring determines $\ell, \deg w$ and $\deg f$ (remark that $\ell \geq 1$, $\deg w \geq 2$ and $\deg f \geq 4$). The proof of Theorem 4.1 divides into the following two cases by the degree of w:

CASE 1: $\deg w = 2$, i.e., 2-dimensional sphere bundle;
CASE 2: $\deg w > 2$, i.e., the dimension of fibre is greater than 2;

moreover, we will divide each case into small parts (see the next section).

5. Proof of the main theorem

In this final section, we prove Theorem 4.1. We divide CASE 1 into 3 parts and CASE 2 into 3 parts by ℓ and $\deg f$ as follows:

CASE 1 (1): $\deg w = 2$ and $\deg f > 4$;
CASE 1 (2): $\deg w = 2$, $\deg f = 4$ and $\ell > 1$;
CASE 1 (3): $\deg w = 2$, $\deg f = 4$ and $\ell = 1$;

CASE 2 (1): $\deg w > 2$ and $\ell \geq 4$;
CASE 2 (2): $\deg w > 2$ and $\ell = 2,\ 3$;
CASE 2 (3): $\deg w > 2$ and $\ell = 1$.

5.1. CASE 1 (1) : $\deg w = 2$ **and** $\deg f > 4$**.** In this case, M must be in $\mathfrak{M}_1$. By Theorem 4.2, the homeomorphism type of M is determined by its cohomology ring.

5.2. CASE 1 (2) : $\deg w = 2$**,** $\deg f = 4$ **and** $\ell > 1$**.** In this case, M must be in $\mathfrak{M}_1$ for $k_1 = k_2 = 1$ or $\mathfrak{M}_2$ for $k = 1$. In fact, M is a 2-sphere bundle over $\mathbb{C}P(\ell)$ for $\ell > 1$. Note that $P(\mathbb{C} \oplus \mathbb{C})$ is homeomorphic to $S(\mathbb{C} \oplus \mathbb{R})$. Thus the total space of $P(\gamma_\rho \oplus \mathbb{C})$ is homeomorphic to the total space of $S(\gamma_\rho \oplus \underline{\mathbb{R}})$. Thus the homeomorphism type of M is determined by ρ up to sign by Theorem 4.2.

We now claimed that the cohomology isomorphism

$$\phi : \mathbb{Z}[x_1, y_1]/ < x_1^{\ell+1}, y_1(y_1 + \rho_1 x_1) > \to \mathbb{Z}[x_2, y_2]/ < x_2^{\ell+1}, y_2(y_2 + \rho_2 x_2) >$$

implies $\rho_1 = \pm\rho_2$. Without loss of generality, $|\rho_2| \leq |\rho_1|$. We may assume

$$\phi \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

where the matrix $\det(\alpha_{ij}) = \pm 1$ and $\alpha_{ij} \in \mathbb{Z}$.

Since

$$\begin{aligned} & \phi(y_1(y_1 + \rho_1 x_1)) \\ = \; & (\alpha_{21}x_2 + \alpha_{22}y_2)(\alpha_{21}x_2 + \alpha_{22}y_2 + \rho_1(\alpha_{11}x_2 + \alpha_{12}y_2)) \\ \in \; & < x_2^{\ell+1}, y_2(y_2 + \rho_2 x_2) >, \end{aligned}$$

we have

$$\alpha_{21}^2 + \rho_1\alpha_{21}\alpha_{11} = 0, \tag{5.1}$$

$$\rho_2(\alpha_{22}^2 + \rho_1\alpha_{22}\alpha_{12}) = 2\alpha_{21}\alpha_{22} + \rho_1\alpha_{12}\alpha_{22} + \rho_1\alpha_{11}\alpha_{22}. \tag{5.2}$$

By (5.1), we get $\alpha_{21} = -\rho_1\alpha_{11}$ or $\alpha_{21} = 0$. Both of them implies α_{21} is divided by ρ_1. From this, in (5.2), $\rho_2\alpha_{22}^2$ is divided by ρ_1. But if $\gcd(\alpha_{22}, \rho_1) = d > 1$, then $\det(\alpha_{ij}) = \pm 1$ is divided by $d > 1$. This is contradiction. Therefore, we have $\rho_1 | \rho_2$. But by the hypothesis, $|\rho_2| \leq |\rho_1|$, we can conclude that $\rho_1 = \pm\rho_2$. This implies that the homeomorphism type of M in CASE 1 (2) is determined by its cohomology ring. Moreover, the homeomorphism types are $P(\gamma_\rho \oplus \mathbb{C})$ for $\rho \in \mathbb{N} \cup \{0\}$.

5.3. CASE 1 (3) : $\deg w = 2$**,** $\deg f = 4$ **and** $\ell = 1$**.** In this case, $M = S^3 \times_{S^1} S(\mathbb{C}_\rho \oplus \mathbb{R}) \cong S^3 \times_{S^1} P(\mathbb{C}_\rho \oplus \mathbb{C})$ is S^2 bundle over S^2. These are classified by $[S^2 : G_3] = \pi_2(G_3) \cong \mathbb{Z}_2$ by Theorem 4.3. Thus there is just one 2-sphere bundle over S^2 except the product bundle. Therefore, the homeomorphism type of M is determined by $\rho \mod 2$. Moreover, we have that $\rho \equiv_2 w_1(M)$ and the homeomorphism type of M is determined by its cohomology ring by Propositions 3.1 and 3.2.

5.4. CASE 2 (1) : $\deg w > 2$ **and** $\ell \geq 4$**.** In this case, M must be in $\mathfrak{M}_2$ or $\mathfrak{M}_3$. Let M_1 and M_2 be in $\mathfrak{M}$. Assume their cohomology rings are isomorphic. Without loss of generality, $M_1 = S((\gamma^{\otimes \rho_1})^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{21}+1})$ and $M_2 = S((\gamma^{\otimes \rho_2})^{\oplus k_{12}} \oplus \mathbb{R}^{2k_{22}+1})$ over $\mathbb{C}P^\ell$ for some $k_{11},\ k_{12} \in \mathbb{N}$ and $k_{21},\ k_{22} \geq 0$, where $(k_{11},\ k_{21}),\ (k_{12},\ k_{22}) \neq (1,\ 0)$ because $\deg w > 2$. Put $H^*(M_1) = \mathbb{Z}[x_1, y_1]/ < x_1^{\ell+1}, f_1(x_1, y_1) >$ and $H^*(M_2) = \mathbb{Z}[x_2, y_2]/ < x_2^{\ell+1}, f_2(x_2, y_2) >$, where $\deg x_i = 2$, $\deg y_i = \deg w = 2k_{1i} + 2k_{2i}$ $(i = 1,\ 2)$ and

$$\begin{aligned} f_i(x_i,\ y_i) &= y_i(y_i + (\rho_i x_i)^{k_{1i}}) \quad \text{for} \quad k_{2i} = 0; \\ f_i(x_i,\ y_i) &= y_i^2 \quad \text{for} \quad k_{2i} > 0. \end{aligned}$$

Let $\phi : H^*(M_1) \to H^*(M_2)$ be an isomorphism. By the assumptions, we can easily show that

$$\phi(x_1) = \pm x_2.$$

We will divide this case into the following three cases by k_{21} and k_{22}.

5.4.1. *The case $k_{21},\ k_{22} > 0$.* In this case, we can easily show that

$$\phi(y_1) = \pm y_2.$$

Therefore, in this case, the cohomology isomorphism does not induce the identity of Pontrjagin classes except their coefficients are same, i.e., $p(M_1) = p(M_2)$. Hence, by taking different Pontrjagin classes, we can easily construct the counter examples of the cohomological rigidity.

Assume Pontrjagin classes of M_1 and M_2 are same. Then, by Proposition 3.2, 3.3 and $\phi(x_1) = \pm x_2$, we have an isomorphic map ϕ preserves their Pontrjagin classes and

$$\begin{aligned}
\phi(p_1(M_1)) = p_1(M_2) \quad &\Leftrightarrow \quad k_{11}\rho_1^2 = k_{12}\rho_2^2, \\
\phi(p_2(M_1)) = p_2(M_2) \quad &\Leftrightarrow \quad \binom{k_{11}}{2}\rho_1^4 = \binom{k_{12}}{2}\rho_2^4.
\end{aligned}$$

Therefore, we have two cases; one is $\rho_1 = \rho_2 = 0$ and the other is $\rho_1 = \pm\rho_2$ and $k_{11} = k_{12}$. Thus the vector bundle $(\gamma^{\otimes \rho_1})^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{21}+1}$ and $(\gamma^{\otimes \rho_2})^{\oplus k_{12}} \oplus \mathbb{R}^{2k_{22}+1}$ are isomorphic. This implies that M_1 and M_2 are homeomorphic. Hence, in this case, we have that if cohomology rings and Pontrjagin classes of M_1 and M_2 are same then M_1 and M_2 are homeomorphic.

5.4.2. *The case $k_{21} > 0$ and $k_{22} = 0$.* Let $\phi(y_1) = ax_2^{k_{12}} + by_2$, where $b = \pm 1$ by $\phi(x_1) = \pm x_2$. By $\phi(y_1)^2 = 0$, we have that

$$\rho_2 = 0 \quad \text{or} \quad k_{12} > \ell.$$

If $k_{12} > \ell$, then we have $\phi(y_1) = \pm y_2$. This is the same situation as the above case (Section 5.4.1). Therefore, it is sufficient to consider the case $\rho_2 = 0$ and $k_{12} \leq \ell$ only, in this case. However, in this case, we also have $\phi(y_1) = \pm y_2$. Hence, this is also the same as above.

5.4.3. *The case $k_{21} = k_{22} = 0$.* If $k_{21} = 0 = k_{22}$ and $k_{11} > \ell$ or $k_{12} > \ell$, then we can easily show that $\phi(y_1) = \pm y_2$ by the similar method demonstrated above.

Assume $k_{21} = 0 = k_{22}$ and $k_{11},\ k_{12} \leq \ell$. Let $\phi(y_1) = ax_2^{k_{12}} + by_2$, where $b = \pm 1$. Then $k_{11} = k_{12}$, and by using the cohomology ring

$$\begin{aligned}
\phi(y_1(y_1 + (\rho_1 x_1)^{k_{11}})) &= (ax_2^{k_{12}} + by_2)^2 + (ax_2^{k_{12}} + by_2)(\pm\rho_1 x_2)^{k_{11}} \\
&= a^2x_2^{2k_{12}} + 2abx_2^{k_{12}}y_2 + b^2y_2^2 + a(\pm\rho_1)^{k_{12}}x_2^{2k_{12}} + b(\pm\rho_1)^{k_{12}}y_2x_2^{k_{12}} \\
&= a^2x_2^{2k_{12}} + 2abx_2^{k_{12}}y_2 - b^2y_2(\rho_2x_2)^{k_{12}} + a(\pm\rho_1)^{k_{12}}x_2^{2k_{12}} + b(\pm\rho_1)^{k_{12}}y_2x_2^{k_{12}} \\
&= ax_2^{2k_{12}}(a + (\pm\rho_1)^{k_{12}}) + bx_2^{k_{12}}y_2(2a - b^2\rho_2^{k_{12}} + b(\pm\rho_1)^{k_{12}}) = 0.
\end{aligned}$$

If $a = 0$, then $|\rho_2| = |\rho_1|$. This implies that the vector bundles $(\gamma^{\otimes\rho_1})^{\oplus k_{11}} \oplus \mathbb{R}$ and $(\gamma^{\otimes\rho_2})^{\oplus k_{12}} \oplus \mathbb{R}$ are same. It follows that the cohomological rigidity holds in this case. If $a \neq 0$, then $a = -(\pm\rho_1)^{k_{12}}$ and by using $b = \pm 1$

$$(b-2)(\pm\rho_1)^{k_{12}} = \rho_2^{k_{12}}.$$

If $b = -1$, we can easily show this gives a contradiction to $k_{12} \geq 2$ and $\rho_1,\ \rho_2 \in \mathbb{Z}$. Therefore, in this case $b = 1$ and $|\rho_1| = |\rho_2|$. Hence, the cohomological rigidity holds.

In summary, the following lemma holds.

LEMMA 5.1. *If the CASE 2-(1) holds, then there are the following two cases:*

(1) *the cohomological rigidity holds $\Leftrightarrow$ the case $k_{21} = 0 = k_{22}$ and $k_{11},\ k_{12} \leq \ell$;*
(2) *the cohomological rigidity does not holds $\Leftrightarrow$ the case $k_{21} > 0$ or $k_{22} > 0$ and the case $k_{21} = 0 = k_{22}$ and $k_{11} > \ell$ or $k_{12} > \ell$,*

where the Pontrjagin class is complete invariant of homeomorphism types for the case that the cohomological rigidity does not holds.

5.5. CASE 2 (2) : $\deg w > 2$ and $\ell = 2,\ 3$. Let $\phi : H^*(M_1) \to H^*(M_2)$ be an isomorphism. With the method similar to that demonstrated in CASE 2 (1), we have an isomorphic map ϕ satisfies $\phi(x_1) = \pm x_2$. Assume $\phi(y_1) = \pm y_2$ and Pontrjagin classes of M_1 and M_2 are same, then

$$\phi(p_1(M_1)) = p_1(M_2) \Leftrightarrow k_{11}\rho_1^2 = k_{12}\rho_2^2.$$

Remark that in this case the second Pontrjagin class p_2 does not appear by its cohomology ring structure.

Because of [**43**], in this case ($\ell = 2,\ 3$) we have $KO(\mathbb{C}P^\ell) \simeq \mathbb{Z}[y_\ell]/ < y_\ell^2 >$, where $y_\ell = r(\gamma)+2$ for the canonical line bundle γ and the realification map $r : K(\mathbb{C}P^\ell) \to KO(\mathbb{C}P^\ell)$. Moreover, we have $r(\gamma^{\otimes n}) = n^2 y_\ell + 2$ by [**43**]. Therefore, we have that

$$\begin{aligned} & (\gamma^{\otimes \rho_1})^{\oplus k_{11}} \oplus \mathbb{R}^{2k_{12}+1} \equiv (\gamma^{\otimes \rho_2})^{\oplus k_{21}} \oplus \mathbb{R}^{2k_{22}+1} \\ \Leftrightarrow \quad & k_{11}\rho_1^2 = k_{12}\rho_2^2 \quad \text{and} \quad k_{11} + k_{12} = k_{21} + k_{22}. \end{aligned}$$

This implies that if two cohomology rings and Pontrjagin classes are same then they are homeomorphic in this case.

If $\phi(y_1) \neq \pm y_2$, then this case is just induced by the case $k_{21} = 0 = k_{22}$ and $k_{11},\ k_{12} \leq \ell$ and the cohomological rigidity holds. Therefore, CASE 2-(2) also holds Lemma 5.1.

5.6. CASE 2 (3) : $\deg w > 2$ and $\ell = 1$. In this case, M must be in $\mathfrak{M}_2$ or $\mathfrak{M}_3$. Thus M is $S^{\deg y}$-bundle over S^2. Moreover, their cohomology rings are always same if we fix their dimension. Since $\pi_2(G_n) \cong \mathbb{Z}_2$, the homeomorphism type of M is determined by $w_2(M)$ if $k \equiv_2 1$ (or $k_1 \equiv_2 1$).

If $k \equiv_2 0$ (or $k_1 \equiv_2 0$), then $\mathbb{C}_\rho^k \oplus \mathbb{R}$ (or $\mathbb{C}_\rho^{k_1} \oplus \mathbb{R}^{2k_2+1}$) is trivial bundle because of their Stiefel-Whitney class is trivial. Hence if $k \equiv_2 0$ (or $k_1 \equiv_2 0$), then $M \cong S^{\deg y} \times S^2$. This implies that homeomorphism types of CASE 2 (3) are determined as follows: if $k \equiv_2 0$, then the cohomology ring completely determines their homeomorphism types; if $k \equiv_2 1$, then the cohomology ring and Stiefel-Whitney class completely determines their homeomorphism types, i.e., this case does not satisfy the cohomological rigidity.

