GKM manifold - survey

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Abstract. Continued from the previous article about GKM manifold -definition [**28**], in this article, we survey some fundamental results in the subject. In particular, we introduce the connection between combinatorics and equivariant topology of GKM manifolds.

1. Introduction

In the previous article [**28**], we introduce two definitions of GKM manifolds and the Goresky-Kottwitz-MacPherson (GKM) Theorem. Because the GKM theorem may be regarded as a variant of the localization theorem of equivariant cohomology (see e.g. [**5, 7**]), the GKM theorem can be applied to compute equivariant cohomology of GKM manifolds. In this article, we survey the connection between GKM manifolds and combinatorics, and introduce how the equivariant cohomology of GKM manifolds can be computed by using combinatorics. As a combinatorial object, in Section 2, we introduce two classes of abstract labeled graphs, called *GKM graphs* introduced by Guillemin-Zara in [**21**] and *torus graphs* introduced by Maeda-Masuda-Panov in [**35**]. Some of these labeled graphs are induced by using the information of torus actions on GKM manifolds and torus manifolds (see Section 3), and convert the geometric data of torus actions to the combinatorial data of labeled graphs. As a typical application, in Section 4 and 5, we introduce some combinatorial formula of equivariant cohomology of torus manifolds with vanishing odd degree cohomology [**37, 35**], called a *face ring* (of induced torus graphs). The face ring may be regarded as the generalization of the *Stanley-Reisner ring* which gives a combinatorial formula of equivariant cohomology of complete non-singular toric varieties (the Danilov-Jurkiewicz theorem, see [**13, 38**]) and that of quasitoric manifolds (the Davis-Januszkiewicz theorem, see [**9, 6**]).

Note that there are many works and generalizations about the relations between equivariant topology (or geometry) of GKM manifolds and combinatorics (e.g. [**1, 14, 15, 16, 18, 19, 20, 22, 25, 26, 33, 34, 39**] etc). In this article, we basically survey a part of the works of Guillemin-Zara [**21**] (the original work introducing the notion of GKM graphs), Braden-MacPherson [**4**] (introducing the notion of sheaves of moment graphs which defines topological or geometrical invariants systematically from GKM graphs or more general labeled graphs) and Maeda-Masuda-Panov [**35**] (introducing the notion of torus graphs and proved the combinatorial formula of its equivariant cohomology).

2. Abstract GKM graph

We first prepare some notation. Let $\Gamma = (V(\Gamma), E(\Gamma))$ (or (V, E) whenever the graph Γ is clear from the context) be an *(abstract) graph* comprising a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of oriented edges; denote its initial vertex by $i(e)$ and its terminal vertex by $t(e)$. The symbol $\overline{e} \in E(\Gamma)$ represents the edge e with its orientation reversed, i.e., $i(e) = t(\overline{e})$ and $t(e) = i(\overline{e})$. In this article, we assume that there are no loops in $E(\Gamma)$, i.e., for any $e \in E(\Gamma)$, $i(e) \neq t(e)$, and Γ is

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connected and finite, i.e., $V(\Gamma)$ and $E(\Gamma)$ are finite sets. Put the subset of all out-going edges as

$$E_p(\Gamma)(= E_p) = \{e \in E(\Gamma) \mid i(e) = p\} \subset E(\Gamma).$$

An abstract graph Γ is called an *m-valent graph* if $|E_p| = m$ for all $p \in V$, where the symbol $|X|$ represents the cardinality of a finite set X.

Let Γ be an m-valent graph. In order to define a GKM graph, we need to introduce a label $\alpha : E \to H^2(BT)$ on edges of Γ, often denoted by (Γ, α), where BT is a classifying space of an n-dimensional torus T. Recall that the cohomology ring of BT is isomorphic to the polynomial ring

$$H^*(BT) \simeq \mathcal{R}[\alpha_1, \ldots, \alpha_n],$$

where $\mathcal{R}$ is a coefficient ring and α_i is a variable with $\deg \alpha_i = 2$ for $i = 1, \ldots, n$. So its degree 2 part $H^2(BT)$ is isomorphic to $\mathcal{R}^n$. Take any function $\alpha : E(\Gamma) \to H^2(BT)$ and set

$$\alpha(E_p)(= \alpha(E_p(\Gamma))) = \{\alpha(e) \mid e \in E_p\} \subset H^2(BT).$$

We call a function $\alpha : E \to H^2(BT^n) \setminus \{0\}$ for $n \leq m$ an *axial function* (resp. *torus axial function*) on Γ if it satisfies the following three conditions:

(1): $\alpha(e) = -\alpha(\overline{e})$ (resp. $\alpha(e) = \pm\alpha(\overline{e})$);

(2): for each vertex $p \in V$, the set $\alpha(E_p)$ spans $H^2(BT^n)$; moreover, $\alpha(E_p)$ is *pairwise linearly independent*, i.e., each pair of elements in $\alpha(E_p)$ is linearly independent in $H^2(BT)$;

(3): for each edge $e \in E$, there exists a bijective map $\nabla_e : E_{i(e)} \to E_{t(e)}$ such that
  (1) $\nabla_{\overline{e}} = \nabla_e^{-1}$,
  (2) $\nabla_e(e) = \overline{e}$, and
  (3) for each $e' \in E_{i(e)}$, $\alpha(\nabla_e(e')) - \alpha(e') \equiv 0 \mod \alpha(e) \in H^2(BT)$; this equation is called a *congruence relation*.

The collection $\nabla = \{\nabla_e \mid e \in E\}$ is called a *connection* on the labelled graph (Γ, α); we denote the labelled graph with connection as (Γ, α, ∇).

Definition 2.1 (GKM graph [**21**] and torus graph [**35**]). If an m-valent graph Γ is labeled by an axial function $\alpha : E(\Gamma) \to H^2(BT^n)$ for some $n \leq m$, then such labeled graph is said to be an (abstract) *GKM graph* (see [**21**]). If an n-valent graph Γ is labeled by a torus axial function $\alpha : E(\Gamma) \to H^2(BT^n)$, then such labeled graph is said to be an (abstract) *torus graph* (see [**35**]). Both labeled graphs are denoted as (Γ, α, ∇) or (Γ, α) (if the connection ∇ is obviously determined).

The two figures in Figure 1 are typical examples of GKM graphs and torus graphs.

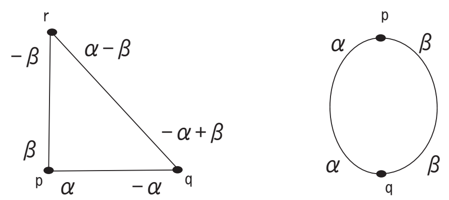

Figure 1. The left graph is a GKM graph (this is also a torus graph) and the right one is a torus graph, where α, β are generators of $H^2(BT^2)$.

By definition, a 2-valent GKM graph (i.e., the boundary of ℓ-gon) is always a torus graph and its connection is uniquely determined (see Figure 1). Moreover, the following combinatorial fact says that connections of most (but not all) of GKM graphs are uniquely determined (see [**21**] and [**35**]).

Proposition 2.2. *Let (Γ, α) be a GKM graph. If $\alpha(E_p)$ is three independent (i.e., each three tuple of elements in $\alpha(E_p)$ is linearly independent) for all vertices $p \in V$, then the connection ∇ on (Γ, α) is unique.*

Furthermore, a connection on a torus graph is uniquely determined.

Therefore, we may always denote a torus graph as (Γ, α) without connection.

3. GKM graphs induced from GKM manifolds

In this section, we recall that some of GKM graphs (resp. torus graphs) are induced from GKM manifolds (resp. torus manifolds) in [**21**] (resp. in [**35**]), also see [**28**, Remark 3.5] and [**30**].

3.1. GKM graphs induced from GKM manifolds. We first recall how GKM manifolds in the sense of [**21**] (i.e., [**28**, Definition 2.1]) define GKM graphs. Let M be a $2m$-dimensional GKM manifold with T^n-action. Then, the GKM graph $(\Gamma_M, \alpha_M, \nabla_M)$ is defined by the following way. The abstract graph Γ_M is the orbit space of one-skeleton of M. Namely, the set of vertices V is the set of fixed points M^T, and an edge connecting $p, q \in V$ is an embedding 2-sphere connecting $p, q \in M^T$, also see [**28**]. The axial function $\alpha_M : E \to H^2(BT^n)$ is defined by using the tangential representations around fixed points. More precisely, using an invariant almost complex structure on M and the differentiable slice theorem, every tangent space T_pM on $p \in M^T$ decomposes into the following irreducible complex T^n-representation spaces:

$$T_pM = \oplus_{i=1}^m V(\alpha_{i,p}), \tag{3.1}$$

where $V(\alpha_{i,p})$ is the complex one-dimensional representation space with $\alpha_{i,p} \in \mathrm{Hom}(T^n, S^1) \simeq H^2(BT^n)$. From this fact, we also know that the graph Γ_M is an m-valent graph. Now $V(\alpha_{i,p})$ may be regarded as the complex tangent space of the embedded 2-sphere $e_{i,p}$ (this 2-sphere is equivariantly diffeomorphic to $\mathbb{C}P^1$ with the standard T^1-action, see [**28**, Example 2.3]). Therefore, for $E_p = \{e_{1,p}, \ldots, e_{m,p}\}$, the axial function is defined as $\alpha_M(e_{i,p}) = \alpha_{i,p} \in H^2(BT^n)$. The connection ∇_M is determined by the splitting of the restricted tangent bundle TM of M to the embedded 2-sphere $e \in E$. More precisely, because e is isomorphic to $\mathbb{C}P^1$, its restricted tangent bundle $TM|_e$ split into the equivariant complex line bundles:

$$TM|_e = \oplus_{i=1}^m \mathbb{L}_{i,e},$$

where $\mathbb{L}_{i,e}$ is the complex line bundle over $\mathbb{C}P^1$ for all $i = 1, \ldots, m$. Recall that the complex line bundle $\mathbb{L}$ over $\mathbb{C}P^1$ can be classified by its 1st Chern class $c_1(\mathbb{L}) \in H^2(\mathbb{C}P^1; \mathbb{Z}) \simeq \mathbb{Z}$. Therefore, the complex line bundle $\mathbb{L}_{i,e}$ is isomorphic to

$$\mathbb{L}_{i,e} \equiv S^3 \times_{S^1} \mathbb{C}_{\rho_i},$$

where S^1 acts on $S^3 \subset \mathbb{C}^2$ by the scaler multiplication and on $\mathbb{C}_{\rho_i}$ by the representation $\rho_i : S^1 \to S^1$. Note that this representation is determined by $\rho_i(z) = z^{a_i}$ for the integer $a_i = c_1(\mathbb{L}_{i,e})$. Then, each line bundle $\mathbb{L}_{i,e}$ satisfies that $\mathbb{L}_{i,e}|_p = V(\alpha_{i,p})$ and $\mathbb{L}_{i,e}|_q = V(\alpha_{i,q})$ for two fixed point $p, q \in V$ connected by the 2-sphere $e \in E$. Therefore, the bijective map $(\nabla_M)_e : E_p \to E_q$ is defined by $(\nabla_M)_e(e_{i,p}) = e_{i,q}$. Consequently, by definition of GKM manifold, $(\Gamma_M, \alpha_M, \nabla_M)$ is a GKM graph. This labeled graph $(\Gamma_M, \alpha_M, \nabla_M)$ is called an *induced GKM graph* from a GKM manifold M.