Part 2

Operations and projective bundles on small covers

CHAPTER 5

Operations on three dimensional small covers

1. Introduction of Part 5

Small covers were introduced by Davis and Januszkiewicz as a real version of quasitoric manifolds in 1991. In their paper [**11**], they showed that there are strong links between small covers M with the orbit projection map $\pi : M \to P$ and combinatorial structures of their orbit polytopes P. For example, some topological invariants (e.g., equivariant cohomologies or $\mathbb{Z}_2$-Betti numbers) of small covers $\pi : M \to P$ are consistent with the combinatorial invariants of their orbit polytopes P (e.g., Stanley-Reisner rings or h-vectors).

Not only topological invariants but also topological operations on M (e.g., equivariant connected sums or equivariant surgeries) correspond with combinatorial operations on the orbit polytope P (see Section 3). Making use of this correspondence, constructions of 3-dimensional small covers M^3 from basic small covers have been studied by Izmestiev, Nishimura, Lü and Yu. In [**19**] Izmestiev studies the class of small covers M^3 called a *linear model*, i.e., small covers over 3-colored polytopes. He proves that 3-dimensional linear models are constructed from one basic small cover (the 3-dimensional torus T^3) by using two operations $\sharp$ and $\natural$ (see Section 3 and Theorem 4.2). In [**42**] Nishimura generalizes the Izmestiev result to *oriented* small covers M^3, i.e., small covers over 3 or 4-colored polytopes. He proves that 3-dimensional oriented small covers are constructed from two basic small covers (T^3 and the real projective space $\mathbb{R}P(3)$) by using three operations $\sharp$, $\natural$ and $\flat$ (see Section 3 and Theorem 4.3). In [**28**] Lü and Yu prove that all 3-dimensional small covers are constructed from five basic small covers by using six operations $\sharp$, $\sharp^e$, $\sharp^{eve}$, $\natural$, $\sharp^{\Delta}$ and $\sharp^C$ (see Section 3 and Theorem 4.5).

In the Nishimura's theorem (Theorem 4.3), the operation $\flat$ appears; however, in the Lü-Yu's theorem (Theorem 4.5), this operation does not appear. On the other hand, in the Lü-Yu's theorem, the operations $\sharp^e$, $\sharp^{eve}$, $\sharp^{\Delta}$ and $\sharp^C$ appear; however, in the Nishimura's theorem, these operations do not appear. So we can naturally ask what relations exist between the Nishimura's theorem and the Lü-Yu's theorem. Motivated by this question, in this chapter, we prove the following theorem (Theorem 4.1).

THEOREM 11. The operations $\flat$, $\sharp^e$, $\sharp^{eve}$ can be obtained by using $\sharp$, $\natural$ as follows:

(1) $\flat = \natural \circ (\sharp\Delta^3)$;
(2) $\sharp^e = \natural \circ (\sharp P^3(3))$;
(3) $\sharp^{eve} = \natural^2 \circ (\sharp P^3_-(3))$,

where $\natural^2$ denotes two times Dehn surgery $\natural \circ \natural$.

Using the above Theorem 11 and the Nishimura's theorem (Theorem 4.3), we have the following corollary (Corollary 4.4).

COROLLARY 12. Each 3-dimensional oriented small cover can be (equivariantly) constructed from the real projective space $\mathbb{R}P(3)$ and the 3-dimensional torus T^3 by using finite times following two operations: the equivariant connected sum $\sharp$; and the equivariant Dehn surgery $\natural$.

Moreover, we prove the following corollary (Corollary 4.8) by making use of Theorem 11 and the Lü-Yu's theorem (Theorem 4.5).

COROLLARY 13. Each 3-dimensional small cover can be (equivariantly) constructed from $\mathbb{R}P(3)$, $M(P^3(3),\ \lambda_2)$ and $M(P^3(3),\ \lambda_3)$ (up to weak $\mathbb{Z}_2^3$-equivariant diffeomorphism) by using finite times following four operations: the equivariant connected sum $\sharp$; the equivariant surgery $\natural$;

the operation $\sharp^{\Delta}$ and the coloring change $\sharp^{C}$, where $(M(P^3(3)),\ \lambda_i)$ denotes $S^1 \times \mathbb{R}P(2)$ with the $\mathbb{Z}_2^3$-action induced by λ_i for $i = 2,\ 3$.

For 2-torus manifolds, the following theorem (Theorem 6.1) holds.

THEOREM 14. The operations $\sharp^{\Delta}$, $\sharp^{C}$ can be obtained by using $\natural$, $\natural_0$ and $\sharp$ as follows:

(1) $\sharp^{\Delta} = \natural_0 \circ \natural \circ \sharp$;
(2) $\sharp^{C} P^3(l) = \natural_0 \circ \natural^{l-2} \circ (\sharp P^3(l))$,

where $P^3(l)$ is the l-sided prism, $l \geq 3$ and $\natural^{l-2}$ denotes $(l-2)$-times $\natural$.

The organization of this chapter is as follows. In Section 2, we recall the basic facts for small covers. In Section 3, we introduce the seven operations on 3-dimensional small covers. In Section 4, we prove the main theorem (Theorem 4.1). As an application of Theorem 4.1, we also prove Corollary 4.4 and 4.8 which improve the Nishimura's theorem (Theorem 4.3) and the Lü-Yu's theorem (Theorem 4.5). In Section 5, we remark a relation between the Nishimura's theorem and Lü-Yu's theorem. In Section 6, we introduce the new operation and prove Theorem 6.1 for 2-torus manifolds.

2. Basics of small covers

In this section, we recall some basic facts for small covers (see [**11**] for detail). We describe the quotient additive group $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}_2$ throughout this chapter.

2.1. Definition of a small cover. First we shall recall the terminology: 2-torus manifolds in [**26, 27**]. A 2-*torus manifold* M^n is an n-dimensional, closed smooth manifold with a non-free effective smooth $\mathbb{Z}_2^n$-action.

Let P^n be a *simple* convex n-polytope, i.e., precisely n codimension-one faces (*facets*) of P^n meet at each vertex. A *small cover* is a 2-torus manifold M^n which satisfies the following two conditions:

(a): the $\mathbb{Z}_2^n$-action is *locally standard*, i.e., locally same as the $\mathbb{Z}_2^n$-action on $\mathbb{R}^n$; and
(b): there is the orbit projection map $\pi : M^n \to P^n$ constant on $\mathbb{Z}_2^n$-orbits which maps every rank k orbit (i.e., orbits which isomorphic to $\mathbb{Z}_2^k$) to a point in the interior of k-dimensional face of P^n, $k = 0,\ \cdots,\ n$.

We can easily show that π sends $\mathbb{Z}_2^n$-fixed points in M^n to vertices of P^n by using the above condition (b). We often call P^n an *orbit polytope* of M.

2.2. Characteristic function and constructions of small covers. On the other hand, for given P^n, small covers M^n with orbit projection $\pi : M^n \to P^n$ can be reconstructed by using a characteristic function $\lambda : \mathcal{F} \to \mathbb{Z}_2^n$, where $\mathcal{F}$ is the set of facets (i.e., codimension-1 subfaces) in P. In this subsection, we shall recall the characteristic function λ and the construction of small covers by using P and λ.

Due to the definition of a small cover $\pi : M \to P$, we have that $\pi^{-1}(int\ F^{n-1})$ consists of $(n-1)$-rank orbits, in other words, the isotropy subgroup at $x \in \pi^{-1}(int\ F^{n-1})$ is $K \subset \mathbb{Z}_2^n$ such that $K \simeq \mathbb{Z}_2$, where $int\ F^{n-1}$ is an interior of a facet F^{n-1}. Hence, the isotropy subgroup at x is determined by a primitive vector $v \in \mathbb{Z}_2^n$ generating a subgroup K. In this way we obtain a function λ from the set of facets of P, denoted by $\mathcal{F}$, to primitive vectors in $\mathbb{Z}_2^n$. We call such $\lambda : \mathcal{F} \to \mathbb{Z}_2^n$ a *characteristic function* or *coloring* on P. By the locally standard property, a characteristic function satisfies the following property (we call *the property* $(\star)$):

$(\star)$**:** if $F_1 \cap \cdots \cap F_n \neq \emptyset$ for $F_i \in \mathcal{F}$ $(i = 1,\ \cdots,\ n)$, then $\{\lambda(F_1),\ \cdots,\ \lambda(F_n)\}$ spans $\mathbb{Z}_2^n$.

Next we mention the construction of small covers by using P^n and λ. Let P^n be a simple convex polytope. Suppose that the characteristic function $\lambda : \mathcal{F} \to \mathbb{Z}_2^n$ which satisfies the above property $(\star)$ is defined on P^n. Small covers can be constructed from P and λ as follows:

$$\mathbb{Z}_2^n \times P/\sim,$$

where $(t, x) \sim (t', y)$ is defined as $x = y \in P$ and

$$t = t' \quad \text{if } x \in intP;$$
$$t^{-1}t' \in \langle \lambda(F_1),\ \cdots,\ \lambda(F_l) \rangle \simeq \mathbb{Z}_2^l \quad \text{if } x \in int(F_1 \cap \cdots \cap F_l),$$

where $\langle \lambda(F_1),\ \cdots,\ \lambda(F_l) \rangle \subset \mathbb{Z}_2^n$ denotes the subgroup generated by $\lambda(F_i)$ for $i = 1,\ \cdots,\ l$. We describe such small cover as $M(P,\ \lambda)$.

2.3. Examples. Let $\{e_1,\ \cdots,\ e_n\}$ be the standard basis in $\mathbb{Z}_2^n$. Figure 1 shows two examples of characteristic functions on polytopes. The polytope of the left example is the 3-simplex Δ^3 with 4-coloring, i.e., $\lambda_0(\mathcal{F}) = \{e_1,\ e_2,\ e_3,\ e_1 + e_2 + e_3\}$. The polytope of the right example is the 3-cube I^3 with 3-coloring, i.e., $\lambda_0^I(\mathcal{F}) = \{e_1,\ e_2,\ e_3\}$.

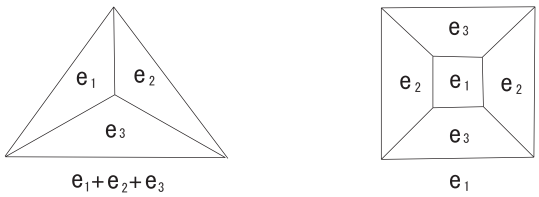

FIGURE 1. The left figure is $(\Delta^3,\ \lambda_0)$ and the right figure is $(I^3,\ \lambda_0^I)$, where the bottom $e_1 + e_2 + e_3$ and e_1 are the colors of the behind facets. We see that $M(\Delta^3,\ \lambda_0) = \mathbb{R}P(3)$ and $M(I^3,\ \lambda_0^I) = T^3$, where $\mathbb{R}P(3)$ and T^3 have the standard $\mathbb{Z}_2^3$-actions.

We can easily show that Δ^3 has the unique characteristic function λ_0 up to $GL(3,\ \mathbb{Z}_2)$, i.e., for all characteristic function λ on Δ^3, there is an element $\sigma \in GL(3,\ \mathbb{Z}_2)$ such that $\sigma \circ \lambda = \lambda_0$. In other words, small covers over Δ^3 are unique up to weak equivariant diffeomorphism. We can assume that there is at least one vertex $v = F_1 \cap \cdots \cap F_n$ in P such that $\lambda(F_i) = e_i$ for all $i = 1,\ \cdots,\ n$ up to $GL(3,\ \mathbb{Z}_2)$ because of the property $(\star)$.

A small cover over P^n with n-coloring, i.e., $\lambda(\mathcal{F}) = \{e_1,\ \cdots,\ e_n\}$ (up to $GL(n,\ \mathbb{Z}_2)$) is called a *linear model*. So the right example in Figure 1 is a linear model. Nakayama and Nishimura show that a 3-dimensional small cover is orientable if and only if the characteristic function on the orbit polytope P^3 is 3 or 4 colored in [**41**]. Therefore, small covers which construct from two examples in Figure 1 are orientable.

2.4. $\mathbb{Z}_2^3$-invariant normal bundle over an invariant S^1. Let $\pi : M \to P$ be a 3-dimensional small cover. In this subsection, we study that the equivariant normal bundle of the inverse of edges in P.

Fix $\{e_1,\ e_2,\ e_3\}$ as a basis (not necessary standard) of $\mathbb{Z}_2^3$. We can easily show, by using the property $(\star)$, that the characteristic functions on neighboring facets around one edge in 3-dimensional polytope are only the four cases in Figure 2.

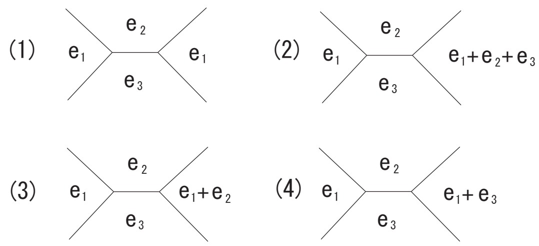

FIGURE 2. The characteristic functions around edges.

The small cover over 1-simplex $\Delta^1(= I^1)$ is identified with $\mathbb{R}P(1)(= S^1)$. Hence, for the small cover $\pi : M \to P$, the inverses $\pi^{-1}(I^1)$ of all edges I^1 in P are invariant submanifolds diffeomorphic to S^1. Moreover, we know an equivariant normal bundle of $\pi^{-1}(I^1)$ from the characteristic functions around I^1. Let $(t_1, t_2, t_3) \in \mathbb{Z}_2^3$ act on $((x_1, x_2), (y_1, y_2)) \in S^1 \times D^2$ by $((x_1, x_2), (y_1, y_2)) \mapsto ((t_1 x_1, x_2), (t_2 y_1, t_3 y_2))$. We can easily show the following proposition by computing the characteristic functions of $S^1 \times_{\mathbb{Z}_2} D^2$'s and using Figure 2 and the construction of small covers in Section 2.2.

Proposition 2.1. Let I be an edge in P and N a small neighborhood of I. For an invariant submanifold $S^1 = \pi^{-1}(I)$ in the 3-dimensional small cover, the normal bundles $\pi^{-1}(N)$ are weak $\mathbb{Z}_2^3$-equivariantly isomorphic to one of the following four disk bundles:

(1) if N satisfies (1) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}})$;
(2) if N satisfies (2) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\mathbb{R} \oplus \mathbb{R})$;
(3) if N satisfies (3) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\mathbb{R} \oplus \underline{\mathbb{R}})$;
(4) if N satisfies (4) in Figure 2, then $\pi^{-1}(N) \simeq S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \mathbb{R})$,

where the non-trivial element in $\mathbb{Z}_2$ acts on S^1 by the antipodal involution, and $D(V \oplus V') = D^2$ denotes a closed disk in $V \oplus V'$ ($V,\ V'$ are 1-dimensional real vector spaces), $\mathbb{Z}_2$ acts on $\mathbb{R}$ canonically and on $\underline{\mathbb{R}}$ trivially.

Remark 2.2. In the above Proposition 2.1, we have that $S^1 \times_{\mathbb{Z}_2} (\underline{\mathbb{R}} \oplus \underline{\mathbb{R}}) \cong \epsilon^1 \oplus \epsilon^1$, $S^1 \times_{\mathbb{Z}_2} (\mathbb{R} \oplus \mathbb{R}) \cong \gamma_1^1 \oplus \gamma_1^1$, $S^1 \times_{\mathbb{Z}_2} (\mathbb{R} \oplus \underline{\mathbb{R}}) \cong \gamma_1^1 \oplus \epsilon^1$ and $S^1 \times_{\mathbb{Z}_2} (\underline{\mathbb{R}} \oplus \mathbb{R}) \cong \epsilon^1 \oplus \gamma_1^1$, where ϵ^1 is the trivial bundle and γ_1^1 is the canonical bundle over $\mathbb{R}P(1) (= S^1)$. Hence, we have that the bundle (1) (resp. (3)) is isomorphic to the bundle (2) (resp. (4)) in Proposition 2.1, by making use of the basic facts of the vector bundle over $\mathbb{R}P(1)$ (see [**37**]). However, these four bundles are different as the $\mathbb{Z}_2^3$-equivariant bundle because their characteristic functions are different (see Figure 2). We also remark that, up to weak $\mathbb{Z}_2^3$-equivariant diffeomorphism, (3) and (4) are same.