EXAMPLE 3.1 (toric manifold). By using the arguments in [**28**, Example 2.3], we have that the induced GKM graph from $\mathbb{C}P^2$ is the left GKM graph in Figure 1. More generally, the GKM graph of a non-singular complete toric variety is determined by the one-skeleton of its T^n-orbit space and the dual of its isotropy weight vectors (i.e., its characteristic functions, see [**31**] for details).

3.2. Torus graphs induced from torus manifolds. We next define a torus graph from a torus manifold introduced in [**35**]. Let M be a torus manifold (see the definition in [**28**, Example 2.5]). By definition of torus manifold, with the method similar to that demonstrated in Section 3.1, the induced abstract n-valent graph Γ_M can be defined as the orbit space of its one-skeleton. Then, a torus axial function $\alpha_M : E(\Gamma_M) \to H^2(BT^n)$ is defined as follows. We first remark that a torus manifold does not have an invariant almost complex structure in general, unless GKM manifolds (see Section 3.1). This means that there is no canonical way to regard the tangent space T_pM on $p \in M^T$ as the complex T^n-representation, i.e., the decomposition (3.1) does not hold canonically. To regard T_pM as the complex representation space, we need the notion of *omniorientation* (see [**24, 6**] or [**8, 31, 32**] also). Let M_i, $i = 1, \ldots, m$, be a *characteristic submanifold* of M, i.e., a codimension 2 torus submanifold in M; equivalently, the fixed point wise connected components of some circle subgroups in T^n. We fix orientations on a torus manifold M and all characteristic submanifolds $M_1, \ldots, M_m$. This fixed orientations on M, $M_1, \ldots, M_m$ is called an *omniorientation* on M and denote it as $\mathcal{O}$. We call a torus manifold M with fixed omniorientation $\mathcal{O}$ an *omnioriented torus manifold* and denote it as $(M, \mathcal{O})$. Because there are exactly 2 orientations on each manifolds, there are exactly 2^{m+1} choices of omniorientations on M, where m is the number of characteristic submanifolds. Let $(M, \mathcal{O})$ be an omnioriented torus manifold. By using the orientations on M_i and M, we may regard the normal bundle of M_i in M as the invariant complex line bundle, say ν_i, for all $i = 1, \ldots, m$. Take a fixed point $p \in M^T$. Denote the restricted bundle of ν_i to p as $\nu_i|_p$ if $p \in M_i^T$. Because $\nu_i|_p$ is a complex one-dimensional representation space, we may write $\nu_i|_p = V(\alpha_{i,p})$ for some non-trivial weight $\alpha_{i,p} : T^n \to S^1$. Therefore, T_pM is isomorphic to

$$\oplus_{i \in I_p} \nu_i|_p = \oplus_{i \in I_p} V(\alpha_{i,p}),$$

where $I_p \subset \{1, \ldots, m\}$ is the subset which satisfies $\cap_{i \in I_p} M_i = \{p\}$; therefore, $|I_p| = n$. Hence, the decomposition (3.1) is canonically determined by the omniorientation $\mathcal{O}$. Moreover, we may also regard the normal bundle of each invariant S^2 as the complex bundle by the restrictions of some ν_i's. Therefore, with the method similar to that demonstrated in Section 3.1, we can define a torus graph $(\Gamma_M, \alpha_M, \nabla_M)$ from $(M, \mathcal{O})$. By Proposition 2.2, the connection ∇_M is uniquely determined, i.e., the decomposition of the restricted tangent bundle over an invariant S^2 is automatically determined. So the labeled graph (Γ_M, α_M) (without connection) defined from an omnioriented torus manifold $(M, \mathcal{O})$ is called an *induced torus graph* from $(M, \mathcal{O})$.

Remark 3.2. If there is an invariant almost complex structure on a torus manifold M, e.g., toric manifolds, then there is the canonical omniorientation $\mathcal{O}_{\mathbb{C}}$ which is determined by the almost complex structure. In this case, the induced torus graph is a GKM graph (also see [**19**]). More generally, there is a torus manifold with an invariant stably complex structure, called a *unitary toric manifold* (see [**36, 32**] and [**40**] for stably complex structure), e.g., quasitoric manifolds. By using the invariant stably complex structure, there is the canonical omniorientation on M. Unless the case when torus manifolds have almost complex structures, a torus graph induced from an invariant stably complex structure on a torus manifold is not always a GKM graph (see Figure 2).

The main difference between torus graphs and GKM graphs is the existence of the torus axial function described in the right figure in Figure 2. The left one in Figure 2 is induced from the omniorientation induced from the canonical S^1-invariant complex structure on $\mathbb{C}P^1$; more precisely, this is induced by the S^1-action on $\mathbb{C}P^1$ by $[z_0 : z_1] \mapsto [z_0 : tz_1]$ for $[z_0 : z_1] \in \mathbb{C}P^1$ and $t \in S^1$. On the other hand, the right one in Figure 2 is induced from the different omniorientation which is obtained by changing the orientation on the north pole of $\mathbb{C}P^1$; this example can be regarded as the S^1-action on $S^2 \subset \mathbb{C} \oplus \mathbb{R}$ by $(z, r) \mapsto (tz, r)$ for $(z, r) \in S^2$ and $t \in S^1$.