3. Operations for 3-dimensional small covers

Henceforth, we assume that $M(P,\ \lambda)$ is a 3-dimensional small cover over a 3-dimensional simple convex polytope with coloring $(P,\ \lambda)$, and $\{e_1,\ e_2,\ e_3\}$ is a basis (not necessary standard) of $\mathbb{Z}_2^3$. In this section, we introduce operations on small covers and orbit polytopes (also see [**19, 28, 42**]).

3.1. The equivariant connected sum $\sharp$. The operation in Figure 3 is called the *equivariant connected sum* $\sharp$ (from left to right) and its inverse $\sharp^{-1}$ (from right to left). The left figure shows two neighborhoods of two same colored vertices in $(P_1,\ \lambda_1)$ and $(P_2,\ \lambda_2)$. We can do the connected sum these vertices (see [**19**, Definition 3]), then we get new polytope with coloring $(P_1 \sharp P_2,\ \lambda)$, vice versa. Remark that $P_1 \sharp P_2$ is a combinatorial simple convex polytope by using *the Steinitz' theorem*: the graph Γ is associated from the 3-dimensional polytope P if and only if Γ is 3-connected and planer (see [**55**, Chapter 4]).

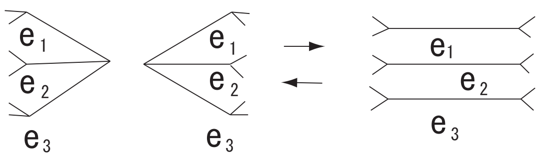

Figure 3. The equivariant connected sum $\sharp$ (from left to right) and its inverse $\sharp^{-1}$ (from right to left). Here, e_3 is the coloring of the behind facet.

From the geometric point of view, this operation $\sharp$ corresponds with the equivariant connected sum $M(P_1,\ \lambda_1) \sharp M(P_2,\ \lambda_2)$ for two fixed points in $M(P_1,\ \lambda_1)$ and $M(P_2,\ \lambda_2)$. We can easily check $M(P_1,\ \lambda_1) \sharp M(P_2,\ \lambda_2)$ is a small cover and its orbit polytope with coloring is $(P_1 \sharp P_2,\ \lambda)$, i.e., $M(P_1,\ \lambda_1) \sharp M(P_2,\ \lambda_2) = M(P_1 \sharp P_2,\ \lambda)$ (also see [**19**, Lemma 2]).

3.2. The cutting edge operation $\sharp^e$. Before we mention the cutting edge operation, we introduce the connected sum along edges. Let P_1 and P_2 be 3-dimensional, simple, convex polytopes. Suppose that the edges $I_1 \subset P_1$ and $I_2 \subset P_2$ are chosen, and a one-to-one correspondence $F_i \mapsto F_i'$ $(i = 1, \cdots, 4)$ is established, where this correspondence is from the facets $\{F_1, F_2, F_3, F_4\}$ containing I_1 as $I_1 = F_2 \cap F_3$ to the facets $\{F_1', F_2', F_3', F_4'\}$ containing I_2 as $I_2 = F_2' \cap F_3'$. The *connected sum along edges* with respect to these data is a polytope combinatorially equivalent to the result of the gluing P_1 and P_2 with small neighborhoods of I_1 and I_2 removed. The corresponding facets must be glued together. The operation in Figure 4 is the special case of this operation.

The operation in Figure 4 is called the *cutting edge operation* $\sharp^e$ (from left to right) and its inverse $(\sharp^e)^{-1}$ (from right to left). The left figure shows two neighborhoods of two same colored edges in (P_1, λ_1) and (P_2, λ_2), where P_2 is the 3-sided prism, i.e., $P_2 = P^3(3) = I^1 \times \Delta^2$. We can do the connected sum along edges for these same colored edges, then we get new polytope (remark that this is combinatorially equivalent to a simple polytope because of the Steinitz' theorem) with coloring $(P_1\sharp^e P_2, \lambda)$, vice versa (also see [**28**, Section 2]).

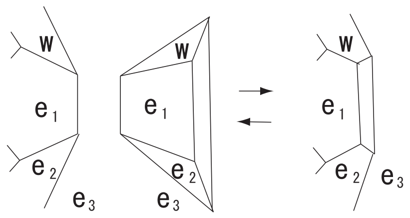

FIGURE 4. The cutting edge operation $\sharp^e$ (from left to right) and its inverse $(\sharp^e)^{-1}$ (from right to left). Here, e_3 is the coloring of the behind facet, and w is an element in $\mathbb{Z}_2^3$ such that the property $(\star)$ holds around vertices (see Figure 2).

From the geometric point of view, the operation $\sharp^e$ corresponds with the following operation. Let $\pi_i : M_i \to P_i$ be a small cover and I_i an edge in P_i for $i = 1, 2$. Suppose that $P_2 = P^3(3)$ and colorings of facets around I_1 and I_2 are same as in Figure 4. Then we see that a closed invariant tubular neighborhood N_1 of $\pi_1^{-1}(I_1)$ is equivariantly isomorphic to a closed invariant tubular neighborhood N_2 of $\pi_2^{-1}(I_2)$ (see Section 2.4). Therefore, two boundaries of $M(P_1, \lambda_1)\backslash intN_1$ and $M(P_2, \lambda_2)\backslash intN_2$ are equivariantly diffeomorphic, where $intN_i$ is an interior of N_i for $i = 1, 2$. Hence, we can glue equivariantly these two boundaries, and get the $\mathbb{Z}_2^3$-manifold $M(P_1, \lambda_1)\backslash intN_1 \cup_\partial M(P_2, \lambda_2)\backslash intN_2$; we describe it as $M(P_1, \lambda_1)\sharp^e M(P_2, \lambda_2)$. We can easily show that $M(P_1, \lambda_1)\sharp^e M(P_2, \lambda_2)$ is small cover and its orbit polytope with coloring is $(P_1\sharp^e P_2, \lambda)$ (also see [**28**, Section 5.2]).

3.3. The cutting edge-vertex-edge operation $\sharp^{eve}$. In this chapter, the operation in Figure 5 is called the *cutting e-v-e operation* $\sharp^{eve}$ (from left to right) and its inverse $(\sharp^{eve})^{-1}$ (from right to left), where e-v-e means edge-vertex-edge. Let 2-*edges* in a simple polytope P be an union of two different edges with the common vertex. The left figure shows two neighborhoods of two same colored 2-edges in (P_1, λ_1) and (P_2, λ_2), where P_2 is the *truncated prism*, i.e., P_2 is the polytope constructed by the connected sum of the 3-simplex Δ^3 and the 3-sided prism P_3^3: $P_2 = P_-^3(3) = P^3(3)\sharp\Delta^3$. We describe the 2-edges in (P_1, λ_1) as $I_1 \vee I_2$ and that in (P_2, λ_2) as $I_1' \vee I_2'$. Then we can establish a one-to-one correspondence $F_i \mapsto F_i'$ $(i = 1, \cdots, 5)$, from the facets $\{F_1, \cdots, F_5\}$ containing $I_1 \vee I_2$ as $I_1 = F_2 \cap F_5$ and $I_2 = F_3 \cap F_5$ to the facets $\{F_1', \cdots, F_5'\}$ containing $I_1' \vee I_2'$ as $I_1' = F_2' \cap F_5'$ and $I_2' = F_3' \cap F_5'$, such that $\lambda_1(F_i) = \lambda_2(F_i')$. The *cutting e-v-e operation* with respect to these data is a polytope combinatorially equivalent to the result of the gluing P_1 and P_2 with small neighborhoods of $I_1 \vee I_2$ and $I_1' \vee I_2'$ removed. The corresponding facets must be glued together. Remark that the cutting e-v-e operation of P_1 and P_2 is combinatorially equivalent to a simple polytope because of the Steinitz' theorem. Therefore, we have the new polytope with coloring $(P_1\sharp^{eve}P_2, \lambda)$ (also see [**28**, Section 2]).

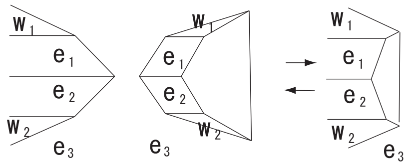

FIGURE 5. The cutting e-v-e operation $\sharp^{eve}$ (from left to right) and its inverse $(\sharp^{eve})^{-1}$ (from right to left). Here, e_3 is the coloring of the behind facet, and $w_1,\ w_2$ are elements in $\mathbb{Z}_2^3$ such that the property $(\star)$ holds around vertices.

From the geometric point of view, this operation $\sharp^{eve}$ corresponds with the following operation. Let $\pi_i : M_i \to P_i$ be a small cover and I_i an edge in P_i for $i = 1,\ 2$. Assume that $P_2 = P^3_-(3)$ and colorings of facets around $I_1 \vee I_2$ in P_1 and $I_1' \vee I_2'$ in P_2 are same as in Figure 5. Now we see that $\pi_1^{-1}(I_1 \vee I_2) \cong S^1 \vee S^1$ and $\pi_2^{-1}(I_1' \vee I_2') \cong S^1 \vee S^1$. Here, $S^1 \vee S^1$ is the $\mathbb{Z}_2^3$-invariant bouquet of two S^1's on the fixed points, i.e., $S^1 \vee S^1 = (S^1 \amalg S^1)/v_1 \sim v_2$, where $S^1 \amalg S^1$ is a disjoint union of two S^1's and v_i is a Z_2^3-fixed point in the i-th S^1 for $i = 1,\ 2$. Moreover, by our assumption, there are $\mathbb{Z}_2^3$-invariant closed neighborhoods N_1 and N_2 of $\pi_1^{-1}(I_1 \vee I_2)$ and $\pi_2^{-1}(I_1' \vee I_2')$, respectively, and they are equivariantly diffeomorphic. Therefore, with a method similar to that demonstrated in the previous section (Section 3.3), we have the $\mathbb{Z}_2^3$-manifold $M(P_1,\ \lambda_1)\backslash intN_1 \cup_\partial M(P_2,\ \lambda_2)\backslash intN_2$; we describe it as $M(P_1,\ \lambda_1)\sharp^{eve}M(P_2,\ \lambda_2)$. We can easily show that $M(P_1, \lambda_1)\sharp^{eve}M(P_2,\ \lambda_2)$ is small cover and its orbit polytope with coloring is $(P_1\sharp^{eve}P_2,\ \lambda)$ (see [**28**, Section 5.3] for detail).

3.4. The equivariant Dehn surgery $\natural$. The operation described in Figure 6 is called the *equivariant Dehn surgery* $\natural$ (from left to right) and its inverse $\natural^{-1}$ (from right to left). If a coloring around edge is 3-coloring (i.e., from geometric point of view, an equivariant normal bundle which is weak equivariantly isomorphic to $S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}})$ by Proposition 2.1), then we can do this operation $\natural$ for $(P,\ \lambda)$. Remark that the obtained object by $\natural$ might not be a convex polytope (also see [**19**]); however, we get the coloring on this object which satisfies the property $(\star)$ around the vertex. We describe the object obtained by the operation $\natural$ as $\natural(P,\ \lambda)$.

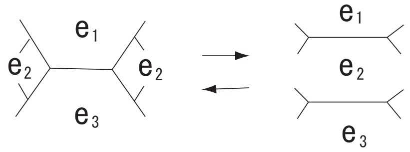

FIGURE 6. The equivariant Dehn surgery $\natural$ (from left to right) and its inverse $\natural^{-1}$ (from right to left).

From the geometric point of view, this operation $\natural$ corresponds with the following operation. Let $\mathbb{Z}_2^3$ act on the last three coordinate in $S^3 \subset \mathbb{R} \oplus \mathbb{R}^3$, naturally. Then there is the $\mathbb{Z}_2^3$-invariant submanifold $S^1 \subset \mathbb{R}^2 \oplus \{0\} \oplus \{0\}$ (remark that there are two fixed points in this submanifold), and we can take its tubular neighborhood $S^1 \times D^2 \cong S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}})$. We next consider $S^3 = S^1 \times D^2 \cup D^2 \times S^1$, and prepare the $\mathbb{Z}_2^3$-invariant part $S^3\backslash S^1 \times D^2 = D^2 \times S^1$ (remark that there is no fixed points in this manifold). Then we remove the invariant neighborhood $S^1 \times D^2 (\simeq S^1 \times_{\mathbb{Z}_2} D(\underline{\mathbb{R}} \oplus \underline{\mathbb{R}}))$ around S^1 from $M(P,\ \lambda)$, i.e., we take $M(P,\ \lambda)\backslash S^1 \times D^2$. Finally we glue these two invariant manifold $M(P,\ \lambda)\backslash S^1 \times D^2 \cup_\partial S^3 \backslash S^1 \times D^2$. This operation is identically the *equivariant surgery* of three dimensional manifold (also see [**19**, Definition 5] or [**28**, Section 5.4]). We describe the manifold obtained by this equivariant surgery as $\natural(M(P,\ \lambda))$. If $\natural(P,\ \lambda)$ is a convex polytope, then $\natural(M(P,\ \lambda)) = M(\natural(P,\ \lambda))$.

3.5. The equivariant Dehn surgery $\flat$. The operation described in Figure 7 is called the *equivariant Dehn surgery* $\flat$ (from left to right) and its inverse $\flat^{-1}$ (from right to left). This move is also called the *bistellar* 1-*move* (see [**7**, Chapter 2]: remark that the operation in Figure 7 corresponds with the dual of the bistellar 1-move in [**7**]). If a coloring around edge is 4-coloring as that in Figure 7 (i.e., from geometric point of view, an equivariant normal bundle which is weak equivariantly isomorphic to $S^1 \times_{\mathbb{Z}_2} D(\mathbb{R} \oplus \mathbb{R})$ by Proposition 2.1) , then we can do this operation $\flat$. We describe the object with coloring obtained by this operation as $\flat(P,\ \lambda)$. Remark that if the polytope P is prime, i.e, P is not decomposed into connected sum of two different polytopes, then this operation $\flat$ does not collapse the convex property (see [**42**]). We also remark that this move is the same as the connected sum along edges between $(P,\ \lambda)$ and $(\Delta^3,\ \sigma \circ \lambda_0)$ for some $\sigma \in GL(3; \mathbb{Z}_2)$.

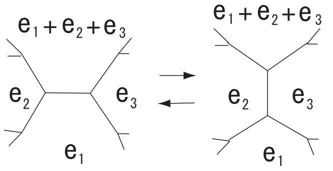

FIGURE 7. The equivariant surgery $\flat$ (from left to right) and its inverse $\flat^{-1}$ (from right to left).

Now we may explain what happen from the geometric point of view. We can regard $\mathbb{R}P(3)$ as $S^3/\mathbb{Z}_2$ by the antipodal involution $\mathbb{Z}_2$. Then we can consider $\mathbb{R}P(3) = S^1 \times_{\mathbb{Z}_2} D^2 \cup D^2 \times_{\mathbb{Z}_2} S^1$, where $S^1 \times_{\mathbb{Z}_2} D^2 \simeq S^1 \times_{\mathbb{Z}_2} D(\mathbb{R} \oplus \mathbb{R})$. We first prepare the $\mathbb{Z}_2^3$-invariant part $\mathbb{R}P(3) \backslash S^1 \times_{\mathbb{Z}_2} D^2 = D^2 \times_{\mathbb{Z}_2} S^1$ (remark that there are two fixed points in this manifold). Next, we remove the invariant neighborhood $S^1 \times_{\mathbb{Z}_2} D^2$ around S^1 from $M(P,\ \lambda)$, i.e., we take $M(P,\ \lambda) \backslash S^1 \times_{\mathbb{Z}_2} D^2$. Finally, we glue these two invariant manifold $M(P,\ \lambda) \backslash S^1 \times_{\mathbb{Z}_2} D^2 \cup_\partial \mathbb{R}P(3) \backslash S^1 \times_{\mathbb{Z}_2} D^2$; We describe the manifold obtained by this operation as $\flat(M(P,\ \lambda))$. If $\flat(P,\ \lambda)$ is a convex polytope, then $\flat(M(P,\ \lambda)) = M(\flat(P,\ \lambda))$.