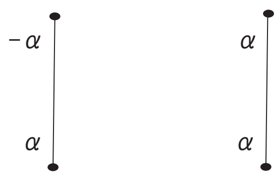

Figure 2. The GKM graph (left) induced from the standard T^1-action on $\mathbb{C}P^1$, and the torus graph (right) induced from the T^1-action on $S^2 \subset \mathbb{C} \oplus \mathbb{R}$.

Example 3.3. From T^n-action on $S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ defined in [**28**, Example 2.5], we can define torus graphs which are not GKM graphs. For example, the right graph in Figure 2 is the torus graph induced from the case when $n = 1$; the right graph in Figure 1 is that of the case when $n = 2$. In general, the torus graph induced from the T^n-action on S^{2n} is the graph $\Gamma = (V, E)$ such that $V = \{p, q\}$ and p, q are connected by n edges $E = \{e_1, \ldots, e_n\}$, and the axial function $\alpha : E \to H^2(BT^n)$ is defined by $\alpha(e_i) = \alpha(\overline{e_i}) = \alpha_i$ for some basis $\{\alpha_1, \ldots, \alpha_n\}$ of $H^2(BT^n)$.

4. Sheaves on GKM (torus) graphs and equivariant cohomology

Let (Γ, α) be a GKM (or torus) graph. In order to introduce the definition of (graph) equivariant cohomology $H^*(\Gamma, \alpha)$ of (Γ, α), we introduce sheaves on (Γ, α) and its global sections introduced in [**4**] (also see [**2, 10, 42**]). Note that in [**4**] the notion of sheaves is defined on the labeled graphs, called *moment graphs*. From a moment graph, it is easy to induce a GKM graph; moreover, the notion of sheaves also can be generalized to torus graphs naturally.

4.1. Sheaves on labeled graphs. Let us define the sheaves on GKM graphs and torus graphs.

Definition 4.1. A *sheaf* $\mathcal{M}$ on (Γ, α) is the datum $\mathcal{M} = (\{M_p\}, \{M_e\}, \{\rho_{p,e}\})$ such that

1. M_p is an $H^*(BT)$-module defined for each $p \in V(\Gamma)$;
2. M_e is an $H^*(BT)$-module with $\alpha(e)M_e = 0$, i.e., this also becomes an $H^*(BT)/\langle\alpha(e)\rangle$-module, for each edge $e \in E(\Gamma)$, where $\langle\alpha(e)\rangle \subset H^*(BT)$ is the ideal generated by $\alpha(e) \in H^2(BT)$;
3. $\rho_{p,e} : M_p \to M_e$ is a homomorphism of $H^*(BT)$-modules for any $p \in V(\Gamma)$ with $i(e) = p$;

Example 4.2. The following datum, called $\mathcal{A} = (\{A_p\}, \{A_e\}, \{\rho_{p,e}\})$, give the typical example of a sheaf on (Γ, α): $A_p = H^*(BT)$ for all $p \in V(\Gamma)$; $A_e = H^*(BT)/\langle\alpha(e)\rangle$ for all $e \in E(\Gamma)$; and $\rho_{p,e} : H^*(BT) \to H^*(BT)/\langle\alpha(e)\rangle$ is the natural projection. This sheaf $\mathcal{A}$ is called a *structure sheaf* on (Γ, α) in [**10**] (or a *sheaf of rings* in [**4**]). Figure 3 shows the structure sheaves of the torus graphs in Figure 1.

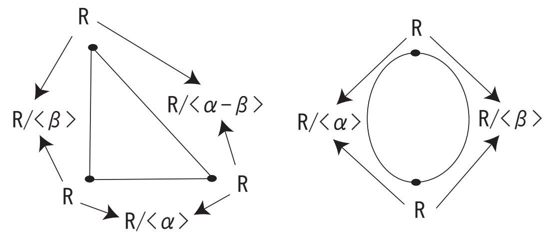

Figure 3. The structure sheaves on the GKM graph and the torus graph in Figure 1. In the figure, $R = H^*(BT^2) \simeq \mathcal{R}[\alpha, \beta]$ and each arrow represents the natural projection.

Note that we may regard a sheaf $\mathcal{M}$ on (Γ, α) as a sheaf on a "topological space" Γ in the usual sense (see [**4**], cf [**10**]). Now we may regard the abstract graph $\Gamma = (V(\Gamma), E(\Gamma))$ as the abstract set $S(\Gamma) = V(\Gamma) \sqcup E(\Gamma)$ (this is a finite set). Define a topology on this abstract set $S(\Gamma)$ as follows: a finite subset $U \subset S(\Gamma)$ is open if and only if $U = \emptyset$, $U \subset E(\Gamma)$ or if $p \in U \cap V(\Gamma)$ then $E_p(\Gamma) \subset U$. We denote $U \cap E(\Gamma) = E(U)$ and $U \cap V(\Gamma) = V(U)$. Under this topology, for the given sheaf $\mathcal{M}$, we define *sections* $\mathcal{M}(U)$ of $\mathcal{M}$ over an open set $U \subset S(\Gamma)$ as follows:

$$\begin{aligned}
\mathcal{M}(U) &:= \left\{ \bigoplus_{p \in V(U)} f_p \oplus \bigoplus_{e \in E(U)} f_e \;\middle|\; \rho_{p,e}(f_p) = f_e \text{ if } p = i(e) \in U \right\} \\
&\subset \bigoplus_{p \in V(U)} M_p \oplus \bigoplus_{e \in E(U)} M_e = \bigoplus_{x \in U} M_x; \\
\mathcal{M}(U) &:= \bigoplus_{e \in E(U)} M_e \text{ if } V(U) = \emptyset; \\
\mathcal{M}(\emptyset) &:= \{0\}.
\end{aligned} \tag{4.1}$$