3.6. The operation $\sharp^\Delta$. The operation described in Figure 8 is called the *operation* $\sharp^\Delta$ (from left to right) and its inverse $(\sharp^\Delta)^{-1}$ (from right to left). If a coloring around triangle facet is 2-independent coloring (i.e., their colorings can be choose from one of $\{e_1,\ e_2,\ e_1 + e_2\}$), then we can do this operation $\sharp^\Delta$. Using the Steinitz' theorem, $P_1 \sharp^\Delta P_2$ is a convex, simple polytope (also see [**28**, Section 2.2]).

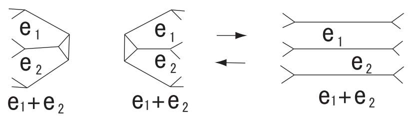

FIGURE 8. The operation $\sharp^\Delta$ (from left to right) and its inverse $(\sharp^\Delta)^{-1}$ (from right to left).

From the geometric point of view, a neighborhood of a triangle facet whose neighboring facets are 2-independent coloring corresponds with an invariant normal bundle which is weak equivariantly isomorphic to $\mathbb{R}P(2) \times I^1$ with the standard $\mathbb{Z}_2^3$-action (i.e., the first $\mathbb{Z}_2^2$ acts on $\mathbb{R}P(2)$ and the last $\mathbb{Z}_2$ acts on $I^1 = [-1, 1] \subset \mathbb{R}$ naturally), by computing its characteristic function. Therefore, we have that the operation $\sharp^\Delta$ corresponds with the following operation. We first remove an open invariant neighborhood $\mathbb{R}P(2) \times intI^1$ from $M(P_i,\ \lambda_i)$, i.e., $M(P_i,\ \lambda_i) \backslash \mathbb{R}P(2) \times intI^1$ for $i = 1,\ 2$. Next we glue these two manifolds along boundaries, i.e., $M(P_1,\ \lambda_1) \backslash \mathbb{R}P(2) \times intI^1 \cup_\partial M(P_2,\ \lambda_2) \backslash \mathbb{R}P(2) \times intI^1$; we describe it as $M(P_1,\ \lambda_1) \sharp^\Delta M(P_2,\ \lambda_2)$. We can easily show $M(P_1,\ \lambda_1) \sharp^\Delta M(P_2,\ \lambda_2)$ is a small cover and its orbit polytope with coloring is $(P_1 \sharp^\Delta P_2,\ \lambda)$ (also see [**28**, Section 5.5]).

3.7. The coloring change $\sharp^C$. We explain the operation in Figure 9. Let F be an l-gon facet whose neighboring facets are 2-independent coloring in $(P,\ \lambda)$ (see the left bottom polytope in Figure 9). Then we can construct a l-sided prism $P^3(l) = F \times I^1$ (see the left above polytope in Figure 9), which naturally admits a coloring such that the coloring of the neighboring facets around the bottom facet (or top facet) is the same as that of F in $(P,\ \lambda)$. Next we glue these two polytopes along the facet F, and we have the new polytope; we describe this new polytope as $P\sharp^C P^3(l)$. Remark that two polytopes P and $P\sharp^C P^3(l)$ are combinatorially equivalent; however, the colorings of the l-gon facet F in P and $P\sharp^C P^3(l)$ are different (also see [**28**, Section 2.3]).

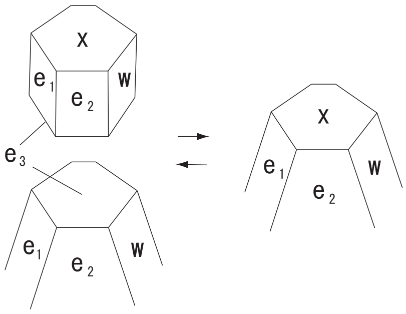

FIGURE 9. The coloring change $\sharp^C$ (from left to right) and its inverse $(\sharp^C)^{-1}$ (from right to left), where $w = e_1$ or $e_1 + e_2$ and the left e_3 means the coloring of the bottom of prism and the top of polytope. We also have $x = e_1 + e_3,\ e_2 + e_3$ or $e_1 + e_2 + e_3$.

From the geometric point of view, this operation corresponds with a (geometric) operation similar to that described in Section 3.6 (see [**28**, Section 5.6] for detail).

4. Main theorem and corollaries

In this section, we shall prove the main theorem of this chapter.

4.1. Main theorem. First we prove the following main theorem.

THEOREM 4.1. *The operations* $\flat$, $\sharp^e$, $\sharp^{eve}$ *can be obtained by using* $\sharp$, $\natural$ *as follows:*

(1) $\flat = \natural \circ (\sharp\Delta^3)$, *i.e.*, $\flat(P,\ \lambda) = \natural((P,\ \lambda)\sharp(\Delta^3,\ \lambda'))$;
(2) $\sharp^e = \natural \circ (\sharp P^3(3))$, *i.e.*, $(P,\ \lambda)\sharp^e(P^3(3),\ \lambda') = \natural((P,\ \lambda)\sharp(P^3(3),\ \lambda'))$;
(3) $\sharp^{eve} = \natural^2 \circ (\sharp P^3_-(3))$, *i.e.*, $(P,\ \lambda)\sharp^{eve}(P^3_-(3),\ \lambda') = \natural^2((P,\ \lambda)\sharp(P^3_-(3),\ \lambda'))$,

where $\natural^2$ *denotes* $\natural \circ \natural$.

PROOF. These relations (1), (2) and (3) are shown by the Figures 10, 11 and 12, respectively.

In these Figures 10, 11 and 12, the upper map from left to right means the connected sum $\sharp$ on two black vertices with Δ, $P^3(3)$ and $P^3_-(3)$ respectively, and the right map from top to bottom means the equivariant Dehn surgery $\natural$ along the black edge. Remark that in Figure 12, we do the operation $\natural$ two times along two black edges. As the result, we get the formulas (1), (2), (3) in the statement.

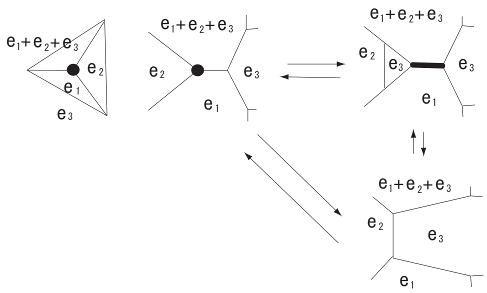

FIGURE 10. $\flat = \natural \circ (\sharp\Delta)$.

$\square$

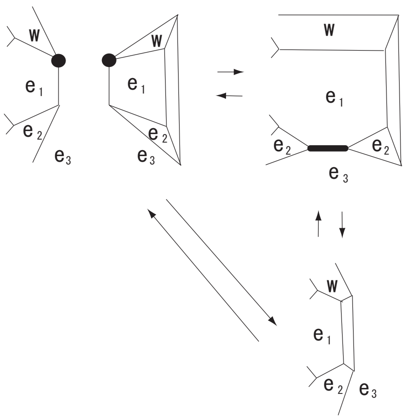

FIGURE 11. $\sharp^e = \natural \circ (\sharp P^3(3))$.

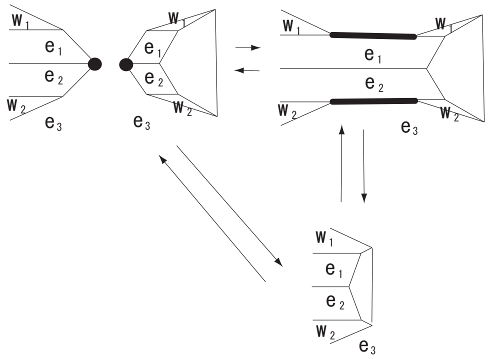

FIGURE 12. $\sharp^{eve} = \natural^2 \circ (\sharp P^3_-(3))$.

4.2. Constructions of oriented small covers. In this and next subsection, we apply our main theorem (Theorem 4.1) to constructions of the three-dimensional small covers. First, we recall the following Izmestiev's theorem ([**19**, Theorem 3]).

THEOREM 4.2 (Izmestiev). *Suppose that the characteristic function λ of a 3-dimensional small cover $M(P,\ \lambda)$ satisfies that $\lambda(\mathcal{F}) = \{e_1,\ e_2,\ e_3\}$, i.e., $M(P,\ \lambda)$ is a 3-dimensional linear model. Then $M(P,\ \lambda)$ can be (equivariantly) constructed from the 3-dimensional torus T^3 by using finite times following two operations: the equivariant connected sum $\sharp$; and the equivariant Dehn surgery $\natural$.*

Nishimura generalizes the above theorem to the following theorem ([**42**, Theorem 1.10]).

THEOREM 4.3 (Nishimura). *Suppose that the characteristic function λ of a 3-dimensional small cover $M(P,\ \lambda)$ satisfies that $\lambda(\mathcal{F}) \subset \{e_1,\ e_2,\ e_3,\ e_1+e_2+e_3\}$, i.e., $M(P,\ \lambda)$ is an oriented small cover. Then $M(P,\ \lambda)$ can be (equivariantly) constructed from T^3 and the 3-dimensional real projective space $\mathbb{R}P(3)$ by using finite times following three operations: the equivariant connected sum $\sharp$; and the equivariant Dehn surgeries $\natural$ and $\flat$.*

By Section 2.3, we see that the small cover over Δ is $\mathbb{R}P(3)$. Therefore we can prove the following corollary by applying Theorem 4.1 (1) to the above Nishimura's theorem.

COROLLARY 4.4. Each 3-dimensional oriented small cover can be (equivariantly) constructed from T^3 and $\mathbb{R}P(3)$ by using finite times following two operations: the equivariant connected sum $\sharp$; and the equivariant Dehn surgery $\natural$.

4.3. Constructions of all small covers. For all 3-dimensional small covers, the following Lü-Yu's theorem are known ([**28**, Theorem 1.1, 1.2]).

Theorem 4.5 (Lü-Yu). *Each 3-dimensional small cover can be (equivariantly) constructed from $\mathbb{R}P(3)$ and $S^1 \times \mathbb{R}P(2)$ with certain four type $\mathbb{Z}_2^3$-actions (up to weak equivariantly diffeomorphic) by using the following six operations: $\sharp$; $\sharp^e$; $\sharp^{eve}$; $\natural$; $\sharp^{\Delta}$ and $\sharp^C$.*

Remark 4.6. In this chapter, we abuse the operations on small covers and on polytopes.

We shall explain the four types $\mathbb{Z}_2^3$-actions on $S^1 \times \mathbb{R}P(2)$ in Theorem 4.5. The manifold $S^1 \times \mathbb{R}P(2)$ is the small cover over the three sided prism $P^3(3) = I^1 \times \Delta^2$. The four types $\mathbb{Z}_2^3$-actions are defined by using the coloring in Figure 13. We call them (from left) $M(P^3(3),\ \lambda_1)$, $M(P^3(3),\ \lambda_2)$, $M(P^3(3),\ \lambda_3)$, $M(P^3(3),\ \lambda_4)$, respectively.

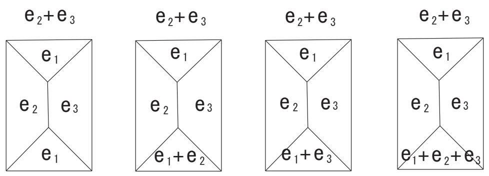

Figure 13. Basic four $\mathbb{Z}_2^3$-actions on $S^1 \times \mathbb{R}P(2)$ in Theorem 4.5; $M(P^3(3),\ \lambda_1)$, $M(P^3(3),\ \lambda_2)$, $M(P^3(3),\ \lambda_3)$, $M(P^3(3),\ \lambda_4)$. Here, $e_2 + e_3$ is the coloring of behind facets.

Remark that colorings of neighboring facets of triangle facets in Figure 13 are 2-independent. Therefore, we can do $\sharp^{\Delta}$ (see Section 3.6) for these manifolds as Figure 14.

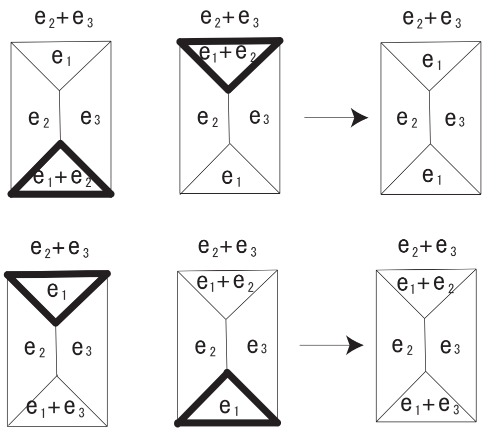

Figure 14. $M(P,\ \lambda_1)$ and $M(P,\ \sigma \circ \lambda_4)$ can be constructed from $M(P,\ \lambda_2)$ and $M(P,\ \lambda_3)$ by using $\sharp^{\Delta}$. This figure shows that we do $\sharp^{\Delta}$ along black facets.

In Figure 14, we remark that the middle two $P^3(3)$'s are equivariantly diffeomorphic to $M(P,\ \lambda_2)$ because there is the combinatorially equivalent to the left-second figure in Figure 13 which preserves the colorings (from the geometric point of view, this equivalence on polytopes corresponds with the equivariant diffeomorphisms on small covers). We can easily show that the coloring of the right bottom figure in Figure 14 is same as the left-fourth figure in Figure 13 up to $GL(3,\ \mathbb{Z}_2)$-equivariant isomorphism, by using the following $\sigma \in GL(3,\ \mathbb{Z}_2)$:

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the above argument, we have the following lemma.

Lemma 4.7. *For the basic small covers in Theorem 4.5 (Figure 13), the following relations hold:*

$$(P^3(3),\ \lambda_1) = (P^3(3),\ \lambda_2)\sharp^{\Delta}(P^3(3),\ \lambda_2);$$
$$(P^3(3),\ \sigma\circ\lambda_4) = (P^3(3),\ \lambda_2)\sharp^{\Delta}(P^3(3),\ \lambda_3).$$

Next we remark that the operation $\sharp^e$ (resp. $\sharp^{eve}$) itself includes the gluing $P^3(3)$ (resp. $P^3_-(3)$). However, the 3-sided prism with coloring $(P^3(3),\ \lambda)$ which is not included in Figure 13 can be constructed as $(\Delta,\ \lambda_0)\sharp(\Delta,\ \lambda_0)$ using the argument in [**28**, Section 3]. Moreover, all truncated prisms with colorings $(P^3_-(3),\ \lambda)$ can be constructed as $(\Delta,\ \lambda_0)\sharp(P^3(3),\ \lambda)$ using the argument in [**28**, Section 3 (a)]. Hence, we have the following corollary (Corollary 4.8) by applying Theorem 4.1 (2), (3) and Lemma 4.7 to the Lü-Yu's theorem (Theorem 4.5).

Corollary 4.8. Each 3-dimensional small cover can be (equivariantly) constructed from $\mathbb{R}P(3)$, $M(P^3(3),\ \lambda_2)$ and $M(P^3(3),\ \lambda_3)$ (up to weak equivariant diffeomorphism) by using the following four operations: $\sharp$; $\natural$; $\sharp^{\Delta}$ and $\sharp^C$.

5. Relation between the Nishimura's theorem and the Lü-Yu's theorem

In this section, we give the relations among the Izmestiev's, Nishimura's and Lü-Yu's theorem.

5.1. Relation of Corollary 4.4 and the operation $\sharp^C$. In this subsection, we apply the operation $\sharp^C$ to Corollary 4.4 for oriented small covers.

The torus with standard $\mathbb{Z}_2^3$-action $T^3 = M(I^3,\ \lambda_0^I)$ can be constructed from $(\Delta^3,\ \lambda_0)$ by using $\sharp$, $\natural$ and $\sharp^C$ as Figure 15.

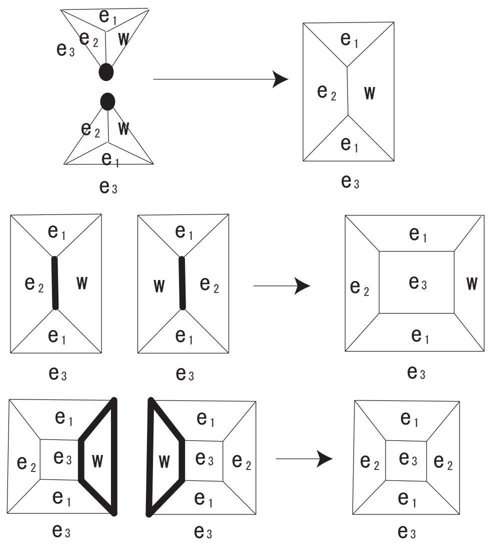

Figure 15. $(I^3,\ \lambda_0^I)$ can be constructed from $(\Delta^3,\ \lambda_0)$ by using $\sharp$, $\natural$ and $\sharp^C$, where $w = e_1 + e_2 + e_3$.

In the Figure 15, the first figure shows that the connected sum on the black vertices; then, we get $((\Delta^3,\ \lambda_0))\sharp(\Delta^3,\ \lambda_0) = (P^3(3),\ \lambda)$. The second figure shows that the connected sum along the black edges: this operation is identically with $\sharp^e = \natural\circ\sharp$, and we get $(P^3(3),\ \lambda)\sharp^e(P^3(3),\ \lambda) = (I^3,\ \lambda')$. In $(I^3,\ \lambda')$ (the second right figure I^3), the colorings around the square facets with coloring w are 2-independent; therefore, we can do $\sharp^C$ along this facets. The third figure shows this coloring change, i.e., the coloring change along the black square facets and we get $(I^3,\ \lambda')\sharp^C(I^3,\ \lambda') = (I^3,\ \lambda_0^I)$.

Therefore, we have the following proposition by applying the above argument to Corollary 4.4.

PROPOSITION 5.1. Each 3-dimensional oriented small cover can be (equivariantly) constructed from $\mathbb{R}P(3)$ by using finite times following three operations: the equivariant connected sum $\sharp$; the equivariant Dehn surgery $\natural$; and the coloring change $\sharp^C$.

5.2. Problem. The above Proposition 5.1 shows that Corollary 4.8 is not the (directly) generalization of Corollary 4.4. Now we may give the relations among Theorem 4.2, Corollary 4.4, 4.8 and Proposition 5.1 as the following list:

linear model	oriented	3-dimensional small covers
Izmestiev (Theorem 4.2)	Nishimura (Corollary 4.4)	??
	Proposition 5.1	Lü-Yu (Corollary 4.8)

Here, the column in the list means the category of 3-dimensional small covers, and this list means that Corollary 4.4 is the generalization of Theorem 4.2 and Corollary 4.8 is the generalization of Proposition 5.1. So we can ask the following problem:

PROBLEM 5.2. What is a generalization of Corollary 4.4 for 3-dimensional small covers? In other wards, what are basic small covers which construct all 3-dimensional small covers using operations $\sharp$, $\natural$ (or $\sharp^{\Delta}$)?

6. On 2-torus manifolds

In this final section, we shall give some remark for 2-torus manifolds. A *2-torus manifold* M^n is an n-dimensional, closed smooth manifold with a non-free effective smooth $\mathbb{Z}_2^n$-action (see Section 2.1). In this chapter, we are interested in the $n = 3$ case. First we give some basic facts for 2-torus manifolds (see [**26, 27**] for detail).

6.1. Basics of 2-torus manifolds. Let $\pi : M^3 \to M^3/\mathbb{Z}_2^3$ be the orbit projection of the 2-torus manifold M. We see that the orbit space $M^3/\mathbb{Z}_2^3$ is a three dimensional closed space. If $\mathbb{Z}_2^3$-action is locally standard then the orbit space is a 3-dimensional manifold with corner. Moreover, we can define the cell decomposition on $\partial M^3/\mathbb{Z}_2^3$ (we call it the *orbit cell decomposition*) induced from the information of $\mathbb{Z}_2^3$-orbits as follows: the information of fixed points as 0-cells (vertices); the information of rank one-orbits (i.e., orbits $\mathbb{Z}_2^3/K \simeq \mathbb{Z}_2$) as 1-cells (edges); the information of rank two-orbits (i.e., orbits $\mathbb{Z}_2^3/K \simeq \mathbb{Z}_2^2$) as 2-cells (facets); we remark that the free orbits correspond with the interior of $M^3/\mathbb{Z}_2^3$. With a method similar to that defines the characteristic function on the small cover, we can define the characteristic function from facets in $\partial M^3/\mathbb{Z}_2^3$ to $\mathbb{Z}_2^3$. For example, $\mathbb{Z}_2^3$ acts canonically on the last three coordinates of $S^3 \subset \mathbb{R} \oplus \mathbb{R}^3$. Then S^3 is a 2-torus manifold (not a small cover), and its orbit space with characteristic function is as that in Figure 16 by computing isotropy subgroups on rank two-orbits.

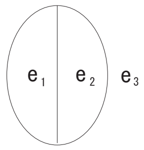

FIGURE 16. The orbit cell decomposition on $S^3/\mathbb{Z}_2^3$. We describe this orbit cell decomposition with coloring as $(D^3,\ \rho)$

Figure 16 shows that $S^3/\mathbb{Z}_2^3$ becomes the 3-disk D^3, and its orbit cell decomposition is as follows: two vertices; three edges; and three facets. We remark that, in Figure 16, e_3 is the coloring of the behind 2-cell.

We describe the orbit cell decomposition with coloring in Figure 16 as $(D^3,\ \rho)$

6.2. The equivariant surgery $\natural_0$. For the 2-torus manifold, we can define the new operation $\natural_0$ introduced from the (geometric) equivariant surgery. First, we explain this operation $\natural_0$.

The *equivariant surgery* $\natural_0$ is the operation described in Figure 17. As we see in Figure 16, generally in the orbit cell decomposition of 2-torus manifolds, there is the multi-edge (i.e., two vertices connected by more than two edges, also see the left figures in Figure 17). The left figures in Figure 17 show the neighborhood around the multi-edge with coloring in the orbit cell decomposition $(Q,\ \lambda)$ and $(D^3,\ \rho)$. First we take two facets $D_1 \subset Q$ and $D_2 \subset D^3$ as the facets surrounded by black edges in Figure 17. The equivariant surgery $\natural_0$ is the gluing Q and D^3 with small neighborhoods of D_1 and D_2 removed. As a result, we get the new orbit cell decomposition with coloring $(Q',\ \lambda')$ as in the right figure in Figure 17, vice versa. $(Q',\ \lambda')$ is denoted by $\natural_0(Q,\ \lambda)$.

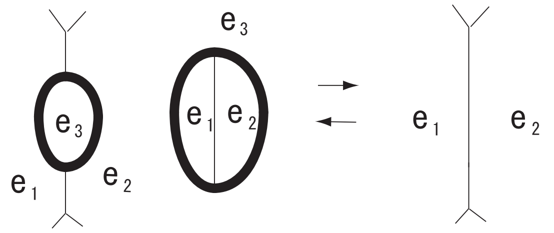

FIGURE 17. The equivariant surgery $\natural_0$ (from left to right) and its inverse $(\natural_0)^{-1}$ (from right to left).

Let D be a facet in the orbit cell decomposition Q whose boundary is 2 vertices and 2 edges, and $\pi : M \to Q$ be an orbit projection. We can easily show that this facet D corresponds with the invariant submanifold diffeomorphic to S^2 in M, that is, $\pi^{-1}(D) = S^2$. So we can understand the geometric meaning of the operation $\natural_0$ as follows. Because $\mathbb{Z}_2^3$ acts on the last three coordinate of $S^3 \subset \mathbb{R} \oplus \mathbb{R}^3$, there is the $\mathbb{Z}_2^3$-invariant submanifold $S^2 \subset \mathbb{R}^3 \oplus \{0\}$ (remark that there are two fixed points in this submanifold). Then its invariant, closed, tubular neighborhood is equivariantly isomorphic to $S^2 \times D^1$ with the standard $\mathbb{Z}_2^3$-action, where $D^1 = I^1$. We next consider $S^3 = S^2 \times D^1 \cup D^3 \times S^0$, and prepare the $\mathbb{Z}_2^3$-invariant part $S^3 \backslash S^2 \times D^1 = D^3 \times S^0$ (remark that there is no fixed points in this disconnected manifold). Next we remove the invariant neighborhood $S^2 \times D^1$ around $S^2 = \pi^{-1}(D)$ from the 2-torus manifold M, i.e., we take $M \backslash S^2 \times D^1$. Finally we glue these two invariant manifold $M \backslash S^2 \times D^1 \cup_\partial S^3 \backslash S^2 \times D^1$. This operation is identically the *equivariant surgery* of three dimensional manifold which is different form the equivariant surgery explained in Section 3.4.

6.3. Remark on the operations $\sharp^\Delta$ and $\sharp^C$. In this chapter, we define the three type different operations which introduced by the (geometric) equivariant surgeries as $\sharp$, $\natural$ and $\natural_0$ (remark that we can easily regard the equivariant connected sum as the equivariant surgery). Finally in this chapter, we prove that $\sharp^\Delta$ and $\sharp^C$ can be constructed by these equivariant surgeries, i.e., we prove the following theorem.

THEOREM 6.1. *The operations $\sharp^\Delta$, $\sharp^C$ can be obtained by using $\natural$, $\natural_0$ and $\sharp$ as follows:*
(1) *$\sharp^\Delta = \natural_0 \circ \natural \circ \sharp$, i.e., $(P_1,\ \lambda)\sharp^\Delta(P_2,\ \lambda') = \natural_0 \circ \natural((P_1,\ \lambda)\sharp(P_2,\ \lambda'))$;*
(2) *$\sharp^C P^3(l) = \natural_0 \circ \natural^{l-2} \circ \sharp$, i.e., $(P,\ \lambda)\sharp^C(P^3(l),\ \lambda') = \natural_0 \circ \natural \circ \cdots \circ \natural((P,\ \lambda)\sharp(P^3(l),\ \lambda'))$,*
where $P^3(l)$ is the l-sided prism, $l \geq 3$ and $\natural^{l-2}$ denotes $(l-2)$-times $\natural \circ \cdots \circ \natural$.

PROOF. These relations (1) and (2) are shown by the Figures 18 and 19, respectively.

In Figures 18, the upper map from left to right means the connected sum $\sharp$ on two black vertices. Next we can do $\natural$ along the black edge; then we have the 2-gon facet as in the left bottom figure. Finally we can do $\natural_0$ for this 2-gon facet. As the result, we have the operation $\sharp^\Delta$ as $\natural_0 \circ \natural \circ \sharp$.

In Figure 19, we explain the $l = 5$ case only. However, we can easily apply the same argument for all $l \geq 3$. In Figure 19, the first map means the connected sum $\sharp$ on two black vertices. Then we have the $(2l-2)$-gon facet (in Figure 19, this facet is the 8-gon). Next we can do $\natural$ along the black edge as in the second figure; then, this facet becomes the $(2l-4)$-gon facet. Iterating this argument ($(l-2)$-times), finally we have the 2-gon facet. Then we can do $\natural_0$ along this 2-gon facet. As the result, we have the operation $\sharp^C P^3(l)$ as $\natural_0 \circ \natural^{l-2} \circ \sharp$. $\square$

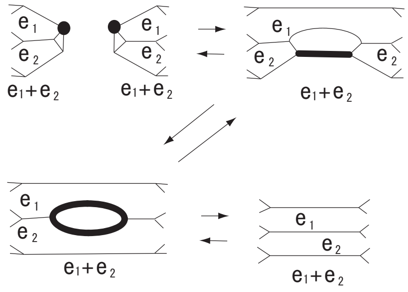

FIGURE 18. $\sharp^{\Delta} = \natural_0 \circ \natural \circ \sharp$.

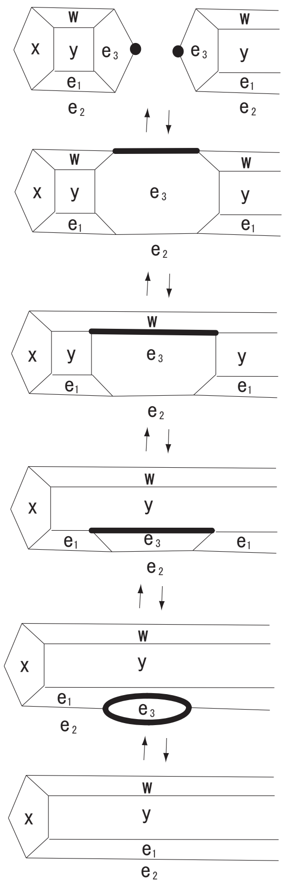

FIGURE 19. $\sharp^C P^3(l) = \natural_0 \circ \natural^{l-2} \circ \sharp$.

CHAPTER 6

On projective bundles over small covers

This is the joint work with Zhi Lü.

1. Introduction of Part 6

In the paper [**32**], Masuda gives a counter example of the cohomological rigidity of small covers, that is, the cohomology ring does not distinguish small covers. He classified homeomorphism types of height 2 real Bott manifolds by using the KO-ring structure of real projective spaces, where the *height* 2 *real Bott manifold* is the projective bundle of the line bundle over real projective space, and he found counter examples of the cohomological rigidity in height 2 real Bott manifolds. We note that if the vector bundle over small cover decomposes into line bundles then its projective bundle is small cover (see Proposition 3.1), and the real projective is small cover. Therefore, from his method, we are naturally led to study projective bundles over every small cover.

Due to the Masuda's method, if we know the KO-ring of small covers then we might be able to classify the homeomorphism types of such projective bundles; however, unfortunately it is difficult to compute KO-ring of small covers in general. So, in this part, we try to study homeomorphism types of such projective bundles from the different direction. Our method is closed to the Orlik-Raymond's method in [**45**]. In [**45**], Orlik and Raymond show that 4-dimensional torus manifolds can be constructed from basic torus manifolds by using connected sums. In fact, the small cover (resp. 2-torus manifold) M^n is the real version of the quasitoric manifold (resp. torus manifold) M^{2n}. Therefore, the Orlik-Raymond's method in [**45**] can be also applied for 2-dimensional small covers. The goal of this chapter is to generalize their method to the projective bundles of line bundles over 2-dimensional small covers, and to show the construction theorem of such projective bundles. The main result of this chapter is as follows:

Theorem 15. Let $P(\xi)$ be a projective bundle over 2-dimensional small cover M^2. Then $P(\xi)$ can be constructed from projective bundles $P(\kappa)$ over $\mathbb{R}P^2$ and $P(\zeta)$ over T^2 by using the projective connected sum $\sharp^{\Delta^{k-1}}$.

The organization of this chapter is as follows. In Section 2, we introduce some general results of projective bundles over spaces. In Section 3, we introduce the basic results for projective bundles over small covers. In Section 4, we show the structure of projective bundles over small covers, that is, all projective bundles over small covers are induced by the trivial projective bundles over moment-angle manifolds. In Section 5, as a toy model, we classify the topological types of projective bundles over S^1. In Section 6, we show the construction theorem of 2-dimensional small covers. In Section 7, we classify the topological types of projective bundles over $\mathbb{R}P(2)$ and T^2. This section is also the main part of this chapter. In Section 8, we introduce the new characteristic function (*projective characteristic function*) to characterize the projective bundles over small covers. In Section 9, we introduce the new operation (*projective connected sum*). Finally, in Section 10, we prove the construction theorem of projective bundles over small covers by using the above new notions.