Then, $\mathcal{M}(U)$ is an $H^*(BT)$-submodule of the $H^*(BT)$-module $\oplus_{x \in U} M_x$. For an inclusion $U' \subset U$ of open subsets in $S(\Gamma)$, there exists the natural projection $\widetilde{\rho}_{UU'} : \oplus_{x \in U} M_x \to \oplus_{x \in U'} M_x$, i.e., the map induced from the restriction maps defined by $\widetilde{\rho}_{UU'}|_{M_x} = \mathrm{Id}$ if $x \in U' \subset U$ and $\widetilde{\rho}_{UU'}|_{M_x} = 0$

if $x \in U \setminus U'$. It is easy to check that this projection induces the restricted homomorphism $\rho_{UU'} : \mathcal{M}(U) \to \mathcal{M}(U')$ (might not be surjective). Hence, the sheaf $\mathcal{M}$ induces the presheaf on the topological space $S(\Gamma) = V(\Gamma) \sqcup E(\Gamma)$. Moreover, it follows from the definition of the topology on $S(\Gamma)$ that this presheaf on $S(\Gamma)$ satisfies the axiom of a sheaf (see e.g. [**29**]).

4.2. Global sections of sheaves and equivariant cohomology. Now we call the section $\mathcal{M}(S(\Gamma))$ *global sections* of the sheaf $\mathcal{M}$ on (Γ, α), and we denote it by $H^0(\Gamma, \alpha; \mathcal{M})$. More precisely,

$$H^0(\Gamma, \alpha; \mathcal{M}) = \left\{ \bigoplus_{p \in V(\Gamma)} f_p \oplus \bigoplus_{e \in E(\Gamma)} f_e \;\middle|\; \rho_{p,e}(f_p) = f_e \text{ for } p = i(e) \right\}.$$

The following lemma is straightforward:

LEMMA 4.3. *Let $\mathcal{M} = (\{M_p\}, \{M_e\}, \{\rho_{p,e}\})$ be a sheaf on (Γ, α). If there exists an $H^*(BT)$-module R such that $M_p = R$ for all $p \in V(\Gamma)$, then the following $H^*(BT)$-module isomorphism holds:*

$$H^0(\Gamma, \alpha; \mathcal{M}) \simeq \{f : V(\Gamma) \to R \mid \rho_{p,e}(f(p)) = \rho_{q,e}(f(q)) \text{ for } p = i(e), q = t(e)\}.$$

EXAMPLE 4.4. Let $\mathcal{R} = (\{R_p\}, \{R_e\}, \{\rho_{p,e}\})$ be the sheaf on (Γ, α) defined by $R_p = R$ for all $p \in V(\Gamma)$, $R_e = \{0\}$ for all $e \in E(\Gamma)$ and $\rho_{p,e}$ is the zero-map. Namely, $\mathcal{R} = (\{R\}, \{\{0\}\}, \{0\})$. Then, by Lemma 4.3,

$$H^0(\Gamma, \alpha; \mathcal{R}) \simeq \bigoplus_{p \in V(\Gamma)} R.$$

Let $\mathcal{A}$ be the structure sheaf on (Γ, α). By Lemma 4.3, its global sections $H^0(\Gamma, \alpha; \mathcal{A})$ is isomorphic to the following $H^*(BT)$-module:

(4.2)

$$\{f : V(\Gamma) \to H^*(BT) \mid f(p) - f(q) \equiv 0 \mod \alpha(e) \text{ for } p = i(e), q = t(e)\} \subset \bigoplus_{p \in V(\Gamma)} H^*(BT).$$

The relation "$f(p) - f(q) \equiv 0 \mod \alpha(e)$" is often denoted as "$f(p) \equiv_{\alpha(e)} f(q)$". Because $\oplus_{p \in V(\Gamma)} H^*(BT)$ naturally has the structure of an $H^*(BT)$-algebra, $H^0(\Gamma, \alpha; \mathcal{A})$ may be regarded as an $H^*(BT)$-subalgebra of $\oplus_{p \in V(\Gamma)} H^*(BT)$. Moreover, by using the graded ring structure of $H^*(BT)$, $H^0(\Gamma, \alpha; \mathcal{A})$ may also be regarded as a graded ring. Namely, $H^0(\Gamma, \alpha; \mathcal{A})$ has the structure of a graded $H^*(BT)$-algebra. We denote it as $H^*(\Gamma, \alpha)$.

EXAMPLE 4.5. Let $(\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2})$ be the GKM graph induced from the standard T^2-action on $\mathbb{C}P^2$, i.e., the left graph in Figure 1. Then, the global sections of its structure sheaf (see the left sheaf in Figure 3) $H^*(\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2})$ is as follows:

$$\{f : \{p, q, r\} \to H^*(BT^2) \mid f(p) \equiv_\alpha f(q),\ f(p) \equiv_\beta f(r),\ f(r) \equiv_{\alpha - \beta} f(q)\}.$$

Let $(\Gamma_{S^4}, \alpha_{S^4})$ be the torus graph induced from the standard T^2-action on $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$, i.e., the right graph in Figure 1. Then, the global sections of its structure sheaf (see the right sheaf in Figure 3) $H^*(\Gamma_{S^4}, \alpha_{S^4})$ is as follows:

$$\{f : \{p, q\} \to H^*(BT^2) \mid f(p) \equiv_\alpha f(q) \text{ and } f(p) \equiv_\beta f(q)\}.$$

By the GKM Theorem (see [**28**, Theorem 3.6]), we have the following theorem:

THEOREM 4.6. *Let (M, T) be an equivariantly formal GKM manifold and (Γ_M, α_M) be its induced GKM graph. Then, $H_T^*(M; \mathbb{Q}) \simeq H^*(\Gamma_M, \alpha_M)$ as a graded $H^*(BT; \mathbb{Q})$-algebra, where the structure sheaf $\mathcal{A}$ of (Γ_M, α_M) is defined by $A_p = H^*(BT; \mathbb{Q})$ for all $p \in V(\Gamma_M)$.*

Motivating by this fact, the graded ring (or the $H^*(BT)$-algebra) $H^*(\Gamma, \alpha)$ defined by (4.2) is called a *(graph) equivariant cohomology* of (Γ, α). Note that for an abstract graph (Γ, α) there might not be a geometric object M which induces (Γ, α). The description (4.2) is also known as a *GKM description* of an equivariant cohomology.