2. General results

First, we introduce some general notations and basic results for projective bundles.

Let ξ be a real vector bundle over M, $P(\xi)$ be its projective bundle. Now we prove the following proposition.

PROPOSITION 2.1. Let M be a closed manifold, and ξ be a k-dimensional vector bundle. Then the following two statements are equivalent:

(1) $H^*(P(\xi);\ \mathbb{Z}_2) \simeq H^*(M \times \mathbb{R}P^{k-1};\ \mathbb{Z}_2)$;
(2) $w(\xi) = (1+X)^k$ for some $X \equiv_2 kw_1(\xi) \in H^1(M;\ \mathbb{Z}_2)$.

PROOF. By using the Borel-Hirzebruch formula (see [**10**, (23.3)]), we have that

$$w_1(\gamma_\xi)^k = \sum_{i=1}^{k} \pi^*(w_i(\xi))w_1(\gamma_\xi)^{k-i}, \tag{2.1}$$

where $\pi^* : H^*(M) \to H^*(P(\xi))$ is the induced injection and γ_ξ is the canonical line bundle of ξ. Therefore, we have that

$$H^*(P(\xi);\ \mathbb{Z}_2) \simeq \mathbb{Z}_2[x_1,\ \cdots,\ x_l,\ w_1(\gamma_\xi)]/\langle f_j(x_1,\ \cdots,\ x_l),\ (2.1)\rangle$$

Assume (1), that is,

$$H^*(P(\xi);\ \mathbb{Z}_2) \simeq H^*(M \times \mathbb{R}P^{k-1};\ \mathbb{Z}_2) = \mathbb{Z}_2[x_1,\ \cdots,\ x_l,\ A]/\langle f_j(x_1,\ \cdots,\ x_l),\ A^k\rangle.$$

Then we can write $A = w_1(\gamma_\xi) + \epsilon_1 x_1 + \cdots + \epsilon_l x_l = w_1(\gamma_\xi) + X$ and $A^k = (w_1(\gamma_\xi) + X)^k = 0$. Comparing with the formula (2.1), we see that $w(\xi) = (1+X)^k$.

Assume (2), that is, $w(\xi) = (1+X)^k$. By using (2.1), we have that $(w_1(\gamma_\xi) + X)^k = 0$. Therefore, there is the following isomorphism from $H^*(P(\xi))$ to $H^*(M \times \mathbb{R}P^{k-1})$:

$$\varphi : x_i \mapsto x_i;$$
$$\varphi : w_1(\gamma_\xi) \mapsto w_1(\gamma_\xi) + X.$$

□

PROPOSITION 2.2. Let M be a paracompact Hausdorff space and ξ its k-dimensional vector bundle. Then there is the following splitting manifold $P_k(\xi)$ and the surjection $g : P_k(\xi) \to M$:

(1) $g^*\xi = \gamma_1 \oplus \cdots \oplus \gamma_k$;
(2) $g^* : H^*(M) \to H^*(P_k(\xi))$ is injective,

where $P_k(\xi)$ is the following tower:

$$P_k(\xi) \xrightarrow{\simeq} P_{k-1}(\xi) \xrightarrow{\mathbb{R}P^1} \cdots \xrightarrow{\mathbb{R}P^{k-3}} P_2(\xi) \xrightarrow{\mathbb{R}P^{k-2}} P_1(\xi) \xrightarrow{\mathbb{R}P^{k-1}} M.$$

PROOF. We only explain about the second stage manifolds $P_2(\xi)$. There is the canonical line bundle γ_ξ over $P_1(\xi) = P(\xi)$ defined by ξ and the pull-back $\pi^*\xi$. Now we take the normal bundle $\gamma_\xi^\perp$ of $\gamma_\xi \subset \pi^*\xi$. Then this bundle $\gamma_\xi^\perp$ is a $(k-1)$-dimensional vector bundle over $P_1(\xi)$. So we can take the projective bundle of $\gamma_\xi^\perp$ and we denote $P(\gamma_\xi^\perp) = P_2(\xi)$. Iterating this argument, we can define $P_k(\xi)$, and the proof of the statement is the same as the splitting lemma of vector bundles. □

3. Basic results for small covers

Next, we introduce some notations and basic results for projective bundles of small covers. From this section, we assume M is an n-dimensional small cover.

PROPOSITION 3.1. Let M be a small cover, and ξ be an k-dimensional vector bundle over M. Then $P(\xi)$ is small cover if and only if $\xi = \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$.

PROOF. If ξ has the $\mathbb{Z}_2^k$-action along fibre, then $P(\xi)$ has the $\mathbb{Z}_2^{k-1}$-action along the fibre, vice versa. Hence, the problem is to determine when ξ has the $\mathbb{Z}_2^k$-action along fibre.

We first recall the definition of vector bundles. Let $\xi = \amalg(U_i \times \mathbb{R}^k)/\sim$ for an open covering $\{U_i\}$ of M, where $(u,x) \sim (u,y)$ is defined as follows: $u \in U_i \cap U_j$ and $x = Ay \in \mathbb{R}^k$ for some $A \in O(k)$. If the natural $\mathbb{Z}_2^k$-action on $\mathbb{R}^k$ extends to the global action on ξ, then A must be in $\mathbb{Z}_2^k \subset O(k)$. This implies that $\xi = \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$. Conversely, if $\xi = \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_k$, then this vector bundle has the $\mathbb{Z}_2^k$-action along fibre. □

By using Proposition 2.2 and 3.1, $P_k(\xi)$ has the following property.

COROLLARY 3.2. Let M be a small cover, and ξ be a Whitney sum of k line bundles over M. Then $P_k(\xi)$ is a small cover.

By using the fact that $P(\xi \otimes \gamma) \cong P(\xi)$ for all line bundle γ (e.g. see [**35**]) and the above Proposition 3.1, we have the following corollary.

COROLLARY 3.3. Let M be a small cover, and ξ be a Whitney sum of k line bundles over M. Then the small cover $P(\xi)$ is homeomorphic to

$$P(\gamma_1 \oplus \cdots \gamma_{k-1} \oplus \epsilon).$$

In this chapter, the *projective bundle over small cover* means the projective bundle in Corollary 3.3.

4. Structures of projective bundles over small covers

In this section, we show the structure of projective bundles of small covers. First, we recall the moment-angle complex of small covers. Let P be a simple, convex polytope and $\mathcal{F}$ a set of facets $\{F_1, \cdots, F_m\}$. We denote $\mathcal{Z}_P$ as the following manifolds:

$$\mathcal{Z}_P = \mathbb{Z}_2^m \times P/\sim,$$

where $(t, p) \sim (t', p)$ is defined by $t^{-1}t' \in \prod_{p\in F_i} \mathbb{Z}_2(i)$ ($\mathbb{Z}_2(i) \subset \mathbb{Z}_2^m$ is the subgroup generated by the i-th factor), and we call it a *moment-angle manifold* of P. We note that if $P = M^n/\mathbb{Z}_2^n$ then there is the subgroup $K \subset \mathbb{Z}_2^m$ such that $K \simeq \mathbb{Z}_2^{m-n}$ and K acts freely on $\mathcal{Z}_P$. Therefore, we can denote the small cover $M = \mathcal{Z}_P/\mathbb{Z}_2^l$ for $l = m - n$.

Because $[X;\, B\mathbb{Z}_2] = H^1(M;\, \mathbb{Z}_2) \simeq \mathbb{Z}_2^l$, we see that all line bundles γ can be written as follows:

$$\gamma \equiv \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}_\rho,$$

where $\mathbb{Z}_2^l$ acts on $\mathbb{R}_\rho = \mathbb{R}$ for some representation $\rho : \mathbb{Z}_2^l \to \mathbb{Z}_2$. Moreover, its Stiefel-Whitney class is $w(\mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}) = 1 + \delta_1 x_1 + \cdots + \delta_l x_l$ where $(\delta_1, \cdots, \delta_l) \in \{0, 1\}^l$ is induced by a representation $\mathbb{Z}_2^l \to \mathbb{Z}_2$, i.e., $(-1, \cdots, -1) \mapsto (-1)^{\delta_1} \cdots (-1)^{\delta_l}$. Therefore, by using Corollary 3.3, all projective bundles of small covers are as follows:

$$P(\xi) = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} (\mathbb{R}^k - \{0\})/\mathbb{R}^* = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}P^{k-1}, \tag{4.1}$$

where

$$\xi = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}^k$$

with the following $\mathbb{Z}_2^l$-action on $\mathbb{R}^k = \mathbb{R}_{\alpha_1} \oplus \cdots \oplus \mathbb{R}_{\alpha_k}$

$$\alpha_i : \mathbb{Z}_2^l \to \mathbb{Z}_2$$

where $i = 1, \cdots, k$ and α_k is the trivial representation. We also denote the projective bundle of small cover as

$$\mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}P^{k-1} = P(\gamma_1 \oplus \cdots \gamma_{k-1} \oplus \epsilon),$$

where $\gamma_i = \mathcal{Z}_P \times_{\mathbb{Z}_2^l} \mathbb{R}_{\alpha_i}$ $(i = 1, \cdots, k-1)$ satisfies that $w(\gamma_i) = 1 + \delta_{1i}x_1 + \cdots + \delta_{li}x_l$ for $(\delta_{1i}, \cdots, \delta_{li}) \in \{0, 1\}^l$ which is induced by the representation $\alpha_i : \mathbb{Z}_2^l \to \mathbb{Z}_2$.

Then we can easily prove the following proposition.

PROPOSITION 4.1. Let $(I_n : \Lambda)$ be the characteristic function of M. Then the characteristic function of $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon)$ is as follows:

$$\begin{pmatrix} I_n & O & \Lambda & \mathbf{0} \\ O & I_{k-1} & \Lambda_\xi & \mathbf{1} \end{pmatrix}, \tag{4.2}$$

where

$$\Lambda_\xi = \begin{pmatrix} \delta_{11} & \cdots & \delta_{l1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k-1} & \cdots & \delta_{l,k-1} \end{pmatrix}.$$

Therefore, we have the following corollary by Proposition 2.1.

COROLLARY 4.2. Let M be an small cover, and ξ a Whitney sum of k line bundles over M. Then the following two statements are equivalent:

(1) $H^*(P(\xi);\ \mathbb{Z}_2) \simeq H^*(M \times \mathbb{R}P^{k-1};\ \mathbb{Z}_2)$;
(2) $w(\xi) = w(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon) = \prod_{j=1}^{k-1}(1 + \sum_{i=1}^{l} \delta_{ij} x_i) = (1 + X)^k$.

5. Topological classification of projective bundles over the 1-dimensional small cover

Before we explain the main result of this part, in this section, as a toy model we explain the topological classification of projective bundles over the 1-dimensional small cover.

It is easy to show that the 1-dimensional small cover is the 1-dimensional circle S^1 and its moment-angle manifold is S^1 itself, i.e., the moment-angle manifold $S^1 \subset \mathbb{R}^2$ has the diagonal free $\mathbb{Z}_2$-action and its quotient is the 1-dimensional small cover S^1. Therefore, the classification problem corresponds with the classification of the following projective bundle of $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\epsilon)$:

$$S^1 \times_{\mathbb{Z}_2} P(q\gamma \oplus (k-q)\epsilon),$$

where γ is the canonical line bundle and ϵ is the trivial line bundle over S^1 and $0 \leq q \leq k-1$ (also see [**32**]).

Because $[S^1,\ BO(k)] = \mathbb{Z}_2$, the vector bundle over S^1 can be classified by its 1st Stiefel-Whitney classes. Therefore, we see that

(1) if $q \equiv_2 0$, then $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\epsilon) \equiv S^1 \times_{\mathbb{Z}_2} k\epsilon \cong S^1 \times \mathbb{R}^k$;
(2) if $q \equiv_2 1$, then $S^1 \times_{\mathbb{Z}_2} (q\gamma \oplus (k-q)\epsilon) \equiv S^1 \times_{\mathbb{Z}_2} (\gamma \oplus (k-1)\epsilon) \cong (S^1 \times_{\mathbb{Z}_2} \mathbb{R}) \times \mathbb{R}^{k-1}$.

Because $P(\xi \otimes \gamma) \cong P(\xi)$ and $H^*(S^1 \times_{\mathbb{Z}_2} P(\xi);\ \mathbb{Z}_2) \simeq \mathbb{Z}_2[x,\ y]/ < x^2,\ y^k + y^{k-1} w_1(\pi^*\xi) >$ for $\deg x = \deg y = 1$ (by the Borel-Hirzebruch formula) and the injectivity of induced homeomorphism $\pi^* : H^*(S^1;\ \mathbb{Z}_2) \to H^*(P(\xi);\ \mathbb{Z}_2)$ of $\pi : P(\xi) \to S^1$ (see [**40**]), we have the following proposition.

PROPOSITION 5.1. Let $P(q\gamma \oplus (k-q)\epsilon)$ be a projective bundle over S^1. Then its topological type is one of the followings:

(1) if $k \equiv_2 1$ or $q \equiv_2 0$, then $P(q\gamma \oplus (k-q)\epsilon) = S^1 \times \mathbb{R}P^{k-1}$;
(2) if otherwise, i.e., $k \equiv_2 0$ and $q \equiv_2 1$, then $P(q\gamma \oplus (k-q)\epsilon) = S^1 \times_{\mathbb{Z}_2} \mathbb{R}P^{k-1}$,

where $\mathbb{Z}_2$ acts freely on S^1 and on the first coordinate of $\mathbb{R}P^{k-1}$.

6. Construction theorem of 2-dimensional small covers

In order to state the main result of this chapter, we give some results for 2-dimensional small covers. First, we remark that the 2-dimensional small cover can be induced by the 2-dimensional polytope and characteristic functions as Figure 1, where $e_1 = (1,\ 0)$, $e_2 = (0,\ 1)$ and $e_1 + e_2 = (1,\ 1)$.

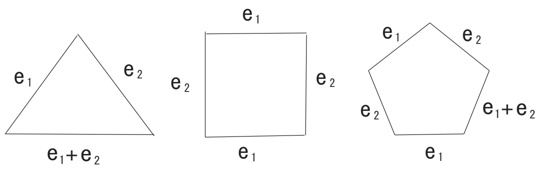

FIGURE 1. The left figure is $(\Delta^2,\ \lambda_0)$ and the middle figure is $(I^2,\ \lambda_0^I)$. We see that $M(\Delta^2,\ \lambda_0) = \mathbb{R}P(2)$ and $M(I^2,\ \lambda_0^I) = T^2$, where $\mathbb{R}P(2)$ and T^2 have the standard $\mathbb{Z}_2^2$-actions. The third figure is the 5-gon with characteristic functions.

Here, the *n-gon* is that the 2-dimensional polytope with n facets (or n vertices).

We can prove the following proposition (the construction theorem of 2-dimensional small covers).

Proposition 6.1. Let M^2 be a 2-dimensional small cover. Then M^2 is equivariantly homeomorphic to equivariant connected sum of $\mathbb{R}P(2)$ and T^2 with standard $\mathbb{Z}_2^2$-actions.

Proof. The small cover M^2 can be defined by the n-gon with characteristic functions in $\{0,\ 1\}^2$. Moreover, all n-gon can be constructed by the connected sum around vertices from Δ^2. We call this a *connected sum of n-gon*.

Now we can easily define the connected sum with characteristic functions of the n-gon with characteristic functions (also see Chapter 5). Moreover, this connected sum with characteristic functions corresponds with the equivariant connected sum in the small cover. Therefore, it is sufficient to prove that all n-gon with characteristic functions are reduced to connected sums with characteristic functions of $(\Delta^2,\ \lambda_0)$ and $(I^2,\ \lambda_0^I)$ (see Figure 1). Hence, we can apply the Orlik-Raymond's method in [**45**] directly and more simply as Figure 2.