Moreover, the following theorem is proved by Maeda-Masuda-Panov [**35**]:

Theorem 4.7. *Let (M,T) be a torus manifold with $H^{odd}(M)=0$ and (Γ_M,α_M) be its induced torus graph. Then, $H_T^*(M;\mathbb{Z})\simeq H^*(\Gamma_M,\alpha_M)$ as a graded $H^*(BT;\mathbb{Z})$-algebra,, where the structure sheaf $\mathcal{A}$ of (Γ_M,α_M) is defined by $A_p=H^*(BT;\mathbb{Z})$ for all $p\in V(\Gamma_M)$.*

Remark 4.8. The GKM descriptions for the other topological invariants, such as equivariant K-theory, equivariant complex cobordism or more general equivariant complex oriented cohomology theory, are studied in [**8, 22, 23**] etc.

5. Combinatorial formula of equivariant cohomology ring

In Section 4, we introduce the graph equivariant cohomology $H^*(\Gamma,\alpha)$ of a GKM (and torus) graph (Γ,α). However, the definition of $H^*(\Gamma,\alpha)$, i.e., the GKM description, does not say anything about the generators and relations as a ring or an algebra. So, for given (Γ,α), finding generators and relations of $H^*(\Gamma,\alpha)$ is the natural question. In this section, we introduce the Maeda-Masuda-Panov theorem [**35**] which states combinatorially the generators and relations of $H^*(\Gamma,\alpha)$ of a torus graph (Γ,α). This theorem recovers some well-known results in toric geometry and toric topology: the equivariant cohomology ring of complete non-singular toric varieties by Danilov-Jurkiewicz [**13**]; that of quasitoric manifolds by Davis-Januszkiewicz [**9**]; that of torus manifolds with vanishing odd degree cohomologies by Masuda-Panov [**37**].

5.1. Thom class of a torus subgraph. Let (Γ,α) be a torus graph. To describe the combinatorial formula of $H^*(\Gamma,\alpha)$, we need the notion of a Thom class of a torus subgraph of (Γ,α).

Let Γ' be an $(n-h)$-valent subgraph of Γ for $0\le h\le n$, and ∇ be the unique connection on (Γ,α). We call Γ' a *torus subgraph*, if Γ' is closed under the connection ∇, i.e., for all $e\in E(\Gamma')$ with $i(e)=p,t(e)=q\in V(\Gamma')$, the restricted bijection $\nabla_e|_{E_p(\Gamma')}:E_p(\Gamma')\to E_q(\Gamma')$ is well-defined. This is equivalent to the following condition: there exists an h-dimensional subtorus $T''\subset T$ such that the composition function $\alpha_{\Gamma'}:=\pi\circ\alpha|_{E(\Gamma')}:E(\Gamma')\to H^2(BT')$ satisfies the axiom of a torus axial function, where $\alpha|_{E(\Gamma')}:E(\Gamma')\to H^2(BT)$ is the restricted torus axial function and $\pi:H^2(BT)\to H^2(BT)/H^2(BT'')\simeq H^2(BT')$ is the projection for an $(n-h)$-dimensional torus $T'=T/T''$. Namely, the restricted labeled graph $(\Gamma',\alpha_{\Gamma'})$ is again a torus graph.

For an $(n-h)$-valent torus subgraph Γ', the symbol $N_p(\Gamma')$ represents the set of all normal edges of Γ' on $p\in V(\Gamma')$, i.e., $N_p(\Gamma')=E_p(\Gamma)\setminus E_p(\Gamma')$. Because Γ' is an $(n-h)$-valent graph, $|N_p(\Gamma')|=h$. Then, we define the function $\tau':V(\Gamma)\to H^{2h}(BT)$ as follows:

$$\tau'(p)=\begin{cases}\prod_{e\in N_p(\Gamma')}\alpha(e) & p\in V(\Gamma')\\ 0 & p\notin V(\Gamma')\end{cases}$$

By definition of torus subgraph, it is easy to check that $\tau'\in H^{2h}(\Gamma,\alpha)$. We call this element τ' a *Thom class* of Γ'. Figure 4 shows examples of Thom classes of GKM subgraphs in Figure 1. Note that we formally define $\tau_\Gamma=1\in H^0(\Gamma,\alpha)$, i.e., $\tau_\Gamma(p)=1$ for all $p\in V(\Gamma)$, and $\tau_\emptyset=0\in H^0(\Gamma,\alpha)$.

Remark 5.1. From geometric point of view, the equivariant Thom class of a codimension $2h$ torus submanifold X of an omnioriented torus manifold $(M,\mathcal{O})$ can be defined by the following way (see [**36**]). Let ν be the normal bundle of X and $Th(\nu)$ be its Thom space, i.e, $Th(\nu)=M/(D(\nu))^c$ is the collapsing space of M on $(D(\nu))^c$, where $D(\nu)^c$ is the complement of the unit disk bundle of ν embedded into M. Because ν has the induced orientation from the omniorientation (in fact, this becomes a complex h-dimensional vector bundle), there is the following isomorphism, called the Thom isomorphism:

$$H_T^*(X)\to H_T^{*+2h}(Th(\nu)).$$

On the other hand, there is the induced homomorphism $H^*(Th(\nu))\to H_T^*(M)$ from the collapsing map $M\to Th(\nu)$. Therefore, by taking the composition of these homomorphisms, we have the following homomorphism:

$$\varphi_X:H_T^0(X)\to H_T^{2h}(M).$$

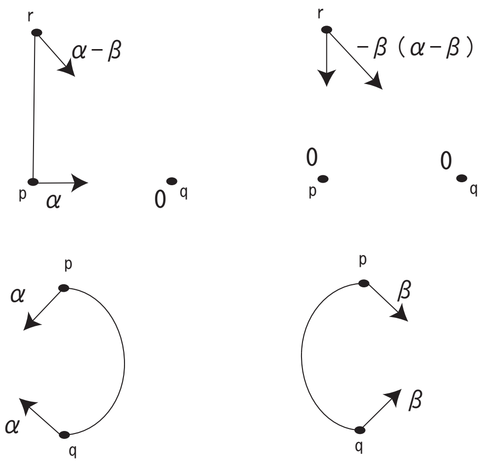

FIGURE 4. Thom classes of some torus subgraphs of torus graphs in Figure 1.

Then, we can define the equivariant Thom class of the codimension $2h$ torus submanifold X by $\varphi_X(1) = \tau_X \in H^{2h}(M)$. Note that $\iota_X^*(\tau_X) = e^T(\nu)(= c_h^T(\nu)) \in H^{2h}(X)$, where $\iota_X^* : H_T^*(M) \to H_T^*(X)$ is the induced homomorphism from the inclusion $\iota_X : X \to M$ and $e^T(\nu)$ (resp. $c_h^T(\nu)$) is the equivariant Euler class (resp. top Chern class) of ν. Because of the definition of a torus graph, the torus subgraph Γ' is induced from some torus submanifold X; therefore, the Thom class τ' of Γ' is the combinatorial interpretation of the Thom class τ_X of X.

5.2. The combinatorial formula of $H^*(\Gamma, \alpha)$ of torus graphs. Let (Γ, α) be a torus graph. To state the Maeda-Masuda-Panov theorem, we prepare some notation. The symbol $\mathbb{Z}[\tau_H \mid H \subset \Gamma]$ represents the polynomial ring with integer coefficient generated by Thom classes of all torus subgraphs $H \subset \Gamma$. We set $\deg \tau_H = 2h$ if H is an $(n-h)$-valent graph, and define $\tau_\Gamma = 1$ and $\tau_\emptyset = 0$. For two torus subgraphs G, H, if $G \cap H \neq \emptyset$, then there exists the unique minimal torus subgraph that contains both G and H (see [**37**, Proposition 5.2]); we denote it as $G \vee H$. Now we may state the Maeda-Masuda-Panov theorem:

THEOREM 5.2 (Maeda-Masuda-Panov). *Let (Γ, α) be a torus graph. Then, its graph equivariant cohomology $H^*(\Gamma, \alpha)$ is isomorphic to the following ring as the graded ring:*

$$\mathbb{Z}[\Gamma, \alpha] := \mathbb{Z}[\tau_H \mid H \subset \Gamma]/\mathcal{I} \tag{5.1}$$

where the ideal $\mathcal{I}$ is generated by the following polynomials:

$$\tau_G \tau_H - \tau_{G \vee H} \sum_{K \in G \cap H} \tau_K,$$

where $K \in G \cap H$ runs through all connected components of $G \cap H$.

Note that if $G \cap H = \emptyset$ then $\tau_G \tau_H = 0$ in $\mathbb{Z}[\Gamma, \alpha]$, because $\tau_\emptyset = 0$. The ring $\mathbb{Z}[\Gamma, \alpha]$ in Theorem 5.2 is called a *face ring* of (Γ, α).

The face ring $\mathbb{Z}[\Gamma, \alpha]$ also has the structure of a graded $H^*(BT)$-algebra by the following fact (also see [**36, 32**]): for all $\alpha \in H^2(BT)$, there exist integers $k_1, \ldots, k_m$ such that the following map induces an injective homomorphism $\pi^* : H^*(BT) \to \mathbb{Z}[\Gamma, \alpha]$:

$$\pi^* : \alpha \mapsto \alpha(:= \alpha\tau_\Gamma) = \sum_{i=1}^{m} k_i \tau_i$$

where m is the number of all $(n-1)$-valent torus subgraphs $\Gamma_1, \ldots, \Gamma_m$ in (Γ, α) and τ_i is the Thom class of Γ_i, i.e., $\deg \tau_i = 2$, for all $i = 1, \ldots, m$. We also note the following fact:

PROPOSITION 5.3. *The isomorphism $H^*(\Gamma, \alpha) \simeq \mathbb{Z}[\Gamma, \alpha]$ in Theorem 5.2 is also a graded $H^*(BT)$-algebra isomorphism.*

Using Theorem 4.7 and Theorem 5.2, we have the following fact proved in [**37**].