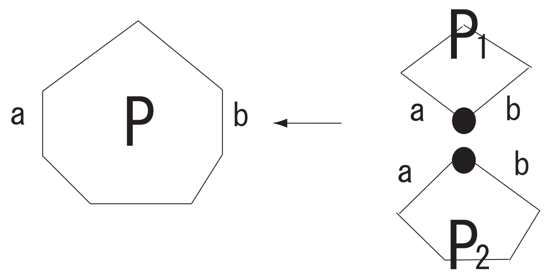

Figure 2. We can find the linearly independent, characteristic functions as a and b (a is separated from b by at least one edge) in the n-gon P. Then we can regard the first polytope P is constructed by the connected sum of P_1 and P_2 on the vertex constructed by two edges whose colors are a and b.

Here, P_1 and P_2 are n_1-gon and n_2-gon ($n_1,\ n_2 < n$), respectively. Iterating the argument in Figure 2, finally we have some $(\Delta^2,\ \lambda_0)$'s and $(I^2,\ \lambda_0^I)$'s. $\square$

Therefore, the real projective space $\mathbb{R}P(2)$ and torus T^2 with standard $\mathbb{Z}_2^2$-actions are the basic small covers in 2-dimensional small covers.

7. Topological classification of projective bundles over $\mathbb{R}P(2)$ and T^2

In this section, we recall the classification of the topological types of projective bundles over basic small covers, i.e., projective bundles over $\mathbb{R}P(2)$ and T^2.

The classification of projective bundles over $\mathbb{R}P(2)$ is known by the Masuda's paper [**32**]. Due to [**32**], we have that $q \equiv_4 q'$ or $k-q'$ if and only if $S^2\times_{\mathbb{Z}_2}P(q\gamma\oplus(k-q)\epsilon) \cong S^2\times_{\mathbb{Z}_2}P(q'\gamma\oplus(k-q')\epsilon)$. Remark that the line bundle over $\mathbb{R}P(2)$ is the canonical line bundle γ or the trivial line bundle ϵ. Therefore, by Proposition 3.1, we see that the projective bundles over $\mathbb{R}P(2)$ are only these types. Hence, we can easily show that the following proposition.

Proposition 7.1. Let $P(q\gamma\oplus(k-q)\epsilon)$ be a projective bundle over $\mathbb{R}P(2)$. Then its topological type is one of the followings.

(1) The case $k \equiv_4 0$:
   (a) if $q \equiv_4 0$, then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P(2) \times \mathbb{R}P(k-1)$;
   (b) if $q \equiv_4 1,\ 3$, then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$;
   (c) if $q \equiv_4 2$, then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\epsilon)$.
(2) The case $k \equiv_4 1$:
   (a) if $q \equiv_4 0,\ 1$, then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P(2) \times \mathbb{R}P(k-1)$;
   (b) if $q \equiv_4 2,\ 3$, then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(2\gamma \oplus (k-2)\epsilon)$.
(3) The case $k \equiv_4 2$:
   (a) if $q \equiv_4 0,\ 2$, then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P(2) \times \mathbb{R}P(k-1)$;
   (b) if $q \equiv_4 1$, then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$;

(c) if $q \equiv_4 3$, then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(3\gamma \oplus (k-3)\epsilon)$.

(4) The case $k \equiv_4 3$:

(a) if $q \equiv_4 0,\ 3$, then $P(q\gamma \oplus (k-q)\epsilon) \cong \mathbb{R}P(2) \times \mathbb{R}P(k-1)$;

(b) if $q \equiv_4 1,\ 2$, then $P(q\gamma \oplus (k-q)\epsilon) \cong S^2 \times_{\mathbb{Z}_2} P(\gamma \oplus (k-1)\epsilon)$.

Remark that the moment-angle manifold over $\mathbb{R}P(2)$ is S^2.

Next we classify projective bundles over T^2. Let γ_i be the pull back of the canonical line bundle over S^1 by the i-th factor projection $\pi_i : T^2 \to S^1$ $(i = 1,\ 2)$. We can easily show that line bundles over T^2 is completely determined by its 1st Stiefel-Whitney classes by $[T^2,\ B\mathbb{Z}_2] \simeq H^1(T^2;\ \mathbb{Z}_2) \simeq \mathbb{Z}_2^2$. Therefore, all of the line bundles over T^2 are ϵ, γ_1, γ_2 and $\gamma_1 \otimes \gamma_2$. By the definition of γ_i, we can easily show that $\gamma_i \oplus \gamma_i = \pi_i^*(\gamma \oplus \gamma) = \pi_i^*(2\epsilon) = 2\epsilon$. Therefore, we also have

$$\begin{aligned} & (\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \gamma_2) \\ = \ & \gamma_1 \otimes (\gamma_2 \oplus \gamma_2) \\ = \ & \gamma_1 \otimes 2\epsilon \\ = \ & \gamma_1 \oplus \gamma_1 = 2\epsilon. \end{aligned}$$

Hence, all of Whitney sums of k line bundles over T^2 is isomorphic to one of the followings by computing their Stiefel-Whitney classes:

$$\begin{aligned} k\epsilon; \\ \gamma_1 \oplus (k-1)\epsilon; \\ \gamma_2 \oplus (k-1)\epsilon; \\ (\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon; \\ \gamma_1 \oplus \gamma_2 \oplus (k-2)\epsilon; \\ \gamma_1 \oplus (\gamma_1 \otimes \gamma_2) \oplus (k-2)\epsilon; \\ \gamma_2 \oplus (\gamma_1 \otimes \gamma_2) \oplus (k-2)\epsilon. \end{aligned}$$

Therefore, by using $P(\xi \otimes \gamma) = P(\xi)$ for all line bundles γ and $H^*(P(\xi);\ \mathbb{Z}_2) \simeq \mathbb{Z}_2[x,\ y,\ z]/ < x^2,\ y^2,\ z^k + z^{k-1}w_1(\pi^*\xi) + z^{k-2}w_2(\pi^*\xi) >$ for $\deg x = \deg y = \deg z = 1$, $\pi : P(\xi) \to T^2$ (by the Borel-Hirzebruch formula), we have the following proposition.

PROPOSITION 7.2. Let $P(\xi)$ be a projective bundle over T^2. Then its topological type is one of the followings.

(1) The case $k \equiv_2 0$:

(a) $P(k\epsilon) \cong T^2 \times \mathbb{R}P(k-1)$;

(b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon) \cong P(\gamma_1 \oplus \gamma_2 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$;

(c) $P(\gamma_1 \oplus (k-1)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;

(d) $P(\gamma_2 \oplus (k-1)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-1})$;

(2) The case $k \equiv_2 1$:

(a) $P(k\epsilon) \cong T^2 \times \mathbb{R}P(k-1)$;

(b) $P((\gamma_1 \otimes \gamma_2) \oplus (k-1)\epsilon) \cong P(\gamma_1 \oplus (k-1)\epsilon) \cong P(\gamma_2 \oplus (k-1)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \underline{\mathbb{R}}^{k-1})$;

(c) $P((\gamma_1 \oplus \gamma_2) \oplus (k-2)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_1 \oplus (k-2)\epsilon) \cong P((\gamma_1 \otimes \gamma_2) \oplus \gamma_2 \oplus (k-2)\epsilon) \cong T^2 \times_{\mathbb{Z}_2^2} P(\mathbb{R}_{\rho_1} \oplus \mathbb{R}_{\rho_2} \oplus \underline{\mathbb{R}}^{k-2})$,

where $\rho_i : \mathbb{Z}_2^2 \to \mathbb{Z}_2$ is the i-th projection and $\underline{\mathbb{R}}$ is the trivial line bundle.

Remark that the moment-angle manifold over T^2 is T^2 itself.

8. New characteristic function of projective bundles over 2-dimensional small covers

In order to show the main result of this part, i.e., the construction theorem of projective bundles over 2-dimensional small covers. We introduce the new characteristic function (in this section) and the new operation of projective bundles over small covers (in the next section).

First, we recall Proposition 4.1 for $n = 2$ case. Let $(I_2 : \Lambda)$ be the characteristic function of M^2, where $\Lambda \in M(2,\ l;\ \mathbb{Z}_2)$ for $l = m - 2$ (m is the number of facets). Then the characteristic function of $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon)$ is as follows:

$$\left(\begin{array}{cccc} I_2 & O & \Lambda & \mathbf{0} \\ O & I_{k-1} & \Lambda_\xi & \mathbf{1} \end{array} \right), \tag{8.1}$$

where

$$\left(\begin{array}{c} \Lambda \\ \Lambda_\xi \end{array} \right) = \left(\begin{array}{ccc} \lambda_{11} & \cdots & \lambda_{l1} \\ \lambda_{12} & \cdots & \lambda_{l2} \\ \delta_{11} & \cdots & \delta_{l1} \\ \vdots & \ddots & \vdots \\ \delta_{1,k-1} & \cdots & \delta_{l,k-1} \end{array} \right) = \left(\begin{array}{ccc} \mathbf{a}_1 & \cdots & \mathbf{a}_l \\ \mathbf{b}_1 & \cdots & \mathbf{b}_l \end{array} \right),$$

where $\mathbf{a}_i \in \{0,\ 1\}^2$ and $\mathbf{b}_i \in \{0,\ 1\}^{k-1}$ for $i = 1, \cdots, l$.

In usual, the characteristic function (8.1) is attached on facets of the polytope $P^2 \times \Delta^{k-1}$. However, because of its formulation, it is sufficient to attach on the polytope P^2 in order to determine $P(\gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \epsilon)$. Therefore, in order to consider the projective bundles, we can reduce the usual characteristic function as follows: for an m-gon P^2 and its facets $\mathcal{F}$, the function

$$\lambda_P : \mathcal{F} = \{F_1,\ \cdots,\ F_m\} \to \{0,\ 1\}^2 \times \{0,\ 1\}^{k-1}$$

satisfies that $\lambda_P(F_1) = e_1 \times \mathbf{0}$, $\lambda_P(F_2) = e_2 \times \mathbf{0}$ (denote them simply $e_1 = (1,\ 0)$, $e_2 = (0,\ 1)$, respectively) and

$$\det(\lambda_P(F_i)\ \lambda_P(F_j)\ X_1\ \cdots\ X_{k-1}) = 1 \tag{8.2}$$

for $F_i \cap F_j \neq \emptyset$ $(i \neq j)$ and all $\{X_1,\ \cdots,\ X_{k-1}\} \subset \{\mathbf{0} \times e'_1,\ \cdots,\ \mathbf{0} \times e'_{k-1},\ \mathbf{0} \times \mathbf{1}\}$, where e'_i $(i = 1,\ \cdots,\ k-1)$ is the canonical basis in $\{0,\ 1\}^{k-1}$. We call this function a *projective characteristic function* (we can easily generalize this notion to the general dimension n, not only 2), and $(P^2,\ \lambda_P)$ corresponds with the projective bundle over the 2-dimensional small cover (over P^2). The Figure 3 is the image of projective characteristic functions.

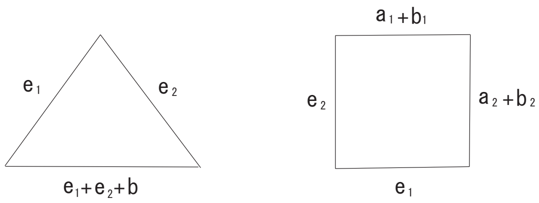

FIGURE 3. The image of projective characteristic functions. In the left triangle and the right square, functions are satisfies (8.2) on each vertex, where $a_1,\ a_2 \in \mathbb{Z}_2^2$.

Remark that in Figure 3, if we put $b = 0$ and $b_1 = b_2 = 0 \in \mathbb{Z}_2^{k-1}$, then this gives ordinary characteristic functions on the triangle and the square. Therefore, we can regard such forgetful map of $\{0,\ 1\}^{k-1}$-part $f : (P^2,\ \lambda_P) \to (P^2,\ \lambda)$ as the projection $P(\xi) \to M^2$.

9. New operations of projective bundles over 2-dimensional small covers

For two polytopes with projective characteristic functions, we can also do the connected sum operation which is compatible with projective characteristic functions as Figure 4.

Remark that, in the geometric level, the inverse image of vertices of polytopes with projective characteristic functions corresponds with the projective space $\mathbb{R}P^{k-1}$. Therefore, the geometric meaning of this operation is the equivariant gluing along the fibre $\mathbb{R}P^{k-1}$. We call this operation a *projective connected sum* and denote it as $\sharp^{\Delta^{k-1}}$.

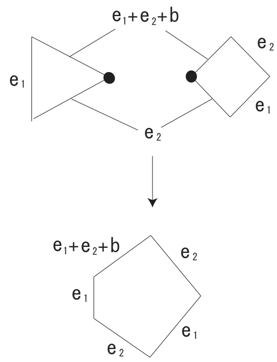

FIGURE 4. We can do connected sum same as the connected sum of polytopes with ordinary characteristic functions (see Chapter 5).

10. Construction theorem of projective bundles over 2-dimensional small covers

Now we may state the main theorem of this chapter.

THEOREM 10.1. *Let $P(\xi)$ be a projective bundle over 2-dimensional small cover M^2. Then $P(\xi)$ can be constructed from one of the projective bundles $P(\kappa)$ over $\mathbb{R}P^2$ in Proposition 7.1 and $P(\zeta)$ over T^2 in Proposition 7.2 by using the projective connected sum $\sharp^{\Delta^{k-1}}$.*

PROOF. We apply the Orlick-Raymond's method in [**45**] (see Section 6). In order to apply this method, it is sufficient to show that there are two edges F_i and F_j in the m-gon P such that $F_i \cap F_j =$ and their projective characteristic functions satisfy (8.2).

Put the projective characteristic function of F_i and F_j as follows:

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_j \\ b_j \end{pmatrix},$$

respectively, where $a_i,\ a_j \in \{0,\ 1\}^2$ and $b_i,\ b_j \in \{0,\ 1\}^{k-1}$. Then we can take $\det(a_i,\ a_j) = 1$. If $(X_1\ \cdots\ X_{k-1}) = I_{k-1}$ then we have that

$$\det \begin{pmatrix} a_i & a_j & O \\ b_i & b_j & I_{k-1} \end{pmatrix} = 1,$$

because $\det(a_i,\ a_j) = 1$. The problem is $X_h = \mathbf{1}$ for $h = 1,\ \cdots,\ k-1$. Assume $X_1 = \mathbf{1}$. Then, by using the basic transformation of matrix, we have that

$$\begin{aligned} & \det \begin{pmatrix} a_i & a_j & \mathbf{0} & O \\ b_i & b_j & \mathbf{1} & I_{k-2} \end{pmatrix} \\ = & \det \begin{pmatrix} a_i & a_j & \mathbf{0} & O \\ b_{1i} & b_{1j} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{k-2} \end{pmatrix} \\ = & \det \begin{pmatrix} a_i & a_j & O \\ \mathbf{0} & \mathbf{0} & I_{k-1} \end{pmatrix} = 1. \end{aligned}$$

For the other cases $X_h = \mathbf{1}$ for $h = 2,\ \cdots,\ k-1$, with the method demonstrated in the above argument, we see that there are two edges F_i and F_j in the m-gon P such that $F_i \cap F_j =$ and their projective characteristic functions satisfy (8.2). $\square$

Part 3

Appendix

CHAPTER 7

On 6-dimensional quasitoric manifolds

1. Introduction of Chapter 7

In the paper [**28**], Lü and Yu proved the construction theorem of the 3-dimensional small covers. From this result, it is easy to imagine that their results can be applied for the study of the construction theorem of the 6-dimensional quasitoric manifolds. However, if you try to apply the Lü-Yu's argument to prove the construction theorem of 6-dimensional quasitoric manifolds, then you will immediately find that there is a big gap in this case because its characteristic function is in $\mathbb{Z}$. In the paper [**36**], McGavran used very closed method with Lü-Yu's method to prove the construction theorem of spin torus manifolds. However, we find the serious mistake in his paper [**36**]. So, the construction theorem of 6-dimensional quasitoric manifolds or torus manifolds with some suitable condition is challengeable problem.