COROLLARY 5.4. *Let M be a torus manifold. If $H^{odd}(M) = 0$, then its equivariant cohomology $H^*(BT;\mathbb{Z})$-algebra is isomorphic to the face ring* (5.1) *which is generated by the Thom classes of all torus submanifolds.*

EXAMPLE 5.5. Let $(\Gamma_{S^4}, \alpha_{S^4})$ be the torus graph induced from the T^2-action on S^4 (see the right graph in Figure 1). Then, Γ_{S^4} is constructed from two edges (1-valent torus subgraphs) e_1 (the left edge), e_2 (the right edge) and two vertices (0-valent torus subgraphs) p, q; we put these Thom classes as τ_1, τ_2, τ_p and τ_q, respectively. By Theorem 5.2, we have that its graph equivariant cohomology $H^*(\Gamma_{S^4}, \alpha_{S^4})$ is isomorphic to the following ring (cf Example 4.5):

$$\begin{aligned}&\mathbb{Z}[\tau_1, \tau_2, \tau_p, \tau_q]/\langle \tau_1\tau_2 - (\tau_p + \tau_q), \tau_p\tau_q\rangle \\ \simeq&\mathbb{Z}[\tau_1, \tau_2, \tau_p]/\langle \tau_1\tau_2 - (\tau_p + \tau_q), \tau_p(\tau_1\tau_2 - \tau_p)\rangle,\end{aligned}$$

where $\deg \tau_1 = \deg \tau_2 = 2$ and $\deg \tau_p = \deg \tau_q = 4$. By Corollary 5.4, this ring is isomorphic to $H^*_T(S^4;\mathbb{Z})$.

Moreover, by definition of Thom classes, $\pi^*(\alpha) = \tau_2$ and $\pi^*(\beta) = \tau_1$ (also see Figure 4). Because $H^{odd}(S^4;\mathbb{Z}) = 0$, its ordinary cohomology $H^*(S^4;\mathbb{Z})$ is obtained by $H^*_T(S^4;\mathbb{Z})/H^{>0}(BT;\mathbb{Z})$. Therefore, we have that

$$H^*(S^4;\mathbb{Z}) \simeq H^*_T(S^4;\mathbb{Z})/H^{>0}(BT) \simeq \mathbb{Z}[\Gamma_{S^4}, \alpha_{S^4}]/\text{Im } \pi^{>0} \simeq \mathbb{Z}[\Gamma_{S^4}, \alpha_{S^4}]/\langle \alpha, \beta\rangle \simeq \mathbb{Z}[\tau_p]/\langle \tau_p^2\rangle.$$

EXAMPLE 5.6. Let $(\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2})$ be the torus graph induced from the T^2-action on $\mathbb{C}P^2$ (see the left graph in Figure 1). Then, $\Gamma_{\mathbb{C}P^2}$ is constructed from three edges pq, pr, qr and three vertices p, q, r; we put these Thom classes as τ_{pq}, τ_{pr}, τ_{qr} and τ_p, τ_q, τ_r, respectively. By the relation of $\mathbb{Z}[\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2}]$ in Theorem 5.2, it is easy to get the following equivalence relations:

$$\tau_{pq}\tau_{pr} = \tau_p,\ \tau_{pq}\tau_{qr} = \tau_q,\ \tau_{qr}\tau_{pr} = \tau_r.$$

Therefore, we may reduce the generators of $\mathbb{Z}[\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2}]$ into only generators with degree two and the graph equivariant cohomology $H^*(\Gamma_{\mathbb{C}P^2}, \alpha_{\mathbb{C}P^2})$ is isomorphic to the following ring (cf Example 4.5):

$$\mathbb{Z}[\tau_{pq}, \tau_{pr}, \tau_{qr}]/\langle \tau_{pq}\tau_{pr}\tau_{qr}\rangle.$$

By Corollary 5.4, this ring is isomorphic to $H^*_T(\mathbb{C}P^2;\mathbb{Z})$.

Moreover, by definition of Thom classes, $\pi^*(\alpha) = \tau_{pr} - \tau_{qr}$ and $\pi^*(\beta) = \tau_{pq} - \tau_{qr}$. Therefore, with the method similar to that demonstrated in Example 5.5, we have that

$$H^*(\mathbb{C}P^2;\mathbb{Z}) \simeq H^*_T(\mathbb{C}P^2;\mathbb{Z})/\langle \alpha, \beta\rangle \simeq \mathbb{Z}[\tau_{pq}]/\langle \tau_{pq}^3\rangle.$$

Note that, in Example 5.6, the generators of $\mathbb{Z}[\Gamma, \alpha]$ reduce to the generators with degree two. The following corollary says that there is a combinatorial reason for this phenomenon.

COROLLARY 5.7. *Let (Γ, α) be a torus graph. Let $\Gamma_1, \ldots, \Gamma_m$ be all $(n-1)$-valent torus subgraphs and $\tau_1, \ldots, \tau_m$ be their Thom classes respectively. Assume that for every pair Γ_i, Γ_j, $\Gamma_i \cap \Gamma_j = \emptyset$ or connected. Then, the graph equivariant cohomology $H^*(\Gamma, \alpha)$ is isomorphic to*

$$\mathbb{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I} \tag{5.2}$$

where the ideal $\mathcal{I}$ is generated by the following monomials:

$$\prod_{i\in I}\tau_i \quad \text{for } I \subset \{1, \ldots, m\} \text{ such that } \bigcap_{i\in I}\Gamma_i = \emptyset.$$

The ring (5.2) is called a *Stanley-Reisner ring*. By Corollary 5.7, we have the following geometric fact:

COROLLARY 5.8. *Let M be a torus manifold. If $H^*(M)$ is generated by the 2nd degree cohomology, then its equivariant cohomology is isomorphic to the Stanley-Reisner ring* (5.2) *generated by Thom classes of all characteristic submanifolds.*

In particular, if M is a complete non-singular toric variety or a quasitoric manifold, then M satisfies the assumption of Corollary 5.8. Therefore, we get the well-known Danilov-Jurkiewicz theorem and Davis-Januszkiewicz theorem from Theorem 5.2.

Remark 5.9. For the other class of GKM graphs (in particular, induced from flag manifolds), the ring structures of their graph equivariant cohomology are studied in [**12, 13, 41, 42, 43**].

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