PROBLEM 1.1. Let M be a 6-dimensional quasitoric manifold (or torus manifold with suitable condition). Is there any basic quasitoric manifolds which yield M by using some operations (such as equivariant connected sum or equivariant surgery)?

For the topological classification, Wall and Jupp proved the following beautiful theorem in [**20**] and [**51**].

THEOREM 1.2 (Wall and Jupp). *The diffeomorphism types of simply connected, closed,* 6*-dimensional manifolds are completely determined by its cohomology rings, Stiefel-Whitney classes and Pontrjagin classes.*

In this chapter, we give the realization of Δ^3 with characteristic functions as quasitoric manifolds in Section 1, and the directly proof when 6-dimensional quasitoric manifolds are spin manifolds in Section 2. Finally, in Section 3, we give the remark of the McGavran's theorem in [**36**]. This part is the detail of the paper [**25**].

2. Geometric realization of the basic parts

The characteristic function of the quasitoric manifolds can realize not only its equivariant structure but also its stably almost complex structure. In this section, we compute the characteristic functions on Δ^3 and their stably almost complex structure. The characteristic functions on Δ^3 is as Figure 1

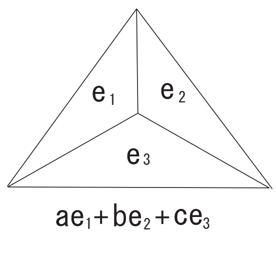

FIGURE 1. By the definition of the characteristic function, $a,\ b,\ c = \pm 1$.

Therefore, there are 8 characteristic functions on Δ^3 corresponding with $(a,\ b,\ c)$. These characteristic functions correspond with the equivariant type of $\mathbb{C}P(3)$ with T^3-actions. Moreover, there is the *weak equivariant diffeomorphism* (or ψ-*equivalence* in [**11**]) as the identity relations. This corresponds with the characteristic functions up to permutation, i.e., $(a,\ b,\ c)\equiv(-1,\ 1,\ 1)\equiv(1,\ -1,\ 1)\equiv(1,\ 1,\ -1)$ and $(a,\ b,\ c)\equiv(-1,\ -1,\ 1)\equiv(1,\ -1,\ -1)\equiv(-1,\ 1,\ -1)$ up to this relation. Hence, there are 4 characteristic functions on Δ^3 up to ψ-equivalence, that is,

$$\begin{aligned}(a,\ b,\ c) &= (-1,\ -1,\ -1);\\ (a,\ b,\ c) &= (1,\ 1,\ 1);\\ (a,\ b,\ c) &= (-1,\ 1,\ 1);\\ (a,\ b,\ c) &= (1,\ -1,\ -1).\end{aligned}$$

The ψ-equivalence corresponds with not only the weak equivariant diffeomorphism but also the stably almost complex structure (see [**7**]). Here, the stably complex structure is the complex structure of $\eta=\tau\oplus\epsilon$ over M where τ is the tangent bundle over M and ϵ is the trivial vector bundle over M.

By [**7**], we can calculate the χ_y-genus of almost complex structures on $\mathbb{C}P(3)$ which corresponds with 4 characteristic functions, where the χ_y-*genus* ($y\in\mathbb{Z}$) is defined as follows:

$$\chi_y(x)=\frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}}.$$

Let $x=1+c_1+c_2+\cdots+c_n$. The following three invariants ($y=-1,\ 0,\ 1$) are very important:

(1) $c_n[M^{2n},\ \eta]=\langle\chi_{-1}(c_1,\ \cdots,\ c_n)^{(n)},\ [M^{2n},\ \eta]\rangle=\sum_{v\in V}\sigma(v)$: the *top Chern number*;
(2) $\mathrm{td}[M^{2n},\ \eta]=\langle\chi_0(c_1,\ \cdots,\ c_n)^{(n)},\ [M^{2n},\ \eta]\rangle=\sum_{v\in V,\ \mathrm{ind}(v)=0}\sigma(v)$: the *Todd genus*;
(3) $\mathrm{sign}[M^{2n},\ \eta]=\langle\chi_1(c_1,\ \cdots,\ c_n)^{(n)},\ [M^{2n},\ \eta]\rangle=\sum_{v\in V}(-1)^{\mathrm{ind}(v)}\sigma(v)$: the *L-genus* (the *signature*),

By using the computation method in [**7**], we have the following list:

TABLE 1

	$\mathbb{C}P^3_{-,-,-}$	$\mathbb{C}P^3_{+,+,+}$	$\mathbb{C}P^3_{-,+,+}$	$\mathbb{C}P^3_{+,-,-}$
c_3	4	-2	0	2
td	1	1	2	0
sign	0	1	2	0

Here, $-$ means -1 and $+$ means $+1$ of $a,\ b,\ c$.

Now we have that $\tau\oplus\epsilon_{\mathbb{C}}=4\gamma$ (up to orientation) for the tangent bundle of $\mathbb{C}P(3)$, where γ is the canonical line bundle (up to orientation). By using the above top Chern number, we have the following decompositions:

- $\mathbb{C}P^3_{-,-,-}=[\mathbb{C}P(3),\ 4\gamma_3^1]$;
- $\mathbb{C}P^3_{+,+,+}=[\mathbb{C}P(3),\ 3\gamma_3^1\oplus\overline{\gamma_3^1}]$;
- $\mathbb{C}P^3_{-,+,+}=[\mathbb{C}P(3),\ 2\gamma_3^1\oplus 2\overline{\gamma_3^1}]$;
- $\mathbb{C}P^3_{-,-,-}=[\mathbb{C}P(3),\ 3\overline{\gamma_3^1}\oplus\gamma_3^1]$,

where γ_3^1 is the canonical line bundle over $\mathbb{C}P(3)$ and $\overline{\gamma_3^1}$ is its orientation reversing line bundle.

Therefore, we have the following proposition.

PROPOSITION 2.1. There are just 4 characteristic functions on the 3-dimensional simplex up to ψ-equivalence. Moreover, these characteristic functions correspond with the following line bundle decomposition:

(1) $\tau(\mathbb{C}P^3_{-,-,-})\oplus\underline{\mathbb{C}}\simeq\gamma_3^1\oplus\gamma_3^1\oplus\gamma_3^1\oplus\gamma_3^1$;

(2) $\tau(\mathbb{C}P^3_{+,+,+})\oplus\underline{\mathbb{C}}\simeq\gamma_3^1\oplus\gamma_3^1\oplus\gamma_3^1\oplus\overline{\gamma_3^1}$;

(3) $\tau(\mathbb{C}P^3_{-,+,+}) \oplus \underline{\mathbb{C}} \simeq \gamma^1_3 \oplus \gamma^1_3 \oplus \overline{\gamma^1_3} \oplus \overline{\gamma^1_3}$;

(4) $\tau(\mathbb{C}P^3_{+,-,-}) \oplus \underline{\mathbb{C}} \simeq \overline{\gamma^1_3} \oplus \overline{\gamma^1_3} \oplus \overline{\gamma^1_3} \oplus \gamma^1_3$,

where γ^1_3 is the canonical line bundle, $\overline{\gamma^1_3}$ is the line bundle with its converse orientation and $\underline{\mathbb{C}}$ is the trivial complex line bundle over $\mathbb{C}P(3)$.

3. Properties of oriented small covers and spin quasitoric manifolds

According to [**11**], if $F_1, \cdots, F_n$ are the codimension-1 faces meeting at some vertex p of P then $\lambda(F_1), \cdots, \lambda(F_n)$ is a basis. This condition is equivalent with the following condition: if $\Lambda(p) = (\lambda(F_1) \cdots \lambda(F_n))$ is an $n \times n$ square matrix whose column vectors consist of images of characteristic functions around a vertex p then $\det \Lambda = 1 \in \mathbb{Z}_2$ for $d = 1$ or $\det \Lambda = \pm 1 \in \mathbb{Z}$ for $d = 2$.

Due to the classical results for the characteristic classes (see [**37**]), we have that the manifold M is oriented iff its first Stiefel-Whitney class satisfies $w_1(M) = 0$, and the manifold M has a spin structure, i.e., M is a spin manifold, iff its second Stiefel-Whitney class satisfies $w_2(M) = 0$.

Using the method of [**41**], we can easily show the following proposition, where $\mathbb{K}_1 = \mathbb{Z}_2$ and $\mathbb{K}_2 = \mathbb{Z}$.

Proposition 3.1. If $M^{dn} = M(P^n, \Lambda(\mathbb{K}_d))$ satisfies $w_d(M^{dn}) = 0$, then each row vector $(\lambda_1, \lambda_2, \cdots, \lambda_n)^t \in \mathbb{K}^n_d$ in $\Lambda'(\mathbb{K}_d)$ satisfies that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n \equiv 1 (\text{mod } 2).$$

Proof. We may only prove the statement for $d = 2$, because the proofs of the statement for $d = 1$ and $d = 2$ are similar.

Let $M = M(P, \Lambda(\mathbb{Z}))$ be a spin quasitoric manifold over a simple polytope P and a characteristic function $\Lambda(\mathbb{Z})$. We use the Davis and Januszkiewicz formula for characteristic classes of quasitoric manifolds in [**11**]. Due to their formula, the Stiefel-Whitney class of M can be calculated using the formula $j^*(1+v_1)\cdots(1+v_m)$, where $j^* : H^*_{T^n}(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ is the induced homomorphism of an embedding $j : M \to ET^n \times_{T^n} M$ and v_i's are generators of $H^*_{T^n}(M;\mathbb{Z})$ $(i = 1, \cdots, m)$. Because $H^*_{T^n}(M;\mathbb{Z})$ is isomorphic to the Stanley-Reisner ring $\mathbb{Z}[P]$ of P (see [**11**]), there is no additive relations on $H^2_{T^n}(M;\mathbb{Z})$. Hence, we have the following equation for the second Stiefel-Whitney classes

$$w_2(M) = j^*(v_1 + \cdots + v_m) \in H^2(M;\ \mathbb{Z}_2). \tag{3.1}$$

Let λ_{ij} be the (i, j)-element of the matrix $\Lambda(\mathbb{Z}) = (I_n \mid \Lambda'(\mathbb{Z})) = (\lambda_{ij})$ $(1 \le i \le n$ and $1 \le j \le m)$, and $\mathcal{J}$ be an ideal in $H^*_{T^n}(M)$ generated by $\theta_i = \lambda_{i1}v_1 + \cdots + \lambda_{im}v_m$ for $i = 1, \cdots, n$. Because of Theorem in [**11**], we have $H^*(M) = H^*_{T^n}(M)/\mathcal{J}$. Because the first n-square matrix of $\Lambda(\mathbb{Z})$ is the identity matrix I_n, we have that $\theta_i = v_i + \lambda_{i,n+1}v_1 + \cdots + \lambda_{im}v_m$ for $i = 1, \cdots, n$. Thus, we have $j^*(v_i) = -(\lambda_{i,n+1}j^*(v_{n+1}) + \cdots + \lambda_{im}j^*(v_m))$ for $i = 1, \cdots, n$. Because of the above Eq.(3.1), we have

$$\begin{aligned} w_2(M) &= j^*(v_1 + \cdots + v_m) \\ &\equiv (1 - \lambda_{1,n+1} - \cdots - \lambda_{n,n+1})j^*(v_{n+1}) + \cdots + (1 - \lambda_{1m} - \cdots - \lambda_{nm})j^*(v_m) \mod 2. \end{aligned}$$

The proof is then completed by employing $w_2(M) = 0$, i.e., the assumption that M is a spin manifold. The proof of the statement for $d = 1$ is similar to the above proof. □

4. Remark on the McGavran's paper

Suppose that M^6 is a spin 6-dimensional quasitoric manifold. We can take the characteristic functions around edges as Figure 2.

We note that the characteristic function satisfies the locally standard property, i.e., a determinant of a matrix whose column vectors consist of characteristic functions around a vertex must be ± 1. Hence $a = \pm 1$. Using Proposition 3.1 and $a = \pm 1$, we also have $b + c \equiv 0$ (mod 2). Moreover, McGavran shows in his paper [**36**, 4.9 Lemma] that the characteristic functions around edges satisfies the following properties: $b, c = 0$ or ± 1. We call it the *McGavran's lemma.* However, the

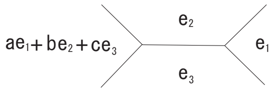

FIGURE 2. Characteristic functions around edges

following example shows that there are quasitoric manifolds which do not hold the McGavran's lemma (also see examples in [**8**]).

EXAMPLE 4.1. Let $M_\rho = S^5 \times_{S^1} P(\mathbb{C}_\rho \oplus \underline{\mathbb{C}})$ be a quotient manifold of $S^5 \times P(\mathbb{C}_\rho \oplus \underline{\mathbb{C}})$ by the following S^1-action: S^1 naturally acts on $S^5 \subset \mathbb{C}^3$; and S^1 acts on $\mathbb{C}_\rho$ by the representation $\rho(t) = t^r$ where $t \in S^1$ and $r \in \mathbb{Z}$, and on $\underline{\mathbb{C}}$ trivially. Here, S^5 is a 5-dimensional sphere and $P(\mathbb{C}_\rho \oplus \underline{\mathbb{C}})$ is a projectify of the complex space $\mathbb{C}_\rho \oplus \underline{\mathbb{C}}$. We define a T^3-action on M_ρ as follows:

$$(t_1, t_2, t_3)[(w_1, w_2, w_3), [z_0 : z_1]] \mapsto [(t_1 w_1, t_2 w_2, w_3), [t_3 z_0 : z_1]],$$

in which case the characteristic function is as the following Figure 3.

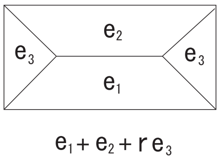

FIGURE 3. The characteristic function of Example 4.1, where $e_1 + e_2 + re_3$ is the characteristic function of the facets behind this polytope.

In the case that r is odd, this manifold is a 6-dimensional, spin quasitoric manifold by making use of Proposition 3.1. Due to the McGavran's lemma, we have that $r = 0$ or ± 1. However, if $|r| > 2$, this example does not satisfy this condition. It follows that there is a mistake in the McGavran's lemma (4.9 Lemma in [**36**]).

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List of publications and preprints in postdoctoral period

1. ACCEPTED PAPERS

(1) Shintarô Kuroki, *Classification of compact transformation groups on complex quadrics with codimension one orbits*, to appear in Osaka J. Math. **46**, no. 1, 2009.

(2) Shintarô Kuroki, *Characterization of homogeneous torus manifolds*, to appear in Osaka J. Math.

2. PREPRINTS

(3) Shintarô Kuroki, *Hypertorus graphs and graph equivariant cohomologies.*

(4) Shintarô Kuroki, *Classification of quasitoric manifolds with codimension one extended actions.*

(5) Shintarô Kuroki, *Classification of torus manifolds with codimension one extended actions.*

(6) Shintarô Kuroki, *Operations on three dimensional small covers.*

(7) Suyoung Choi, Shintarô Kuroki, *Topological classification of torus manifolds which have codimension one extended G-actions*, in preparation.

(8) Shintarô Kuroki, Zhi Lü, *On projective bundles over small covers*, in preparation.

3. PROCEEDINGS

(9) Shintarô Kuroki, *On* 8*-manifolds with* $SU(3)$*-actions*, RIMS kokyuroku **1569**, 81–93, 2007.

(10) Shintarô Kuroki, *Remarks on McGavran's paper and Nishimura's result*, Trends in Math. - New Series, **10**, No. 1, ICMS in KAIST, 77–79, 2008.

4. OTHERS

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